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Global Attractor for Damped Wave Equations with Nonlinear Memory

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Abstract Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. We consider the longtime dynamics of a class of damped wave equations with a nonlinear memory term

$$u_{tt} + \alpha u_t - \Delta u - \int_0^t \mu(t-s)|u(s)|^\beta u(s)\mathrm{d}s + g(u) = f.$$

Based on a time-uniform priori estimate method, the existence of the compact global attractor is proved for this model in the phase space $H_0^1(\Omega) \times L^2(\Omega)$.

Keywords global attractor; nonlinear memory term; damped wave equation.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary Γ, and α and β be positive constants. We consider the following damped wave equations with nonlinear memory term:

$$u_{tt} + \alpha u_t - \Delta u - \int_0^t \mu(t-s)|u(s)|^\beta u(s)\mathrm{d}s + g(u) = f \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \tag{2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for} \quad x \in \Omega. \tag{3}$$

Many physical phenomena are properly described by partial differential equations where the dynamics is influenced by the past history of one or more variables. A wave equation of hyperbolic type with a convolution term describes simple viscoelastic materials with fading memory. This equation has arisen from the theory of isothermal viscoelastic [3, 16], which describes a homogeneous and isotropic viscoelastic solid.

A problem similar to (1)–(3) with linear memory has been studied in [7, 8, 14, 15, 17]. For the nonlinear one-dimensional equation, Dafermos [4], exploring the dissipative properties of the equation, showed that the system is well posed provided the initial data are small enough,

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whereas for the n-dimensional linear system the author proved the asymptotic stability of the solutions. For 3-dimensional isotropic and homogeneous materials, Dassios and Zafiropoulus [5], using an asymptotic analysis, proved that the solution of the viscoelastic system of memory type has a uniform decay to zero provided that the relaxation kernel is the exponential function. Problem (1)–(3), without the memory term μ and with $f = 0$, has been proposed by Feireisl [7], who studied the longtime behavior of solutions and showed the existence of a universal attractor with the nonlinear term g to satisfy the so-called subcritical exponent growth condition. Ma and Zhong [12] investigated the existence of global attractors of strong solutions for the hyperbolic equations with linear memory using the semigroup approach.

Few have ever considered the nonlinear memory term. Cavalcanti [2] and Park [13] proved the existence and uniform decay of solutions for the wave equations with nonlinear boundary damping and boundary memory source term. That is, they only showed the existence and uniform decay with nonlinear memory term. In this paper, we consider the global attractor problem with nonlinear memory term. The contribution of this paper is that the authors deal with damped wave equations with nonlinear memory term.

The rest of the paper is organized as follows. In Section 2, we introduce the various assumptions, notations, definitions, propositions and theorems which will be needed later. In Section 3, we show the existence of absorbing sets in the phase space $\mathcal{H}$. Finally, we prove that the semigroup associated to the problem possesses a global attractor in $\mathcal{H}$.

2. Preliminaries

With usual notations, we introduce two Hilbert spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. The scalar product and the norm in $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ are denoted by $(\cdot,\cdot), |\cdot|; ((\cdot,\cdot)), \|\cdot\|$, respectively. The symbol $\mathcal{H}$ denotes the product space $V \times H$. We consider the strictly positive Laplace-Dirichlet operator on H

$$A = -\triangle, \quad \mathrm{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

generating, for $s \in \mathbb{R}$, the scale of Hilbert spaces

$$V_s = \mathrm{dom}(A^{\frac{s}{2}}), \quad \langle u, v\rangle_s = \langle A^{\frac{s}{2}}u, A^{\frac{s}{2}}v\rangle.$$

In particular, $V_0 = H$, $V_1 = V$. Whenever $s_1 > s_2$, the imbeding $V_{s_1} \subset V_{s_2}$ is compact and

$$\|u\|_{V_{s_1}} \geq \lambda_1^{(s_1-s_2)/2}\|u\|_{V_{s_2}}, \quad \forall u \in V_{s_1},$$

where λ_1 is the first eigenvalue of the operator $-\Delta$.

In the following, we consider an abstract damped linear equation in a Hilbert space $\bar{H}$

$$u_{tt} + \alpha u_t + \bar{A}u = f, \quad \alpha > 0, \tag{4}$$

$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \tag{5}$$

where $\bar{A}$ is an unbounded, self-adjoint, positive linear operator from $\mathrm{dom}\bar{A} \subset \bar{H}$ to $\bar{H}$. Assume that the injection $\bar{V} = \mathrm{dom}(\bar{A}^{1/2}) \subset \bar{H}$ is continuous and dense.

