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A Complete Solution to the Chromatic Equivalence Class of Graph $\overline{B_{n-8,1,4}}$

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Abstract Two graphs are defined to be adjointly equivalent if and only if their complements are chromatically equivalent. Using the properties of the adjoint polynomials and the fourth character $R_4(G)$, the adjoint equivalence class of graph $B_{n-8,1,4}$ is determined. According to the relations between adjoint polynomial and chromatic polynomial, we also simultaneously determine the chromatic equivalence class of $\overline{B_{n-8,1,4}}$ that is the complement of $B_{n-8,1,4}$.

Keywords chromatic equivalence class; adjoint polynomial; the smallest real root; the fourth character.

MR(2010) Subject Classification 05C15; 05C60; 05C31

1. Introduction

The graphs considered in this paper are finite undirected and simple graphs. We follow the notation of Bondy and Murty [1], unless otherwise stated. For a graph G, let $V(G)$, $E(G)$, $p(G)$, $q(G)$ and $\overline{G}$ be the set of vertices, the set of edges, the order, the size and the complement of G, respectively.

For a graph G, we denote by $P(G,\lambda)$ the chromatic polynomial of G. A partition $\{A_1,A_2,\ldots,A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of graph G if every A_i is nonempty independent set of G. We denote by $\alpha(G,r)$ the number of r-independent partitions of G. Thus the chromatic polynomial G is $P(G,\lambda)=\sum_{r\geq 1}\alpha(G,r)(\lambda)_r$, where $(\lambda)_r=\lambda(\lambda-1)\cdots(\lambda-r+1)$ for all $r\geq 1$. The readers can turn to [17] for details on chromatic polynomials.

Two graphs G and H are said to be chromatically equivalent, denoted by $G\sim H$, if $P(G,\lambda)=P(H,\lambda)$. By $[G]$ we denote the equivalence class determined by G under "$\sim$". It is obvious that "$\sim$" is an equivalence relation on the family of all graphs. A graph G is called chromatically unique (or simply $\chi-unique$) if $H\cong G$ whenever $H\sim G$. See [4,5] for many results on this field.

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Definition 1.1 ([7]) *Let G be a graph with p vertices. The polynomial*

$$h(G,x)=\sum_{i=1}^{p}\alpha(\overline{G},i)x^{i}$$

is called its adjoint polynomial.

Definition 1.2 ([7]) *Let G be a graph and $h_1(G,x)$ the polynomial with a nonzero constant term such that $h(G,x)=x^{\rho(G)}h_1(G,x)$. If $h_1(G,x)$ is an irreducible polynomial over the rational number field, then G is called irreducible graph.*

Two graphs G and H are said to be adjointly equivalent, denoted by $G\sim^h H$, if $h(G,x)=h(H,x)$. Evidently, "$\sim^h$" is an equivalence relation on the family of all graphs. Let $[G]_h=\{H|H\sim^h G\}$. A graph G is said to be adjointly unique (or simply h-unique) if $G\cong H$ whenever $G\sim^h H$.

Theorem 1.1 ([3]) *(1) $G\sim^h H$ if and only if $\overline{G}\sim\overline{H}$; (2) $[G]_h=\{H|\overline{H}\in[\overline{G}]\}$; (3) G is χ-unique if and only if $\overline{G}$ is h-unique.*

The graphs with order n used in this paper are drawn as follows (see Figure 1).

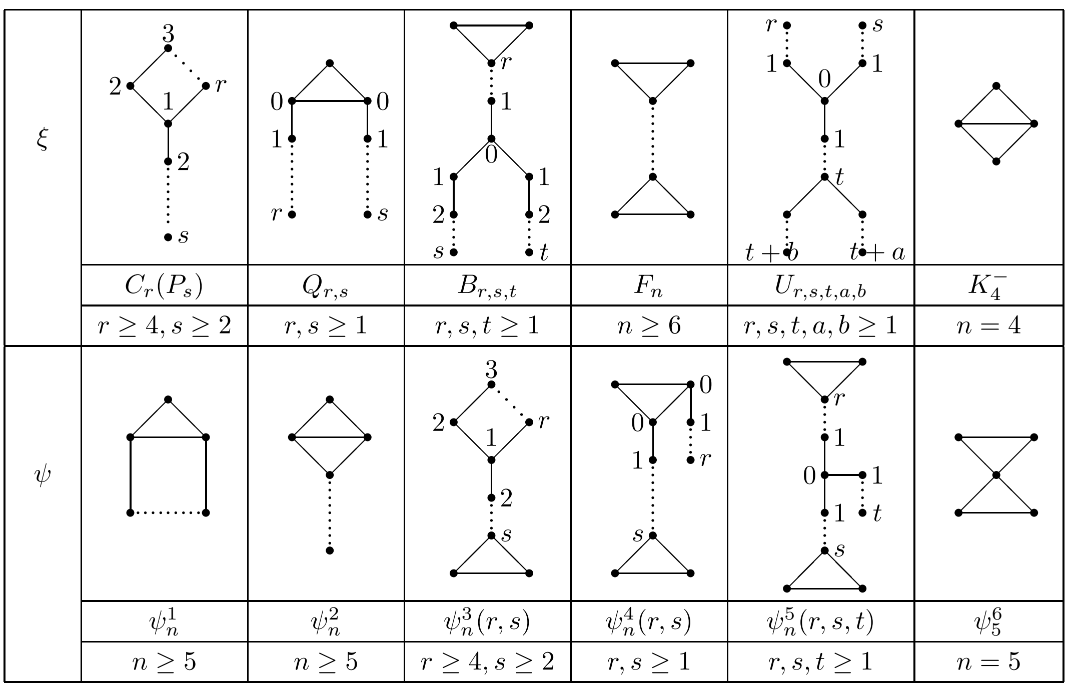

Figure 1 Families of ξ and ψ

Now we define some classes of graphs with order n, which will be used throughout the paper.

(1) C_n(resp., P_n) denotes the cycle (resp., the path) of order n, and write $\mathcal{C}=\{C_n|n\ge 3\}$, $\mathcal{P}=\{P_n|n\ge 2\}$ and $\mathcal{U}=\{U_{1,1,t,1,1}|t\ge 1\}$.

(2) $D_n(n\ge 4)$ denotes the graph obtained from C_3 and P_{n-2} by identifying a vertex of C_3 with a pendent vertex of P_{n-2}.

(3) T_{l_1,l_2,l_3} is a tree with a vertex v of degree 3 such that $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$ and $l_3 \geq l_2 \geq l_1$, write $\mathcal{T}_0 = \{T_{1,1,l_3} | l_3 \geq 1\}$ and $\mathcal{T} = \{T_{l_1,l_2,l_3} | (l_1, l_2, l_3) \neq (1, 1, 1)\}$.

(4) $\vartheta = \{C_n, D_n, K_1, T_{l_1,l_2,l_3} | n \geq 4\}$.

(5) $\xi = \{C_r(P_s), Q(r,s), B_{r,s,t}, F_n, U_{r,s,t,a,b}, K_4^-\}$.

(6) $\psi = \{\psi_n^1, \psi_n^2, \psi_n^3(r,s), \psi_n^4(r,s), \psi_n^5(r,s,t), \psi_5^6\}$.

For convenience, we simply denote $h(G,x)$ by $h(G)$ and $h_1(G,x)$ by $h_1(G)$. By $\beta(G)$ and $\gamma(G)$ we denote the smallest real root of $h(G)$, respectively. Let $d_G(v)$, simply denoted by $d(v)$, be the degree of vertex v. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies. By K_n we denote the complete graph with order n. Let $n_G(K_3)$ and $n_G(K_4)$ denote the number of subgraphs isomorphic to K_3 and K_4, respectively. On the real field, let $g(x)|f(x)$ (resp., $g(x) \nmid f(x)$) denote $g(x)$ divides $f(x)$ (resp., $g(x)$ does not divide $f(x)$) and $\partial(f(x))$ denote the degree of $f(x)$. By $(f(x), g(x))$ we denote the largest common factor of $f(x)$ and $g(x)$.