Next we introduce the existence and uniform decay of the above problem which will be needed later.

Proposition 1 ([18]) *Under the above assumptions, let f, u_0, u_1 be given such that*

$$f \in L^2(0,T;\bar{H}), \quad u_0 \in \bar{V}, \quad u_1 \in \bar{H}.$$

Then there exists a unique solution u to (4), (5), satisfying

$$u \in C(0,T;\bar{V}), \quad u_t \in C(0,T;\bar{H}).$$

Furthermore, if we assume that

$$f \in C(0,T;\bar{H}), \quad \frac{\partial f}{\partial t} \in L^2(0,T;\bar{H}), \quad u_0 \in \mathrm{dom}(\bar{A}), \quad u_1 \in \bar{V},$$

then u satisfies

$$u \in C(0,T;\mathrm{dom}(\bar{A})), \quad u_t \in C(0,T;\bar{V}), \quad u_{tt} \in C(0,T;\bar{H}).$$

Proposition 2 ([18]) *The hypotheses are those of Proposition 1 and we assume that $f \in C(0,T;\bar{H}), u_0 \in \bar{V}, u_1 \in \bar{H}$. Then if u is the solution of (4), (5), $\{u, u_t\} \in C(0,T;\bar{V}\times\bar{H})$, and if*

$$0 < \varepsilon \le \varepsilon_0, \quad \varepsilon_0 = \min\{\frac{\alpha}{4}, \frac{\lambda_1}{2\alpha}\},$$

then, for any $0 < t < T$, u satisfies the following relation

$$\|u(t)\|^2 + |u_t(t) + \varepsilon u(t)|^2 \le \{\|u_0\|^2 + |u_1 + \varepsilon u_0|^2\} \exp(-\frac{\varepsilon t}{2}) + \frac{2}{\varepsilon^2}|f|^2_{L^\infty(\mathbb{R}^+;\bar{H})}(1 - \exp(-\frac{\varepsilon t}{2})).$$

In order to establish the existence of attractors, we need the following related concepts of absorbing sets and Kuratowski measure.

Definition 1 *Let $\mathcal{B}_0$ be an open subset of a matric space $\mathcal{M}$, and $S(t)$ be a semigroup on $\mathcal{M}$. We say that $\mathcal{B}_0$ is an absorbing set of $S(t)$ if the orbit of any bounded set $\mathcal{B}$ enters into $\mathcal{B}_0$ after a certain time (which may depend on the set $\mathcal{B} \subset \mathcal{M}$). Namely, for any bounded set $\mathcal{B}$, there exists $t_1(\mathcal{B})$, such that*

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \quad \forall t \ge t_1(\mathcal{B}).$$

Definition 2 *Let $\mathcal{M}$ be a metric space. The Kuratowski measure of a noncompact set $\mathcal{B}$ in $\mathcal{M}$ is defined by*

$$\alpha_{\mathcal{M}}(\mathcal{B}) = \inf\{d : \mathcal{B} \text{ has a finite cover of open balls of } \mathcal{M} \text{ of diameter less than } d\}.$$

Theorem 1 ([18]) *Let $S(t)$ be a semigroup on a complete metric space $\mathcal{M}$. If the following hold*

(I) There exists a bounded absorbing subset $\mathcal{B}_0$ of $\mathcal{M}$;

(II) For any bounded set $\mathcal{B} \subset \mathcal{M}$ and number $\varepsilon > 0$, there exists $t_1(\varepsilon)$, such that for $t \ge t_1(\varepsilon)$, $\alpha_{\mathcal{M}} S(t)(\mathcal{B}) < \varepsilon$.

Then the ω-limit set of $\mathcal{B}_0$ is the connected and compact global attractor of $S(t)$.