It is an important problem to determine $[G]$ for a given graph G. From Theorem 1.1, it is obvious that the goal of determining $[G]$ can be realized by determining $[\overline{G}]_h$. Thus, if $q(G)$ is large, it may be easier to study $[\overline{G}]_h$ rather than $[G]$. The related topics have been partially discussed in this respect by Dong et al in [3, 14, 15]. In this paper, using the properties of adjoint polynomials, we determine the $[B_{n-8,1,4}]_h$ of graph $B_{n-8,1,4}$, simultaneously, $[\overline{B_{n-8,1,4}}]$ is also determined, where $n \geq 7$.

2. Preliminaries

For a polynomial $f(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n$, we define

$$R_1(f(x)) = \begin{cases} -\binom{b_1}{2} + 1, & \text{if } n = 1. \\ b_2 - \binom{b_1 - 1}{2} + 1, & \text{if } n \geq 2. \end{cases}$$

For a graph G, we write $R_1(G)$ instead of $R_1(h(G))$.

Definition 2.1 ([2, 7]) *Let G be a graph with q edges.*

(1) The first character of a graph G is defined as

$$R_1(G) = \begin{cases} 0, & \text{if } q = 0. \\ b_2(G) - \binom{b_1(G) - 1}{2} + 1, & \text{if } q > 0. \end{cases}$$

(2) The second character of a graph G is defined as

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G) - 2)\left(b_2(G) - \binom{b_1(G)}{2}\right) - b_1(G),$$

where $b_i(G)$ $(0 \leq i \leq 3)$ is the first four coefficients of $h(G)$.

Lemma 2.1 ([2, 7]) *Let G be a graph with k components of $G_1, G_2, \ldots, G_k$. Then $h(G) = \prod_{i=1}^{k} h(G_i)$ and $R_j(G) = \sum_{i=1}^{k} R_j(G_i)$ for $j = 1, 2$.*

It is obvious that $R_j(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_j(G) = R_j(H)$ for $j = 1, 2$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

Lemma 2.2 ([7,8]) *Let G be a graph with p vertices and q edges. Denote by M the set of the triangles in G and by $M(i)$ the number of triangles which cover the vertex i in G. If the degree sequence of G is $(d_1, d_2, \dots, d_p)$, then the first four coefficients of $h(G)$ are, respectively,*

(1) $b_0(G) = 1$, $b_1(G) = q$.

(2) $b_2(G) = \binom{q+1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2 + n_G(K_3)$.

(3) $b_3(G) = \frac{q}{6}(q^2 + 3q + 4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 - \sum_{ij\in E(G)} d_i d_j - \sum_{i\in M} M(i)d_i + (q+2)n_G(K_3) + n_G(K_4)$, *where* $b_i(G) = \alpha(\overline{G}, p-i)$ $(i = 0, 1, 2, 3)$.

For an edge $e = v_1v_2$ of a graph G, the graph $G*e$ is defined as follows: the vertex set of $G*e$ is $(V(G) - \{v_1, v_2\})\bigcup\{v\}(v \notin G)$, and the edge set of $G*e$ is $\{e'|e' \in E(G)$, e' is not incident with v_1 or $v_2\} \cup \{uv|u \in N_G(v_1) \cap N_G(v_2)\}$, where $N_G(v)$ is the set of vertices of G which are adjacent to v.

Lemma 2.3 ([7]) *Let G be a graph with $e \in E(G)$. Then*

$$h(G, x) = h(G - e, x) + h(G*e, x),$$

where $G - e$ denotes the graph obtained by deleting the edge e from G.

Lemma 2.4 ([7]) *(1) For $n \geq 2$, $h(P_n) = \sum_{k\leq n}\binom{k}{n-k}x^k$.*

(2) *For $n \geq 4$, $h(D_n) = \sum_{k\leq n}\left(\frac{n}{k}\binom{k}{n-k} + \binom{k-2}{n-k-3}\right)x^k$.*

(3) *For $n \geq 4$, $m \geq 6$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$, $h(D_m) = x(h(D_{m-1}) + h(D_{m-2}))$.*

Lemma 2.5 ([18]) *Let $\{g_i(x)\}$, simply denoted by $\{g_i\}$, be a polynomial sequence with integer coefficients and $g_n(x) = x(g_{n-1}(x) + g_{n-2}(x))$. Then*

(1) $g_n(x) = h(P_k)g_{n-k}(x) + xh(P_{k-1})g_{n-k-1}(x)$.

(2) $h_1(P_n)|g_{k(n+1)+i}(x)$ *if and only if* $h_1(P_n)|g_i(x)$, *where* $0 \leq i \leq n$, $n \geq 2$ *and* $k \geq 1$.

Lemma 2.6 ([6,10]) *Let G be a nontrivial connected graph with n vertices. Then*

(1) $R_1(G) \leq 1$, *and the equality holds if and only if* $G \cong P_n(n \geq 2)$ *or* $G \cong K_3$.

(2) $R_1(G) = 0$ *if and only if* $G \in \vartheta$.

(3) $R_1(G) = -1$ *if and only if* $G \in \xi$, *especially,* $q(G) = p(G) + 1$ *if and only if* $G \in \{F_n|n \geq 6\} \cup \{K_4^-\}$.

(4) $R_1(G) = -2$ *if and only if* $G \in \psi$ *for* $q(G) = p(G) + 1$ *and* $G \cong K_4$ *for* $q(G) = p(G) + 2$.

Lemma 2.7 ([11]) *Let G be a connected graph. Then*

(1) *If* $R_1(G) = 0, -1, -2$, *then* $q(G) - p(G) \leq |R_1(G)|$;

(2) *If* $R_1(G) = -3$, *then* $q(G) - p(G) \leq |R_1(G) + 1|$.

Lemma 2.8 ([18]) *Let G be a connected graph and H be a proper subgraph of G. Then*

$$\beta(G) < \beta(H).$$

Lemma 2.9 ([18]) *Let G be a connected graph. Then*

(1) $\beta(G) = -4$ *if and only if*

$$G \in \{T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, C_4(P_2), Q(1,1), K_4^-, D_8\} \cup \mathcal{U}.$$

(2) $\beta(G) > -4$ *if and only if*

$$G \in \{K_1, T(1,2,i)(2 \le i \le 4), D_i(4 \le i \le 7)\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_l.$$

Lemma 2.10 ([18]) *Let* G *be a connected graph. Then* $-(2+\sqrt{5}) \le \beta(G) < -4$ *if and only if* G *is one of the following graphs:*

(1) T_{l_1,l_2,l_3} *for* $l_1 = 1, l_2 = 2, l_3 > 5$ *or* $l_1 = 1, l_2 > 2, l_3 > 3$ *or* $l_1 = l_2 = 2, l_3 > 2$ *or* $l_1 = 2, l_2 = l_3 = 3$.

(2) $U_{r,s,t,a,b}$ *for* $r = a = 1$, $(r,s,t) \in \{(1,1,2),(2,4,2),(2,5,3),(3,7,3),(3,8,4)\}$, *or* $r = a = 1$, $s \ge 1$, $t \ge t^*(s,b)$, $b \ge 1$, *where* $(s,b) \ne (1,1)$ *and*

$$t^* = \begin{cases} s+b+2, & \text{if } s \ge 3; \\ b+3, & \text{if } s = 2; \\ b, & \text{if } s = 1. \end{cases}$$

(3) D_n *for* $n \ge 9$.

(4) $C_n(P_2)$ *for* $n \ge 5$.

(5) F_n *for* $n \ge 9$.

(6) $B_{r,s,t}$ *for* $r = 5, s = 1$ *and* $t = 3$, *or* $r \ge 1, s = 1$ *if* $t = 1$, *or* $r \ge 4, s = 1$ *if* $t = 2$, *or* $b \ge c+3, s = 1$ *if* $t \ge 3$.

(7) $G \cong C_4(P_3)$ *or* $G \cong Q(1,2)$.

Corollary 2.1 ([14]) *If graph* G *satisfies* $R_1(G) \le -2$, *then* $\beta(G) < -2-\sqrt{5}$.

3. The algebraic properties of adjoint polynomials

3.1. The divisibility of adjoint polynomials and the fourth characters of graphs

Lemma 3.1 ([18]) *For* $n, m \ge 2$, $h(P_n) \mid h(P_m)$ *if and only if* $(n+1)|(m+1)$.

Theorem 3.1 *(1) For* $n \ge 9$, $\rho(B_{n-8,1,4}) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even;} \\ \frac{n-1}{2}, & \text{otherwise.} \end{cases}$

(2) For $n \ge 9$, $\partial(B_{n-8,1,4}) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even;} \\ \frac{n+1}{2}, & \text{otherwise.} \end{cases}$

(3) For $n \ge 9$, $h(B_{n-8,1,4}) = x(h(B_{n-9,1,4}) + h(B_{n-10,1,4}))$.

Proof (1) Choosing a pendent edge $e = uv \in E(B_{n-8,1,4})$ whose deletion brings about a single vertex and a proper subgraph D_{n-1} of $B_{n-8,1,4}$, and by Lemma 2.3, we have $h(B_{n-8,1,4}) = xh(D_{n-1}) + xh(P_4)h(D_{n-6})$. It follows, from Lemma 2.4, that

$\rho(K_1 \cup D_{n-1}) = 1 + \lfloor \frac{n-1}{2} \rfloor$ and $\rho(K_1 \cup P_4 \cup D_{n-6}) = 3 + \lfloor \frac{n-6}{2} \rfloor$.

If n is even, then $\rho(K_1 \cup D_{n-1}) = \rho(K_1 \cup P_4 \cup D_{n-6}) = \frac{n}{2}$, which implies that $\rho(B_{n-8,1,4}) = \frac{n}{2}$. If n is odd, then we arrive at $\rho(K_1 \cup D_{n-1}) = \frac{n+1}{2} > \frac{n-1}{2} = \rho(K_1 \cup P_4 \cup D_{n-6})$, which implies that $\rho(B_{n-8,1,4}) = \frac{n-1}{2}$.

(2) It obviously follows from (1).

(3) Choosing a pendent edge $e = uv \in E(B_{n-8,1,4})$ whose deletion brings about a single

vertex and a proper subgraph D_{n-1} of $B_{n-8,1,4}$. By Lemma 2.4, We have

$$\begin{aligned}h(B_{n-8,1,4}) &= xh(D_{n-1}) + xh(P_4)h(D_{n-6})\\ &= x(xh(D_{n-2}) + xh(D_{n-3})) + xh(P_4)(xh(D_{n-7}) + xh(D_{n-8}))\\ &= x(xh(D_{n-2}) + xh(P_4)h(D_{n-7})) + x(xh(D_{n-3}) + xh(P_4)h(D_{n-8}))\\ &= x(h(B_{n-9,1,4}) + h(B_{n-10,1,4})).\end{aligned}$$

Theorem 3.2 *For $n \geq 2$, $m \geq 9$, $h(P_n) \mid h(B_{m-8,1,4})$ if and only if $n = 4$ and $m = 5k + 4$ for $k \geq 1$.*

Proof Let $g_0(x) = -x^6 - 10x^5 - 37x^4 - 63x^3 - 50x^2 - 18x - 2$, $g_1(x) = x^6 + 9x^5 + 29x^4 + 41x^3 + 25x^2 + 8x + 1$ and $g_m(x) = x(g_{m-1}(x) + g_{m-2}(x))$. We can deduce that

$$\begin{aligned}g_0(x) &= -x^6 - 10x^5 - 37x^4 - 63x^3 - 50x^2 - 18x - 2,\\ g_1(x) &= x^6 + 9x^5 + 29x^4 + 41x^3 + 25x^2 + 8x + 1,\\ g_2(x) &= -x^6 - 8x^5 - 22x^4 - 25x^3 - 10x^2 - x,\\ g_3(x) &= x^6 + 7x^5 + 16x^4 + 15x^3 + 7x^2 + x,\\ g_4(x) &= -x^6 - 6x^5 - 10x^4 - 3x^3,\\ g_5(x) &= x^6 + 6x^5 + 12x^4 + 7x^3 + x^2,\\ g_6(x) &= 2x^5 + 4x^4 + x^3,\\ g_7(x) &= x^7 + 8x^6 + 16x^5 + 8x^4 + x^3,\\ g_8(x) &= x^8 + 8x^7 + 18x^6 + 12x^5 + 2x^4,\\ g_m(x) &= h(B_{m-8,1,4}),\ \text{if } m \geq 9.\end{aligned} \tag{3.1}$$

Let $m = (n+1)k + i$, where $0 \leq i \leq n$. It is obvious that $h_1(P_n)|h(B_{m-8,1,4})$ if and only if $h_1(P_n)|g_m(x)$. From Lemma 2.5, it follows that $h_1(P_n)|g_m(x)$ if and only if $h_1(P_n)|g_i(x)$, where $0 \leq i \leq n$. We consider the following two cases:

Case 1 $n \geq 9$.

If $0 \leq i \leq 8$, from (3.1), it is not difficult to verify that $h_1(P_n) \nmid g_i(x)$. If $i \geq 9$, from $i \leq n$, Lemma 2.4 and Theorem 3.1, we have that

$$\partial(h_1(P_n)) = \lfloor \frac{n}{2} \rfloor \text{ and } \partial(h_1(B_{i-8,1,4})) = \lfloor \frac{i+1}{2} \rfloor. \tag{3.2}$$

The following cases are taken into account:

Subcase 1.1 $i = n$.

It follows from (3.2) that $\partial(h_1(B_{i-8,1,4})) = \partial(h_1(P_n)) = \lfloor \frac{n}{2} \rfloor$ if n is even and $\partial(h_1(B_{i-8,1,4})) = \partial(h_1(P_n)) + 1 = \lfloor \frac{n+1}{2} \rfloor$ if n is odd.

Subcase 1.1.1 $\partial(h_1(B_{i-8,1,4})) = \partial(h_1(P_n))$.

Suppose that $h_1(P_n)|h_1(B_{i-8,1,4})$, we have $h_1(P_n) = h_1(B_{i-8,1,4})$, which implies $R_1(P_n) = R_1(B_{i-8,1,4})$. By Lemma 2.6, we know it is impossible. Hence $h_1(P_n) \nmid h_1(B_{i-8,1,4})$, together with $(h_1(P_n), x^{\alpha(B_{i-8,1,4})}) = 1$, we have $h_1(P_n) \nmid h(B_{i-8,1,4})$.

Subcase 1.1.2 $\partial(h_1(B_{i-8,1,4})) = \partial(h_1(P_n)) + 1$.