In order to formulate problem (1)–(3) in a proper functional setting and describe the long-time behavior of solutions, we require some conditions on the nonlinear term, namely, we take

$g \in C^1(\mathbb{R})$ and denote

$$G(s) = \int_0^s \bar{g}(y)\mathrm{d}y.$$

There exist positive constants $\gamma \geq 0$, $\rho > \frac{(\gamma+2)}{\beta+2}(\int_\Omega 1\mathrm{d}x)^{1/\gamma-1/\beta}$, such that $g(s) = \rho|u(s)|^\gamma u(s)+\bar{g}$, where $\bar{g}$ satisfies the following conditions:

(g1) $\liminf_{|s|\to\infty} \frac{G(s)}{s^2} \geq 0$;

(g2) $\liminf_{|s|\to\infty} \frac{s\bar{g}(s)-c_1G(s)}{s^2} \geq 0$;

(g3) $|\bar{g}'(s)| \leq c_2(1+|s|^\nu)$ for some $\nu \geq 0$.

The above conditions are in fact redundant to prove the existence and uniqueness of finite energy solutions, but they are necessary to prove the asymptotic behavior results. This is due to the fact that the external force satisfies more restrictive conditions. We infer from (g1), (g2) that, for every $\eta > 0$, there exist positive constants C_η, C'_η such that

$$G(s) + \eta s^2 \geq -C_\eta, \quad \forall s \in \mathbb{R}, \tag{6}$$

$$s\bar{g}(s) - c_1G(s) + \eta s^2 \geq -C'_\eta, \quad \forall s \in \mathbb{R}. \tag{7}$$

A proper choice of η will be made when necessary.

The memory kernel μ is required to satisfy the following hypotheses:

(h1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$;

(h2) $\mu(s) \geq 0$, $\mu'(s) \leq 0$, $\forall s \in \mathbb{R}^+$;

(h3) $\int_0^\infty \mu(s)\mathrm{d}s = 1$;

(h4) There exist $m_1, m_2 > 0$, such that $m_1\mu(s) \leq -\mu'(s) \leq m_2\mu(s)$, $\forall s \in \mathbb{R}^+$.

Notice that the condition (h4) implies the exponential decay of μ. This requirment seems to be unavoidable in order to have the exponential decay of the associated problem [4, 6, 11].

For convenience we denote $\bar{g}$ by g and the equation (1)–(3) are written in the following forms

$$u_{tt} + \alpha u_t - \Delta u + \rho|u(t)|^\gamma u(t) - \int_0^t \mu(t-s)|u(s)|^\beta u(s)\mathrm{d}s + g(u) = f \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{8}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \tag{9}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for} \quad x \in \Omega. \tag{10}$$

The existence of weak and strong global solution of the above equation has been shown in Han [10]. This equation generates a semigroup $S(t)$ on $\mathcal{H}$.

In the following, we give our main result:

Theorem 2 *Assume that (h1)–(h4) and (g1)–(g3) hold, and ν, β satisfy the following conditions*

$$\begin{array}{lll} 0 \leq \nu, \beta, \gamma < \infty, \quad \beta \leq \gamma & \textit{when} & n = 1, 2, \\ 0 \leq \nu, \beta, \gamma < 2, \quad \beta \leq \gamma & \textit{when} & n = 3, \\ \beta = \gamma = \nu = 0, & \textit{when} & n \geq 4. \end{array} \tag{11}$$

Then the semigroup $S(t)$ of the equation (8)–(10) possesses a global attractor $\mathcal{A}$ which is compact, connected, and maximal in $\mathcal{H}$.

3. Existence of global attractor

In this section we are going to establish a time-uniform prior estimate of solutions u in $\mathcal{H}$. Then we apply the estimate to obtain the existence of the global solutions and bounded absorbing set. Finally, we establish the existence of the global attractor for the dynamical system $S(t)$ in the phase space $\mathcal{H}$.

3.1 Absorbing set in $\mathcal{H}$

We are now in a position to deal with the estimates on solutions of problem (8)–(10).