Assume that $h_1(P_n)|h_1(B_{i-8,1,4})$, it follows that $h_1(B_{i-8,1,4}) = (x+a)h_1(P_n)$. Note that $R_1(B_{i-8,1,4}) = -1$ and $R_1(P_n) = 1$, so $R_1(x+a) = -2$, which brings about $a = 4$. This implies that $\beta(B_{i-8,1,4}) = -4$, which contradicts (6) of Lemma 2.10. Hence $h_1(P_n) \nmid h_1(B_{i-8,1,4})$, together with $(h_1(P_n), x^{\alpha(B_{i-8,1,4})}) = 1$, we have $h_1(P_n) \nmid h(B_{i-8,1,4})$.

Subcase 1.2 $i \le n-1$.

It follows by (3.2) that $\partial(h_1(B_{i-8,1,4})) \le \partial(h_1(P_n))$. Assume that $h_1(P_n)|h_1(B_{i-8,1,4})$, we have that $\partial(h_1(B_{i-8,1,4})) = \partial(h_1(P_n))$ and $h_1(P_n) = h_1(B_{i-8,1,4})$. So we can turn to Subcase 1.1.1 for the same contradiction.

Case 2 $2 \le n \le 8$.

From (1) of Lemma 2.4 and (3.1) , we can verify that $h_1(P_n) = g_i(x)$ if and only if $n = 4$ and $i = 4$ for $0 \le i \le n \le 8$. From Lemma 2.5, we have that $h_1(P_n)|h(B_{i-8,1,4})$ if and only if $n = 4$ and $m = 5k+4$. From $\rho(P_4) = 1$ and $\rho(B_{m-8,1,4}) = \lfloor \frac{m}{2} \rfloor \ge 4$ for $m \ge 8$, we obtain that the result holds.

Theorem 3.3 *For* $m \ge 9$, $h^2(P_4) \nmid h(B_{m-8,1,4})$.

Proof Suppose $h^2(P_4) \mid h(B_{m-8,1,4})$. From Theorem 3.2, we have that $m = 5k+4$, where $k \ge 1$. Let $g_m(x) = h(B_{m-8,1,4})$ for $m \ge 9$. By (3) of Theorem 3.1, (1) of Lemma 2.5, it follows that

$$\begin{aligned}
g_m(x) =& h(P_4)g_{m-4}(x) + xh(P_3)g_{m-5}(x)\\
=& h^2(P_4)g_{m-8}(x) + 2xh(P_3)h(P_4)g_{m-9}(x) + (xh(P_3))^2 g_{m-10}(x)\\
=& h^2(P_4)(g_{m-8}(x) + 2xh(P_3)g_{m-13}(x)) + 3(xh(P_3))^2h(P_4)g_{m-14}(x) + (xh(P_3))^3 g_{m-15}(x)\\
=& h^2(P_4)(g_{m-8}(x) + 2xh(P_3)g_{m-13}(x) + 3(xh(P_3))^2 g_{m-18}(x)) +\\
& 4(xh(P_3))^3h(P_4)g_{m-19}(x) + (xh(P_3))^4 g_{m-20}(x)\\
=& \cdots\\
=& h^2(P_4)\sum_{s=1}^{k-2} s(xh(P_3))^{s-1}g_{m-5s-3}(x) + (k-1)(xh(P_3))^{k-2}h(P_4)g_{m+1-(5k-1)}(x) +\\
& (xh(P_3))^{k-1}h(P_4)g_{m-(5k-1)}(x).
\end{aligned}$$

According to the assumption and $m = 5k+4$, we arrive at, by (3.1), that

$$h^2(P_4) \mid (k-1)(xh(P_3))^{k-2}h(P_4)g_{10}(x) + (xh(P_3))^{k-1}h(P_4)g_9(x)$$

that is

$$h(P_4) \mid (k-1)g_{10}(x) + x^3h(P_3)(x^3 + 6x^2 + 7x + 1).$$

By direct calculation, we obtain that $k = 0$, which contradicts $k \ge 1$.

Definition 3.1 ([14]) *Let* G *be a graph with* p *vertex and* q *edges. The fourth character of a* *graph G is defined as follows:*

$$R_4(G) = R_2(G) + p - q.$$

From Lemmas 2.1 and 2.2, we obtain the following two lemmas:

Lemma 3.2 ([14]) *Let graph G have k components $G_1, G_2, \dots, G_k$. Then*

$$R_4(G) = \sum_{i=1}^{k} R_4(G_k).$$

Lemma 3.3 ([14]) *Let graph G and H satisfy that $h(G) = h(H)$ or $h_1(G) = h_1(H)$. Then*

$$R_4(G) = R_4(H).$$

From Definitions 3.1 and 2.1, we have the following lemmas:

Lemma 3.4 ([14]) *(1) $R_4(C_n) = 0$ for $n \geq 4$; $R_4(C_3) = -2$; $R_4(K_1) = 1$.*

(2) $R_4(B_{r,1,1}) = 3$ for $r \geq 1$; $R_4(B_{r,1,t}) = 4$ for $r, t > 1$.

(3) $R_4(F_6) = 4$; $R_4(F_n) = 3$ for $n \geq 7$; $R_4(K_4^-) = 2$.

(4) $R_4(D_4) = 0$; $R_4(D_n) = 1$ for $n \geq 5$; $R_4(T_{1,1,1}) = 0$.

(5) $R_4(T_{1,1,l_3}) = 1$, $R_4(T_{1,l_2,l_3}) = 2$; $R_4(T_{l_1,l_2,l_3}) = 3$ for $l_3 \geq l_2 \geq l_1 \geq 2$.

(6) $R_4(C_r(P_2)) = 3$ for $r \geq 4$; $R_4(C_4(P_3)) = R_4(Q_{1,2}) = 4$.

(7) $R_4(P_2) = 0$; $R_4(P_n) = -1$ for $n \geq 3$.

Lemma 3.5 ([12]) *Let graph $G \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$. Then*

(1) $R_4(G) = 3$ if and only if $G \in \{C_{n-1}(P_2) | n \geq 5\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1} | n \geq 7\}$.

(2) $R_4(G) = 4$ if and only if $G \in \{C_r(P_s) | r \geq 4, s \geq 3\} \cup \{Q_{1,n-4} | n \geq 6\} \cup \{B_{r,1,t}, B_{1,1,1} | r, t \geq 2\}$.

(3) $R_4(G) = 5$ if and only if $G \in \{Q_{r,s} | r, s \geq 2\} \cup \{B_{1,1,t}, B_{r,s,t} | r, s, t \geq 2\}$.

(4) $R_4(G) = 6$ if and only if $G \in \{B_{1,s,t} | s, t \geq 2\}$.

Corollary 3.1 *Let graph $G \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$. Then $R_4(G) \geq 3$.*

3.2 The smallest real roots of adjoint polynomials of graphs

An internal x_1x_k-path of a graph G is path $x_1x_2x_3 \cdots x_k$ (possibly $x_1 = x_k$) of G such that $d(x_1)$ and $d(x_k)$ are at least 3 and $d(x_2) = d(x_3) = \cdots = d(x_{k-1}) = 2$ (unless $k = 2$).

Lemma 3.6 ([18]) *Let T be a tree. If uv is an internal path of T and $T \ncong U(1,1,t,1,1)$ for $t \geq 1$, then $\beta(T) < \beta(T_{xy})$, where $\beta(T_{xy})$ is the graph obtained from T by inserting a new vertex on the edge xy of T.*

Lemma 3.7 ([14, 15, 18]) *(1) For $n \geq 5$, $m \geq 4$, $\beta(C_n(P_2)) < \beta(C_{n-1}(P_2)) \leq \beta(D_m) \leq \beta(C_m)$.*

(2) For $n \geq 6$, $m \geq 6$, $\beta(F_n) = \beta(B_{m-5,1,1})$ if and only if $n = 2k+1$ and $m = k+2$.