Define
$$(\mu\Box u)(t)=\int_0^t \mu(t-s)||u(s)|^{\frac{\beta}{2}}u(s)-|u(s)|^{\frac{\beta}{2}}u(t)|^2\mathrm{d}s,$$
then a simple computation gives
$$\int_0^t \mu(t-s)(|u(s)|^{\beta}u(s),u_t(t))\mathrm{d}s=\frac{1}{2}(\mu'\Box u)(t)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mu\Box u)(t)-\frac{1}{2}\mu(0)\|u(t)\|_{\beta+2}^{\beta+2}+$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t \mu(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s\Big)-\frac{1}{2}\int_0^t \mu'(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s, \tag{12}$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t \mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s\Big)=\frac{1}{2}\mu(0)\|u(t)\|_{\beta+2}^{\beta+2}+\frac{1}{2}\int_0^t \mu'(t-s)\|u(s)|_{\beta+2}^{\beta+2}\mathrm{d}s. \tag{13}$$
Now we take the scalar product in H of equation (8) with $v=u_t+\varepsilon u$, and with (12), (13) we find
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(|v|^2+\|u\|^2+(\mu\Box u)(t)+\frac{2\rho}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}+\int_0^t \mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s-\frac{1}{2}(\mu'\Box u)(t)-$$
$$\int_0^t \mu(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s+2\int_\Omega G(u)\mathrm{d}x\Big)+(\alpha-\varepsilon)|v|^2+\varepsilon\|u\|^2-\varepsilon(\alpha-\varepsilon)(u,v)+$$
$$\varepsilon\rho\|u(t)\|_{\gamma+2}^{\gamma+2}-\frac{1}{2}\int_0^t \mu'(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\frac{1}{2}\int_0^t \mu'(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s+\varepsilon(g(u),u)$$
$$=(f,v)+\varepsilon\int_0^t \mu(t-s)(|u(s)|^{\beta}u(s),u(t))\mathrm{d}s. \tag{14}$$
By Young's inequality, (7) and Poincaré inequality, we obtain the following results:
$$(f,v)\le C''(\eta)|f|^2+\eta|v|^2, \tag{15}$$
$$-\frac{\varepsilon}{4}\lambda_1^2|u|^2-\frac{\varepsilon(\alpha-\varepsilon)^2}{\lambda_1^2}|v|^2\le-\varepsilon(\alpha-\varepsilon)(u,v), \tag{16}$$
$$c_1\int_\Omega G(u)\mathrm{d}x-\eta\|u\|^2-C'_\eta\le(g(u),u), \tag{17}$$
$$\int_0^t \mu(t-s)(|u(s)|^{\beta}u(s),u(t))\mathrm{d}s$$
$$\le\frac{1}{2}\int_0^t \mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\frac{1}{2}\int_0^t \mu(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s. \tag{18}$$

Now combining with (h4), (7) and (15)–(18), we deduce from (14) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(|v|^2+\|u\|^2+(\mu\Box u)(t)+\frac{2\rho}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}+\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s-\\&\quad\int_0^t\mu(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s+2\int_\Omega G(u)\mathrm{d}x\Big)+\big(\alpha-\varepsilon-\frac{\varepsilon(\alpha-\varepsilon)^2}{\lambda_1^2}-\eta\big)|v|^2+\\&\quad(\frac{3\varepsilon}{4}-\eta\varepsilon)\|u\|^2+\frac{m_1}{2}(\mu\Box u)(t)+\varepsilon\rho\|u(t)\|_{\gamma+2}^{\gamma+2}+\frac{m_1-\varepsilon}{2}\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s-\\&\quad\frac{m_2+\varepsilon}{2}\int_0^t\mu(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s+c_1\varepsilon\int_\Omega G(u)\mathrm{d}x\le C''(\eta)|f|^2+C'_\eta\varepsilon.\end{aligned}\tag{19}$$

We take the numbers ε,η sufficiently small so that all the numbers $\{(\alpha-\varepsilon-\frac{\varepsilon(\alpha-\varepsilon)^2}{\lambda_1^2}-\eta),(\frac{3\varepsilon}{4}-\eta\varepsilon),(\frac{m_1-\varepsilon}{2})\}$ are positive. And let

$$\delta=2\min\big\{(\alpha-\varepsilon-\frac{\varepsilon(\alpha-\varepsilon)^2}{\lambda_1^2}-\eta),(\frac{3\varepsilon}{4}-\eta\varepsilon),\frac{m_1}{2},\frac{\varepsilon(\gamma+2)}{2},(\frac{m_1-\varepsilon}{2}),(\frac{m_2+\varepsilon}{2}),\frac{c_1\varepsilon}{2}\big\}.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\delta E(t)\le 2(C''(\eta)|f|^2+C'_\eta\varepsilon),$$

where

$$\begin{aligned}E=&|v|^2+\|u\|^2+(\mu\Box u)(t)+\frac{2\rho}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}+\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s-\\&\int_0^t\mu(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s+2\int_\Omega G(u)\mathrm{d}x.\end{aligned}$$

Considering $\rho>\frac{(\gamma+2)}{\beta+2}(\int_\Omega 1\mathrm{d}x)^{1/\gamma-1/\beta}$, and taking $\eta\le\frac{\lambda_1^2}{4}$ in (6), we deduce

$$\begin{aligned}E+2C(\eta)=&|v|^2+\|u\|^2+(\mu\Box u)(t)+\frac{2\rho}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}+\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s-\\&\int_0^t\mu(t-s)||u(s)|^{\beta/2}u(t)|^2\mathrm{d}s+2\int_\Omega G(u)\mathrm{d}x+2C(\eta)\\\ge&|v|^2+\|u\|^2+(\mu\Box u)(t)+\frac{2\rho}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}+\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s-\\&\frac{\beta}{\beta+2}\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s-\frac{2}{\beta+2}\int_0^t\mu(s)\mathrm{d}s\|u(t)\|_{\beta+2}^{\beta+2}-2\eta|u(t)| \qquad (20)\\\ge&|v|^2+\frac{1}{2}\|u\|^2+(\mu\Box u)(t)+\Big(\frac{2\rho}{\gamma+2}-\frac{\beta}{\beta+2}\Big(\int_\Omega 1\mathrm{d}x\Big)^{\frac{1}{\gamma}-\frac{1}{\beta}}\Big)\|u(t)\|_{\gamma+2}^{\gamma+2}+\\&\frac{2}{\beta+2}\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s\ge 0. \qquad (21)\end{aligned}$$

Hence by virtue of the Gronwall lemma

$$E(t)+2C(\eta)\le(E(0)+2C(\eta))e^{-\delta t}+\frac{2(C(\eta')|f|^2+C'_\eta\varepsilon+\delta C(\eta))}{\delta}(1-\exp(-\delta t)),\ \forall t\ge 0,\tag{22}$$

therefore

$$\limsup_{t\to\infty}E(t)+2C(\eta)\le R_0^2,\quad R_0^2=\frac{2}{\delta}(C(\eta')|f|^2+C'_\eta\varepsilon+\delta C(\eta)).$$

On the other hand, from (7) we deduce that $G(u)$ is a bounded operator from V to H. In addition, by (11) we have $\gamma + 2 < 2n + 2/n - 2$, hence $H_0^1(\Omega) \hookrightarrow L^{\gamma+2}(\Omega)$. Therefore if $\mathcal{B}$ is bounded set of $\mathcal{H}$, then

$$\begin{aligned} R &= \sup_{(u_0,u_1)\in\mathcal{B}} (E(0) + 2C(\eta)) \\ &= \sup_{(u_0,u_1)\in\mathcal{B}} \{\|u(0)\|^2 + |u_t(0) + \varepsilon u(0)|^2 + \frac{2\rho}{\gamma+2}\|u(0)\|_{\gamma+2}^{\gamma+2} + 2G(u(0)) + 2C(\eta)\} < \infty. \end{aligned}$$

Now we set $R_1 = R_0 + 1$, then it readily follows from (22) that for $t \geq t_0 = t_0(R, R_1)$,

$$t_0(R, R_1) = \frac{1}{\delta} \ln(\frac{R}{R_1^2 - R_0^2}). \tag{23}$$

We have $E(t) + 2C(\eta) \leq R_1^2$, and

$$\|u(t)\|^2 + |u_t(t)|^2 \leq (1 + \frac{\varepsilon^2}{\lambda_1^2})\|u\|^2 + |v|^2 \leq 2(1 + \frac{\varepsilon^2}{\lambda_1^2})(E(t) + C(\eta)) \leq 2(1 + \frac{\varepsilon^2}{\lambda_1^2})R_1^2. \tag{24}$$

If we set $R_2^2 = 2(1 + \frac{\varepsilon^2}{\lambda_1^2})R_1^2$, then we can immediately conclude the following lemma.