(3) For $n \geq 4$, $m \geq 6$, $\beta(F_m) < \beta(F_{m-1}) < \beta(D_n)$ and $\beta(B_{m-5,1,1}) < \beta(B_{m-4,1,1}) < \beta(D_n)$.

(4) For $n \geq 7$, $m \geq 6$, $\beta(B_{n-6,1,2}) = \beta(F_m)$ if and only if $m = n-1$.

(5) For $n \geq 7$, $m \geq 6$, $\beta(B_{n-6,1,2}) < \beta(D_m)$; $\beta(B_{n-7,1,3}) < \beta(D_m)$.

(6) For $n \geq 8$, $\beta(B_{n-7,1,3}) = \beta(Q_{1,2}) = \beta(C_4(P_3))$ *if and only if* $n = 13$.

Lemma 3.8 ([13, 14]) *(1)* $\beta(B_{1,1,4}) = \beta(C_8(P_2))$, $\beta(B_{1,1,4}) = \beta(\psi_5^1)$, $\beta(B_{1,1,4}) = \beta(\psi_5^2)$.

(2) $\beta(B_{8,1,4}) = \beta(Q_{2,4})$, $\beta(B_{1,1,4}) = \beta(Q(1,2)) = \beta(C_4(P_3))$.

(3) For $r, t \geq 1$, $\beta(B_{r,1,t}) < \beta(B_{r+1,1,t})$.

(4) $\beta(T_{1,3,6}) = \beta(C_5(P_2))$, $\beta(T_{1,3,11}) = \beta(B_{8,1,2})$.

(5) For $r, t \geq 1$, $\beta(U_{1,2,r,1,t}) = \beta(B_{r,1,t})$ *and* $\beta(B_{t,1,2}) = \beta(F_{t+5})$.

Theorem 3.4 *(1) For* $m \geq 11$, $n \geq 19$, $\beta(B_{1,1,4}) < \beta(B_{2,1,4}) < \beta(B_{3,1,4}) < \beta(B_{4,1,4}) < \beta(B_{5,1,4}) < \beta(B_{6,1,4}) < \beta(C_m(P_2)) < \beta(B_{7,1,4}) < \beta(C_{10}(P_2)) < \beta(C_9(P_2)) < \beta(C_8(P_2)) = \beta(B_{8,1,4}) = \beta(B_{6,1,3}) < \beta(B_{9,1,4}) < \beta(B_{10,1,4}) < \beta(C_7(P_2)) < \beta(B_{11,1,4}) < \beta(C_6(P_2)) < \beta(B_{n-8,1,4}) < \beta(C_5(P_2)) < \beta(C_4(P_2))$.

(2) $m \geq 11$, $n \geq 19$, $\beta(B_{1,1,4}) < \beta(B_{2,1,4}) = \beta(F_6) < \beta(B_{3,1,4}) < \beta(F_7) < \beta(B_{4,1,4}) < \beta(B_{5,1,4}) < \beta(F_8) < \beta(B_{6,1,4}) < \beta(B_{7,1,4}) < \beta(F_9) = \beta(B_{8,1,4}) < \beta(B_{9,1,4}) < \beta(B_{10,1,4}) < \beta(B_{11,1,4}) < \beta(B_{n-8,1,4}) < \beta(F_{m-1}) = \beta(B_{m-6,1,2})$.

(3) For $n \geq m$, $t \geq 4$, $\beta(B_{m-t-4,1,t}) < \beta(B_{n-8,1,4})$.

(4) For $n \geq 9$, $m \geq 4$, $\beta(B_{n-8,1,4}) < \beta(D_m)$.

(5) For $n \geq 9$, $\beta(Q(1,2)) = \beta(C_4(P_3)) = \beta(B_{n-8,1,4})$ *if and only if* $n = 12$.

(6) For $n \geq 9$, $m \geq 6$, $\beta(B_{n-8,1,4}) = \beta(B_{m-5,1,1})$ *if and only if* $m = 6$, $n = 16$.

(7) For $n \geq 9$, $m \geq 7$, $\beta(B_{m-6,1,2}) = \beta(B_{n-8,1,4})$ *if and only if* $m = 7$, $n = 10$ *or* $m = 10$, $n = 16$.

(8) For $n \geq 9$, $m \geq 8$, $\beta(B_{m-7,1,3}) = \beta(B_{n-8,1,4})$ *if and only if* $m = 13$, $n = 16$.

Proof (1) For $n \geq 19$, it is obvious that $T_{1,3,6}$ is a proper subgraph of $B_{n-8,1,4}$. From Lemma 2.8 and (4) of Lemma 3.8, it follows that $\beta(B_{n-8,1,4}) < \beta(T_{1,3,6}) = \beta(C_5(P_2))$. By (1) of Lemma 3.8 and (1) of Lemma 3.7, the result holds.

(2) Using software Mathematica and by calculation, we have that

$\beta(B_{1,1,4}) = -4.49086 < \beta(B_{2,1,4}) = \beta(B_{1,1,2}) = \beta(F_6) = -4.39026 < \beta(B_{3,1,4}) = -4.32931 < \beta(F_7) = -4.30278 < \beta(B_{4,1,4}) = -4.28896 < \beta(B_{5,1,4}) = -4.26076 < \beta(F_8) = \beta(B_{3,1,2}) = -4.24978 < \beta(B_{6,1,4}) = -4.24039 < \beta(B_{7,1,4}) = -4.22541 < \beta(F_9) = \beta(B_{8,1,4}) = \beta(B_{4,1,2}) = \beta(B_{6,1,3}) = -4.21432 < \beta(B_{9,1,4}) = -4.20612 < \beta(B_{10,1,4}) = -4.2001 < \beta(B_{11,1,4}) = -4.19576 < \beta(B_{n-8,1,4}) < \beta(F_{m-1}) = \beta(B_{m-6,1,2})$. For $n \geq 22$, it follows, from Lemma 2.8 and (4) of Lemma 3.8, that $\beta(B_{n-8,1,4}) < \beta(T_{1,3,11}) = \beta(B_{8,1,2})$. From (5) of Lemma 3.8 and (4) of Lemma 3.7, the result holds.

(3) Since $n \geq m$ and $t \geq 4$, from (3) of Lemma 3.8 and Lemma 2.8, we have that $\beta(B_{m-t-4,1,t}) < \beta(B_{n-t-4,1,t}) < \beta(B_{n-8,1,t}) < \beta(B_{n-8,1,4})$.

(4) From (2) of the theorem and (3) of Lemma 3.7, the result evidently holds.

(5) Applying (2) of Lemma 3.8, we can get the result.

(6) From (2) of Lemma 3.7 and (2) of the theorem, the result evidently holds.

(7) Using (4) of Lemma 3.7 and (2) of the theorem easily yields the result.

(8) By (6) of Lemma 3.7 and (5) of the theorem, the result evidently holds.

4. The chromaticity of graph $\overline{B_{n-8,1,4}}$

Corollary 4.1 ([16]) *For $n \geq 4$, D_n is adjointly unique if and only if $n \neq 4, 8$.*

Theorem 4.1 *Let G be a graph such that $G \sim^h B_{n-8,1,4}$, where $n \geq 9$. Then G contains at most one component whose first character is 1, furthermore, it is P_4 or C_3.*

Proof Let G_1 be one of the components of G such that $R_1(G) = 1$. From Lemma 2.6 and Theorem 3.3, it follows that $h(G_1)|h(B_{n-8,1,4})$ if and only if $G_1 \cong P_4$ and $n = 5k+4$. According to (1) of Lemma 2.5, we obtain the following equality:

$$h(B_{5k+4,1,4}) = h(P_5)h(B_{5(k-1)+4,1,4}) + xh(P_4)h(B_{5(k-1)+3,1,4}).$$

Note that $h(P_4) \mid h(B_{5(k-1)+4,1,4})$ implies that $h(P_4) \mid h(B_{5k+12,1,4})$. From this together with Theorem 3.3, the theorem holds.