Lemma 1 *The ball of $\mathcal{H}$, $\mathcal{B}_0 = B(0, R_2)$, centered at 0 of radius R_2, is an absorbing set in $\mathcal{H}$ for the semigroup $S(t)$. For any bounded set $\mathcal{B}$ of $\mathcal{H}$, there exists $t_0 > 0$, such that $S(t)\mathcal{B} \subset \mathcal{B}_0$, for $t \geq t_0$.*

3.2 Asymptotic compactness of $S(t)$

In this subsection we shall split $S(t)$ so that we can make use of the so-called Kuratowski α-measure of noncompact set to prove the asymptotic compactness of $S(t)$. More precisely, we split $S(t)$ into two parts: $S_1(t)$ and $S_2(t)$, where $S_2(t)$ decays exponentially, and $S_1(t)$ is uniformly compact in $\mathcal{H}$. Thus for any bounded set $\mathcal{B} \subset \mathcal{H}$, $\alpha(S(t)\mathcal{B}) \leq \alpha(S_1(t)\mathcal{B}) + \alpha(S_2(t)\mathcal{B}) = \alpha(S_2(t)\mathcal{B}) \to 0$ as $t \to \infty$.

In the following we split the solution u into the sum $\bar{u} + \widetilde{u}$, where $\bar{u}$ is the solution of the problem:

$$\begin{aligned} &\bar{u}_{tt} + \alpha\bar{u}_t - \Delta\bar{u} = \int_0^t \mu(t-s)|u(s)|^\beta u(s)\mathrm{d}s - \rho|u(t)|^\gamma u(t) - g(u) + f \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ &\bar{u}(x,t) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \\ &\bar{u}(x,0) = 0, \quad \bar{u}_t(x,0) = 0 \quad \text{for} \quad x \in \Omega, \end{aligned} \tag{25}$$

and $\widetilde{u}$ is the solution of the following problem:

$$\begin{aligned} &\widetilde{u}_{tt} + \alpha\widetilde{u}_t - \Delta\widetilde{u} = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ &\widetilde{u}(x,t) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \\ &\widetilde{u}(x,0) = u_0, \quad \widetilde{u}_t(x,0) = u_1 \quad \text{for} \quad x \in \Omega. \end{aligned} \tag{26}$$

Since $f = 0$ in the equation (26), by the Proposition 2, we obtain the uniform decay of the solution of equation (26), i.e., the solution $\widetilde{u}$ of equation (26) with $(u_0, u_1) \in \mathcal{B}_0$ satisfies

$$\{\|\widetilde{u}(t)\|^2 + |\widetilde{u}_t(t)|^2\} \leq C(\mathcal{B}_0)\exp(-\alpha_1 t). \tag{27}$$

In the following, we will show that the semigroup $S_1(t)$ is uniformly compact for $t>0$ in $\mathcal{H}$.

Lemma 2 *Under the hypotheses (11), there exists $\sigma>0$ such that for any $(u,u_t)\in\mathcal{C}_b(R^+,\mathcal{H})$, we have*

$$\int_0^t \mu'(t-s)|u(s)|^\beta u(s)\mathrm{d}s+\mu(0)|u(t)|^\beta u(t)-\rho(\gamma+1)|u(t)|^\gamma u_t(t)-g'(u)u_t\in L^\infty(\mathcal{R}^+;V_{\sigma-1}).$$

Furthermore, for any given bounded set $\mathcal{B}\subset\mathcal{H}$, there exists $C(\mathcal{B})>0$ such that if $(u_0,u_1)\in\mathcal{B}$, then

$$\begin{aligned}\sup_{(u,u_t)\in\mathcal{B}}\Big\|\int_0^t \mu'(t-s)|u(s)|^\beta u(s)\mathrm{d}s+\mu(0)|u(t)|^\beta u(t)-\\ \rho(\gamma+1)|u(t)|^\gamma u_t(t)-g'(u)u_t\Big\|_{L^\infty(\mathcal{R}^+;V_{\sigma-1})}\le C(\mathcal{B}).\end{aligned}\tag{28}$$

Proof For $n=1,2$, in this case we take $\sigma>0$ satisfying

$$\frac{n\tau}{2\tau+4}<(1-\sigma)<\frac{n}{2},\tag{29}$$

where $\tau=\max\{\gamma,\nu\}$. Then we infer $\beta+2\le\gamma+2\le\tau+2<2n/(n-2(1-\sigma))$, hence we have

$$L^{2n/(n-2(1-\sigma))}(\Omega)\subset L^{\tau+2}(\Omega)\subset L^{\gamma+2}(\Omega)\subset L^{\beta+2}(\Omega).\tag{30}$$