Lemma 4.2 *Let G be a graph such that $G \sim^h B_{n-8,1,4}$, where $n \geq 9$. Then G does not contain K_4 as one of its components.*

Proof Suppose that $h(K_4^-)|h(B_{n-8,1,4})$, from Lemma 2.3, we know that $h(K_4^-) = x^2(x+1)(x+4)$, which implies that $h_1(P_2)|h(B_{n-8,1,4})$. It contradicts to Theorem 3.2.

Theorem 4.2 *Let G be a graph such that $G \sim^h B_{n-8,1,4}$, where $n \geq 9$. Then*

(1) If $n = 9$, then $[G]_h = \{Q(2,4), B_{1,1,4}, P_4 \cup \psi_5^1, P_4 \cup \psi_5^2\}$;

(2) If $n = 16$, then $[G]_h = \{C_8(P_2) \cup D_7, Q(1,2) \cup C_6, C_4(P_3) \cup C_6\}$;

(3) If $n \neq 9, 16$, then $[G]_h = \{B_{n-8,1,4}\}$.

Proof (1) When $n = 9$, let graph G satisfy $h(G) = h(B_{1,1,4})$. From Lemmas 2.1, 2.2 and 2.6, we obtain that $p(G) = q(G) = 9$ and $R_1(G) = -1$. By direct calculation, we arrive at $h(G) = h(B_{1,1,4}) = x^4(x^5 + 9x^4 + 26x^3 + 28x^2 + 10x + 1)$. We consider the following cases:

Case 1 G is a connected graph.

From $R_4(G) = R_4(B_{1,1,4}) = 5$ and (3) of Lemma 3.5, it follows that $G \in \{Q(2,4), Q(3,3), B_{1,1,4}, B_{2,2,2}\}$. By calculation, we have that $Q(2,4), B_{1,1,4} \in [G]_h$.

Case 2 G is not a connected graph.

By calculation, we have $h(G) = h(B_{1,1,4}) = x^4 f_1(x)f_2(x)$, where $f_1(x) = x^2 + 3x + 1$ and $f_2(x) = x^3 + 6x^2 + 7x + 1$. Thus, $R_1(f_1(x)) = 1$. Noting that $b_1(f_1(x)) = 3$, we obtain that $f_1(x) = h_1(P_4)$ or $f_1(x) = h_1(C_3)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph.

Subcase 2.1 Neither P_4 nor C_3 is a component of G.

Since G is not connected, the expression of G is $G = aK_1 \cup G_1$, where $a \geq 1$ and G_1 is connected. It is not difficult to obtain that $q(G_1) - p(G_1) \geq 1$. We conclude, from Lemma 2.7, that $q(G_1) - p(G_1) \leq 1$. Thus, $q(G_1) - p(G_1) = 1$. From Lemma 2.6, it follows that $G_1 \cong F_8$ and $G = K_1 \cup F_8$. By calculation, we arrive at $h(G) = h(K_1 \cup F_8) \neq h(B_{1,1,4})$.

Subcase 2.2 Either P_4 or C_3 is a component of G.

Subcase 2.2.1 P_4 is a component of G.

Let $G = P_4 \cup G_1$, where $h_1(G_1) = x^3 + 6x^2 + 7x + 1$. The following subcases are taken into account:

Subcase 2.2.1.1 G_1 is a connected graph.

Noting that $R_1(G_1) = -2$ and $q(G_1) = p(G_1)+1 = 6$, we have from Lemma 2.6, that $G_1 \in \psi$. Since the order of G_1 is 5 and $p(\psi_p^3) \geq 6$, $p(\psi_p^4) \geq 6$, $p(\psi_p^5) \geq 6$, we have $G_1 \in \{\psi_5^1, \psi_5^2, \psi_5^6\}$. By calculation, $P_4 \cup \psi_5^1, P_4 \cup \psi_5^2 \in [G]_h$.

Subcase 2.2.1.2 G_1 is not a connected graph.

It follows that $G = P_4 \cup aK_1 \cup G_1$, where $a \geq 1$ and $h_1(G_1) = x^3 + 6x^2 + 7x + 1$. It is not difficult to get that $q(G_1) - p(G_1) \geq 2$. Remarking that $R_1(G_1) = -2$, we obtain, from Lemma 2.7, that $q(G_1) - p(G_1) \leq 2$, which results in $q(G_1) - p(G_1) = 2$. Thus we conclude, from Lemma 2.6, that $G_1 \cong K_4^-$ and $a = 1$. By calculation, $G = P_4 \cup K_1 \cup K_4^- \notin [G]_h$.

Subcase 2.2.2 C_3 is a component of G.

Let $G = C_3 \cup G_1$, where $h_1(G_1) = x^3 + 6x^2 + 7x + 1$. We have the following subcases to be considered.

Subcase 2.2.2.1 G_1 is a connected graph.

Note that $R_1(G_1) = -2$ and $q(G_1) = p(G_1) = 6$. It contradicts to Lemma 2.6.

Subcase 2.2.2.2 G_1 is not a connected graph.

It follows that $G = C_3 \cup aK_1 \cup G_1$, where $a \geq 1$ and $h_1(G_1) = x^3 + 6x^2 + 7x + 1$. It is not difficult to get that $q(G_1) - p(G_1) \geq 1$. Remarking that $R_1(G_1) = -2$, we conclude, from Lemma 2.6, that $1 \leq q(G_1) - p(G_1) \leq 2$. If $q(G_1) - p(G_1) = 1$, or $q(G_1) - p(G_1) = 2$. Then we can turn to Subcase 2.2.1 for the same contradiction.

(2) When $n = 10$, let G be a graph such that $h(G) = h(B_{2,1,4})$, which brings $p(G) = q(G) = 10$ and $R_1(G) = -1$. We distinguish the following cases:

Case 1 G is a connected graph.

From $R_4(G) = R_4(B_{2,1,4}) = 4$ and (2) of Lemma 3.5, it follows that $G \in \{C_4(P_7), C_5(P_6), C_6(P_5), C_7(P_4), C_8(P_3), Q_{1,6}, B_{2,1,4}\}$. By calculation, we have that $h(G) = h(B_{2,1,4})$ if and only if $G \cong B_{2,1,4}$, which implies that $B_{2,1,4}$ is adjoint uniqueness.

Case 2 G is not a connected graph.

By calculation, we obtain that $h(B_{2,1,4}) = x^5 f_1(x) f_2(x)$, where $f_1(x) = x + 3$ and $f_2(x) = x^4 + 7x^3 + 13x^2 + 7x + 1$. Remarking that $R_1(f_1(x)) = 1$ and $b_1(f_1(x)) = 2$. Since $f_1(x)$ is not a factor of adjoint polynomial of some graph G with $R_1(G) = 1$, it means that $B_{2,1,4}$ is adjoint uniqueness.

(3) When $n = 11$, using the similar method to that of (2), we can show that $B_{3,1,4}$ is adjoint uniqueness. The details of the proof are omitted.

(4) When $n \geq 12$, let $G = \bigcup_{i=1}^{t} G_i$. From Lemma 2.1, we have that

$$h(G) = \prod_{i=1}^{t} h(G_i) = h(B_{n-8,1,4}), \tag{4.1}$$

which results in $\beta(G) = \beta(B_{n-8,1,4}) \in [-2-\sqrt{5}, -4)$ by Lemma 2.10. Let s_i denote the number of components G_i such that $R(G_i) = -i$, where $i \geq -1$. From Theorem 4.1, Lemmas 2.1 and 2.2, it follows that $0 \leq s_{-1} \leq 1$ and

$$R_1(G) = \sum_{i=1}^{t} R_1(G_i) = -1, q(G) = p(G) \tag{4.2}$$

which results in

$$s_{-1} = s_1 + 2s_2 - 1. \tag{4.3}$$

We distinguish the following cases by $0 \leq s_{-1} \leq 1$:

Case 1 $s_{-1} = 0$.

It follows, from (4.3), that $s_2 = 0, s_1 = 1$ with $R_1(G_1) = -1$. Without loss of generality, we set

$$G = G_1 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.4}$$

where $R_1(G_1) = -1$, $\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\cup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\cup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\cup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} | l_3 \geq 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} | l_3 \geq l_2 \geq 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} | l_3 \geq l_2 \geq l_1 \geq 2\}$, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree T_{l_1,l_2,l_3} is denoted by $\mathcal{T}$ for short, $A = \{i | i \geq 4\}$ and $B = \{j | j \geq 5\}$.

From Lemmas 3.2, 3.3 and 3.4, we arrive at

$$R_4(G) = R_4(B_{n-8,1,4}) = 4 = R_4(G_1) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.5}$$

From (1) of Lemma 2.7, it follows that $q(G_1) - p(G_1) \leq 1$. Combining this with (4.2), we know that $0 \leq q(G_1) - p(G_1) \leq 1$. Thus, we consider the following subcases:

Subcase 1.1 $q(G_1) = p(G_1) + 1$.

From Lemmas 2.6 and 4.2, we have $G_1 \cong F_m$. Recalling that $q(G) = p(G)$, we obtain the following equality:

$$a + b + |\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = 1. \tag{4.6}$$

If $m \geq 9$, from (3) of Lemma 3.4, (4.5) and (4.6), we arrive at $|B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$, which leads to $|B| + a + |\mathcal{T}_1| = 1$, $|\mathcal{T}_2| = |\mathcal{T}_3| = 0$ and $a + b + |\mathcal{T}_1| = 1$. Then we have the following three cases to be considered:

If $|B| = 1$, then $a = |\mathcal{T}_1| = 0$ and $b = 1$, which results in

$$G = F_m \cup (\cup_{i\in A} C_i) \cup D_j \cup fD_4 \cup T_{1,1,1}.$$

If $a = 1$, then $|B| = |\mathcal{T}_1| = b = 0$, which leads to

$$G = F_m \cup (\cup_{i\in A} C_i) \cup fD_4 \cup K_1.$$

If $|\mathcal{T}_1| = 1$, then $|B| = a = b = 0$, which brings about

$$G = F_m \cup (\cup_{i\in A} C_i) \cup fD_4 \cup T_{1,1,l_3}.$$

From the above arguments, we have, from Lemmas 2.9 and 2.10, that $\beta(G) = \beta(F_m)$. From (2) of Theorem 3.4 and $\beta(G) = \beta(B_{n-8,1,4})$, it follows that $\beta(F_m) = \beta(B_{n-8,1,4})$ if and only if $m = 6$, $n = 10$, or $m = 9$, $n = 16$. Note that $p(G) = p(B_{n-8,1,4}) = n$, so we only have $G = F_9 \cup C_6 \cup K_1$, or $G = F_9 \cup T_{1,1,4}$, which contradicts to $h(G) = h(B_{8,1,4})$ by direct calculation.

Subcase 1.2 $q(G_1) = p(G_1)$.

Recalling that $q(G) = p(G)$, we arrive at, from (4.4), $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, which leads to

$$G = G_1 \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4. \tag{4.7}$$

From (3) of Lemmas 2.6 and 2.10, it follows that

$$G_1 \in \{B_{m-t-4,1,t}, C_r(P_2), Q(1,2), C_4(P_3)\}, \tag{4.8}$$

where $m - t - 4, t$ and r satisfy the conditions of Lemma 2.10.

We distinguish the following subcases by (4.8):

Subcase 1.2.1 $G_1 \cong C_r(P_2)$.

From Lemmas 2.9, 2.10 and (1) of Lemma 3.7, it follows that $\beta(G) = \beta(C_r(P_2))$. Since $\beta(G) = \beta(B_{n-8,1,4})$, we have, from (1) of Theorem 3.4, that $\beta(G) = \beta(C_r(P_2))$ if and only if $p(G) = n = 16$, $r = 8$. From (4.7) and $p(G) = 16$, we only have that $G = C_8(P_2) \cup C_7$ or $G = C_8(P_2) \cup D_7$. By calculation, we arrive at $C_8(P_2) \cup D_7 \in [G]_h$.

Subcase 1.2.2 $G_1 \cong Q(1,2)$ or $G_1 \cong C_4(P_3)$.

From (4) and (5) of Theorem 3.4 and Lemma 2.9, we have that $\beta(G) = \beta(G_1) = \beta(B_{n-8,1,4})$ if and only if $p(G) = n = 12$, which brings about $G_1 \in \mathscr{G}_1 = \{Q(1,2) \cup C_6, C_4(P_3) \cup C_6\}$ by (4.7). By calculation, we have $\mathscr{G}_1 \subseteq [G]_h$.

Subcase 1.2.3 $G_1 \cong B_{m-t-4,1,t}$.

We distinguish the following subcases:

Subcase 1.2.3.1 $t = 1$.

From (3) of Lemma 3.7 and Lemma 2.9, we obtain that $\beta(G) = \beta(B_{m-5,1,1})$. According to (6) of Theorem 3.4, $\beta(B_{m-5,1,1}) = \beta(B_{n-8,1,4})$ if and only if $m = 6$, $n = 16$, which leads to $G \in \mathscr{G}_2 = \{B_{1,1,1} \cup C_{10}, B_{1,1,1} \cup D_{10}, B_{1,1,1} \cup C_4 \cup C_6, B_{1,1,1} \cup D_4 \cup D_6, B_{1,1,1} \cup C_4 \cup D_6, B_{1,1,1} \cup D_4 \cup C_6\}$ from (4.7). By direct calculation, $\mathscr{G}_2 \nsubseteq [G]_h$.

Subcase 1.2.3.2 $t = 2$.

From (3) of Lemma 3.7 and Lemma 2.9, (7) of Theorem 3.4, it follows that $\beta(G) = \beta(B_{m-6,1,2}) = \beta(B_{n-8,1,4})$ if and only if $m = 7$, $n = 10$ or $m = 10$, $n = 16$, which leads to $G \in \{B_{4,1,2} \cup C_6, B_{4,1,2} \cup D_6$ from (4.7). By calculation, we know that it contradicts to $h(G) = h(B_{8,1,4})$.

Subcase 1.2.3.3 $t = 3$.

From (8) of Theorem 3.4, it follows that $\beta(G) = \beta(B_{m-7,1,3}) = \beta(B_{n-8,1,4})$ if and only if $m = 13$, $n = 16$, which contradicts $h(G) = h(B_{8,1,4})$.

Subcase 1.2.3.4 $t \geq 5$.

From Lemma 2.9, (3), (4) of Theorem 3.4 and (3) of Theorem 3.4, we arrive at $\beta(G) = \beta(B_{m-t-4,1,t}) < \beta(B_{n-8,1,4})$, which contradicts to $\beta(G) = \beta(B_{n-8,1,4})$ by direct calculation.

As analyzed above, we obtain that $t = 4$. From (4) of Theorem 3.4 and Lemma 2.9, it follows that $\beta(G) = \beta(B_{m-8,1,4})$, together with $\beta(G) = \beta(B_{n-8,1,4})$ and (3) of Lemma 3.8, we arrive at $m = n$. Hence $G \cong B_{n-8,1,4}$.