On the other hand, due to the interpolation and the Sobolev embedding theorem we obtain the following injections

$$V_{1-\sigma}\subset H^{1-\sigma}(\Omega)\subset L^{2n/n-2(1-\sigma)}(\Omega)\tag{31}$$

with all the injections being continuous. We combine (30) with (31) to obtain

$$V_{1-\sigma}\subset L^{\tau+2}(\Omega)\subset L^{\gamma+2}(\Omega)\subset L^{\beta+2}(\Omega).\tag{32}$$

Observe that, for any $1\le q<\infty$, $H_0^1(\Omega)\subset L^q(\Omega)$. If we take $q=n\nu/(1-\sigma)$, then for any $u\in H_0^1(\Omega)$, $\|u\|\le M$ we have

$$\|u(t)\|_{\gamma n/1-\sigma}\le C\|u(t)\|,\tag{33}$$

and by virtue of (g3), we have

$$\|g'(u)\|_{L^{q/\nu}}\le C(M).\tag{34}$$

Let $w(t)\in V_{1-\sigma}$. Applying the Hölder inequality with exponents $2n/2(1-\sigma),2,2n/(n-2(1-\sigma))$, (32) and (33), we deduce the following result

$$\begin{aligned}&\Big|\Big(\int_0^t \mu'(t-s)|u(s)|^\beta u(s)\mathrm{d}s,w(t)\Big)+\mu(0)(|u(t)|^\beta u(t),w(t))-\\&\quad(\rho(\gamma+1)|u(t)|^\gamma u_t(t),w(t))-(g'(u(t))u_t(t),w(t))\Big|\\&\le m_1\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+1}\|w(t)\|_{\beta+2}\mathrm{d}s+\rho(\gamma+1)\|u(t)\|_{\gamma n/1-\sigma}^{\gamma}|u(t)_t|\|w(t)\|_{2n/(n-2(1-\sigma))}+\\&\quad\mu(0)\|u(t)\|_{\beta+2}^{\beta+1}\|w(t)\|_{\beta+2}+\|g'(u(t))\|_{L^{2n/2(1-\sigma)}}|u_t(t)|\|w(t)\|_{2n/(n-2(1-\sigma))}\\&\le\Big(m_1C\int_0^t\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+1}\mathrm{d}s+\mu(0)C\|u(t)\|_{\beta+2}^{\beta+1}+\end{aligned}$$

$$
\begin{aligned}
&\rho(\gamma+1)C\|u(t)\|^{\gamma}|u_t(t)|+C\|g'(u(t))\|_{L^{q/\nu}}|u_t(t)|\Big)\|w(t)\|_{V_{1-\sigma}}\\
&\leq \Big(\frac{m_1C(\beta+1)}{\beta+2}\int_0^t \mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\frac{\mu(0)(\beta+1)C}{\beta+2}\|u(t)\|_{\beta+2}^{\beta+2}+\frac{m_1C}{\beta+2}+\\
&\quad\rho(\gamma+1)C\|u(t)\|^{\gamma}|u_t(t)|+\frac{\mu(0)}{\beta+2}+C\|g'(u(t))\|_{L^{q/\nu}}|u_t(t)|\Big)\|w(t)\|_{V_{1-\sigma}}\\
&\leq C(\mathcal{B})\|w(t)\|_{V_{1-\sigma}},
\end{aligned}
\tag{35}
$$

where

$$
\begin{aligned}
C(\mathcal{B})=\sup_{\{u_0,u_1\}\in\mathcal{B},t\in\mathcal{R}^+}&\Big\{\frac{m_1C(\beta+1)}{\beta+2}\int_0^t \mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\frac{\mu(0)(\beta+1)C}{\beta+2}\|u(t)\|_{\beta+2}^{\beta+2}+\\
&\frac{m_1C}{\beta+2}+\rho(\gamma+1)C\|u(t)\|^{\gamma}|u_t(t)|+\frac{\mu(0)}{\beta+2}+C\|g'(u(t))\|_{L^{q/\nu}}|u_t(t)|\Big\}.
\end{aligned}
$$

By (22) and (34), $C(\mathcal{B})$ is bounded. That shows $\int_0^t \mu'(t-s)|u(s)|^{\beta}u(s)\mathrm{d}s+\mu(0)|u(t)|^{\beta}u(t)-\rho(\gamma+1)|u(t)|^{\gamma}u_t(t)-g'(u)u_t$ is in the dual space $V_{\sigma-1}$ of $V_{1-\sigma}$ and its norm in $V_{\sigma-1}$ is bounded by $C(\mathcal{B})$, and so the inequality (28) holds.

For $n=3$, if we take $q=6$, then (34) is still true. If we take $\sigma=1-\tau/2$, then $0\leq 2(1-\sigma)=\tau<2<6/2$ and $\tau+2\leq 2n/(n-(1-\sigma))$. Thus we also have the injections (30) and (31). Since $\gamma n/1-\delta=2\gamma n/\tau<2n$, we have (33). Proceeding exactly as above, from (35) we can conclude the inequality (28).

For $n\geq 4$, due to (11), $g'(u)$ is in $L^{\infty}(\Omega)$ for any $u\in H$, and the result is obvious in this case. □

Lemma 3 *For $t>0$, the operator $S_1(t)$ is uniformly compact in $\mathcal{H}$. Therefore, for any bounded set $\mathcal{B}\subset\mathcal{H}$ and $T>0$, the set $\bigcup_{t\geq T}S(t)\mathcal{B}$ is relatively compact in $\mathcal{H}$.*

Proof If (u_0,u_1) belongs to a bounded set $\mathcal{B}$, then for $t\geq t_0$ (given by (23)), Lemma 1 implies that $(u(t),u_t(t))$ is in $\mathcal{B}_0$ and $\|u(t)\|^2+|u_t(t)|^2\leq R_2^2,\forall t>t_0$. Using Proposition 1, we deduce $(u(t),u_t(t))\in C(0,T;\mathcal{H}),\forall T>0$, hence $(u(t),u_t(t))\in C_b(\mathbb{R}^+;\mathcal{H})$. Therefore for all $t>0$, $\int_0^t\mu(t-s)|u(s)|^{\beta}u(s)\mathrm{d}s-\rho|u(t)|^{\gamma}u(t)-g(u)$ is bounded in H. It follows from Proposition 2, $\{\bar{u},\bar{u}_t\}$ is also bounded. By differentiation of equation (25), we find that $w=\bar{u}_t$ is a solution of

$$
\begin{aligned}
w_{tt}+\alpha w_t-\Delta w=&\int_0^t\mu'(t-s)|u(s)|^{\beta}u(s)\mathrm{d}s+\\
&\mu(0)|u(t)|^{\beta}u(t)-\rho(\gamma+1)|u(t)|^{\gamma}u_t(t)-g'(u)u_t,\\
w(0)=0,\quad w_t(0)=&f-g(u_0).
\end{aligned}
$$

Due to Lemma 2, if we replace $\bar{V},\bar{H}$ with $V_{\sigma},V_{-1+\sigma}$ in Proposition 1, respectively, then we obtain that $(w,w_t)=(\bar{u}_t,\bar{u}_{tt})\in C_b(\mathbb{R}^+;V_{\sigma}\times V_{\sigma-1})$.

Now let us return to equation (25). Since $\int_0^t\mu(t-s)|u(s)|^{\beta}u(s)ds-\rho|u(t)|^{\gamma}u(t)-g(u)+f\in C_b(\mathbb{R}^+;H)$, we find that $A\bar{u}\in C_b(\mathbb{R}^+;V_{\sigma-1})$, i.e., $\bar{u}\in C_b(\mathbb{R}^+;V_{\sigma+1})$. Thus $(\bar{u},\bar{u}_t)\in C(\mathbb{R}^+;V_{\sigma+1}\times V_{\sigma})$ and $\bigcup_{t\geq 0}S_1(t)\mathcal{B}$ is included in a bounded set of $V_{\sigma+1}\times V_{\sigma}$. Since the injection of V_{σ_1} into V_{σ_2} is compact $\forall\sigma_1>\sigma_2$. From the fact $V_{\sigma+1}\times V_{\sigma}\hookrightarrow\mathcal{H}$, we can obtain that $\bigcup_{t\geq 0}S_1(t)\mathcal{B}$ is compact in $\mathcal{H}$. □

The proof of Theorem 2 is now obvious. Since we have established the existence of a bounded absorbing set in $\mathcal{H}$ and the asymptotic compactness of $S(t)$, Theorem 2 follows from Theorem 1.

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