Case 2 $s_{-1} = 1$.

It follows, from (4.3), that $s_1 + 2s_2 = 2$, which leads to

$$s_2 = 1, s_1 = 0, \text{ or } s_2 = 0, s_1 = 2. \tag{4.9}$$

We distinguish the following cases by (4.9):

Subcase 2.1 $s_2 = 1$, $s_1 = 0$.

Without loss of generality, let G_1 be the component such that $R_1(G_1) = -2$. From Corollary 2.1, we know that $\beta(G_1) < -2 - \sqrt{5}$, which contradicts to $\beta(B_{n-8,1,4}) \in [-2 - \sqrt{5}, -4)$.

Subcase 2.2 $s_2 = 0$, $s_1 = 2$.

Without loss of generality, let

$$G = G_1 \cup G_2 \cup G_3 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.10}$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_2) = R_1(G_3) = -1$, $\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\cup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\cup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\cup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} | l_3 \geq 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} | l_3 \geq l_2 \geq 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} | l_3 \geq l_2 \geq l_1 \geq 2\}$, $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree T_{l_1,l_2,l_3} is denoted by T for short, $A = \{i | i \geq 4\}$ and $B = \{j | j \geq 5\}$.

From Lemmas 3.2, 3.3 and 3.4, we arrive at

$$R_4(G) = R_4(B_{n-8,1,4}) = 4 = \sum_{i=1}^{3} R_4(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.11}$$

Subcase 2.2.1 $G_1 \cong P_4$.

In terms of Lemmas 2.6, 2.7, (4.2) and (4.10), we have that $1 \leq \sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 2$, which implies the following subcases:

Subcase 2.2.1.1 $q(G_2) - p(G_2) = 1$, $q(G_3) - p(G_3) = 1$.

From Lemmas 2.6, 4.2 and (4.10), it follows that $G_i \cong F_m (i = 2, 3)$ and $a + b + |\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = 1$. Thus

if $b = 0$, then we obtain, from (4.11), that $4 = -1 + 2R_4(F_m) + |B| + 1$, which contradicts $R_4(F_m) = 3$ by Lemma 3.4.

if $b = 1$, then we have, from (4.11), that $4 = -1 + 2R_4(F_m) + |B|$, which also contradicts to $R_4(F_m) = 3$ by Lemma 3.4.

Subcase 2.2.1.2 $q(G_2) = p(G_2)$, $q(G_2) - p(G_2) = 1$.

It is obvious that $G_2 \in \xi$, $G_3 \cong F_m$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemmas 2.6, 4.2 and (4.10). From (4.11), we arrive at $R_4(G_2) = 5 - R_4(F_m) - |B| \leq 2 - |B| \leq 2$, which contradicts $G_2 \in \xi$ by Corollary 3.1.

Subcase 2.2.2 $G_1 \cong C_3$.

From Lemmas 2.6, 2.7, (4.2) and (4.10), we get that $0 \le \sum_{i=2}^{3}(q(G_i) - p(G_i)) \le 2$, which brings about the following subcases:

Subcase 2.2.2.1 $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 2$.

Applying Lemmas 2.6, 4.2, and (4.10), we have that $G_i \cong F_m (i = 2, 3)$ and $a + b + |\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = 2$. From these together with (4.11), we know that

If $b = 0$, then $4 = -2 + 2R_4(F_m) + |B| + 2$, which contradicts to $R_4(F_m) = 3$ by Lemma 3.4.

If $b = 1$, then $4 = -2 + 2R_4(F_m) + |B| + 1$, which also contradicts to $R_4(F_m) = 3$ by Lemma 3.4.

If $b = 2$, then we have, from (4.11), that $4 = -2 + 2R_4(F_m) + |B|$, which results in

$$G = C_3 \cup F_m \cup F_m \cup (\cup_{i\in A} C_i) \cup f D_4 \cup 2T_{1,1,1}.$$

In terms of Lemmas 2.9 and 2.10, we have that $\beta(G) = \min\{\beta(F_{m_1}), \beta(F_{m_2})\} = \beta(F_{m_1})$ if $m_1 \ge m_2$. By (2) of Theorem 3.4, it follows that $\beta(G) = \beta(F_{m_1}) = \beta(B_{n-8,1,4})$ if and only if $m_1 = 6$, $n = 10$ or $m_1 = 9$, $n = 16$. This contradicts $p(G) = p(B_{n-8,1,4})$.

Subcase 2.2.2.2 $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 1$.

From Lemmas 2.6, 4.2 and (4.10), it follows that $G_2 \in \xi$, $G_3 \cong F_m$ and $a+b+|\mathcal{T}_1|+|\mathcal{T}_2|+|\mathcal{T}_3| = 1$. Thus

if $b = 0$, then we obtain, from (4.11), that $4 = -2 + R_4(G_2) + R_4(F_m) + |B| + 1$, which results in $R_4(G_2) \le 2 - |B| \le 2$. It contradicts $G_2 \in \xi$.

if $b = 1$, then we have, from (4.11), that $4 = -2 + R_4(G_2) + R_4(F_m) + |B|$, which leads to $R_4(G_2) = 3$ and $|B| = 0$. Thus

$$G = C_3 \cup G_2 \cup F_m \cup (\cup_{i\in A} C_i) \cup f D_4 \cup T_{1,1,1}.$$

In terms of (1) of Lemma 3.5, we have that $G_2 \in \{C_{n-1}(P_2)\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1}\}$.

If $G_2 \cong C_r(P_2)$, then we obtain, from (1) of Theorem 3.4, that $\beta(G) = \beta(B_{n-8,1,4}) = \beta(C_r(P_2)) = \beta(F_m)$ if and only if $r = 8$, $m = 9$, $n = 16$. It contradicts to $p(G) = 16$.

If $G_2 \cong B_{s,1,1}$, then we get, from (6) of Theorem 3.4, that $\beta(G) = \beta(B_{n-8,1,4}) = \beta(B_{s,1,1}) = \beta(F_m)$ if and only if $s = 1$, $m = 9$, $n = 16$. This contradicts to $p(G) = 16$.

If $G_2 \cong Q_{1,1}$, then from (2) of Theorem 3.4 we arrive at $\beta(G) = \beta(B_{n-8,1,4}) = \beta(F_m)$ if and only if $m = 9$, $n = 16$ or $m = 6$, $n = 10$. It also contradicts to $p(G) = 16$.

Subcase 2.2.2.3 $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 0$.

It is easy to see that $G_i \in \xi (i = 2, 3)$ and $a + b + |\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = 1$ by Lemmas 2.6, 4.2 and (4.10). From (4.11), it follows that $4 = -2 + R_4(G_2) + R_4(G_3) + |B|$. Combining with Corollary 3.1, we have $|B| = 0$ and $R_4(G_i) = 3(i = 2, 3)$. Then

$$G = C_3 \cup G_2 \cup G_3 \cup F_m \cup (\cup_{i\in A} C_i) \cup f D_4.$$

In terms of Lemma 3.5, we have that $G_i \in \{C_{n-1}(P_2)\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1}\}(i = 2, 3)$. With the same methods as that of Subcase 2.2.2.2, we can get a contradiction.

This completes the proof of the theorem. □

Corollary 4.1 *If $n \geq 9$, graph $B_{n-8,1,4}$ is adjoint uniqueness if and only if $n \neq 9, 16$.*

Corollary 4.2 *If $n \geq 9$, the chromatic equivalence class of $\overline{B_{n-8,1,4}}$ only contains the complements of graphs that are in Theorem 4.2.*

Corollary 4.3 *If $n \geq 9$, graph $\overline{B_{n-8,1,4}}$ is chromatic uniqueness if and only if $n \neq 9, 16$.*

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