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Chromatic Uniqueness of K_4-Homeomorphs with Girth 8

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Abstract In this paper, we determine all graphs of K_4-homeomorphs of girth 8 which are chromatically unique.

Keywords chromatic polynomial; chromatically unique graph; K_4-homeomorph.

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1. Introduction

All graphs considered here are finite and simple. For notations and terminology not defined here, we refer to [1]. Let G be a graph and $P(G;\lambda)$ be the chromatic polynomial of G. Two graphs G and H are chromatically equivalent, denoted by $G \sim H$, if $P(G) = P(H)$. A graph G is chromatically unique (or simply χ-unique) if G is isomorphic to H whenever $G \sim H$.

A K_4-homeomorph is a subdivision of the complete graph K_4 which is denoted by $K_4(a,b,c,d,e,f)$. $K_4(a,b,c,d,e,f)$ is a graph that the six edges of K_4 are replaced by the six paths of length a,b,c,d,e,f, respectively, as shown in Figure 1. The study of the chromaticity of K_4-homeomorphs which have girth 3, 4, 5, 6 or 7 has been settled (see [8] and the references therein). When referring to the chromaticity of K_4-homeomorphs with girth 8, there are 13 types altogether, which are $K_4(1,1,6,d,e,f)$, $K_4(1,1,c,1,e,5)$, $K_4(1,1,c,2,e,4)$, $K_4(1,2,c,1,e,4)$, $K_4(1,1,c,3,e,3)$, $K_4(1,3,c,1,e,3)$, $K_4(2,3,3,d,e,f)$, $K_4(1,2,5,d',e',f')$, $K_4(1,3,4,d',e',f')$, $K_4(1,2,c,2,e,3)$, $K_4(1,2,c,3,e,2)$, $K_4(2,3,3,d,e,f)$, $K_4(2,2,4,d,e,f)$, $K_4(2,2,c,2,e,2)$. As we know, only the chromaticity of the ones with at least 2 paths of length 1 have been obtained among all those K_4-homeomorphs with girth 8 (see [4, 7, 10]). In this article, we will discuss the chromaticity of the others. If we write all the whole, the paper will be too long. Therefore we only write one case $K_4(2,3,3,d,e,f)$ (as Figure 2) of them here and the details of the left cases will be given in other papers.

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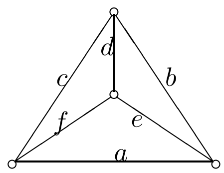

Figure 1 $K_4(a,b,c,d,e,f)$

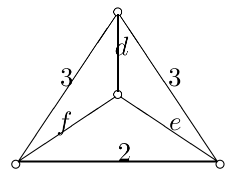

Figure 2 $K_4(2,3,3,d,e,f)$

2. Preparation

The following are some known results.

Proposition 2.1 ([2, 5]) *Let G and H be chromatically equivalent. Then*

(i) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$;

(ii) G and H have the same girth and same number of cycles with the length equal to their girth;

(iii) If G is a K_4-homeomorph, then H is a K_4-homeomorph as well;

(iv) If G and H are homeomorphic to K_4, then both the minimum values of parameters and the number of parameters equal to this minimum value of the graphs G and H coincide.

Proposition 2.2 ([6]) *The graph $K_4(a,b,c,d,e,f)$ is chromatically unique if exactly four numbers among $\{a,b,c,d,e,f\}$ are the same.*

Proposition 2.3 ([7]) *Suppose that $G = K_4(a,b,c,d,e,f)$ and $H = K_4(a',b',c',d',e',f')$. If $G \sim H$ and $\{a,b,c,d,e,f\} = \{a',b',c',d',e',f'\}$ as multisets, then $G \cong H$.*

Proposition 2.4 ([9]) *G and H are both in the type of $K_4(2,3,3,d,e,f)$, then $P(G) = P(H)$ if and only if $G \cong H$.*

3. Main results and proofs

In the following, the girth of any graph we mentioned is 8.

Lemma 3.1 *If G is in the type of $K_4(2,3,3,d,e,f)$, and H is in the type of $K_4(1,2,5,d',e',f')$, then $G \sim H$ if G is isomorphic to $K_4(2,3,3,1,6,f)$, $K_4(2,3,3,1,4,6)$, or $K_4(2,3,3,1,5,6)$.*

Proof Let G and H be two graphs such that $G \cong K_4(2,3,3,d,e,f)$ and $H \cong K_4(1,2,5,d',e',f')$. Since the girth of G is 8, there is at most one 1 among d, e and f.

Let

$$Q(K_4(a,b,c,d,e,f)) = -(x+1)(x^a + x^b + x^c + x^d + x^e + x^f) + x^{a+d} + x^{b+f} + x^{c+e} + x^{a+b+e} + x^{b+c+d} + x^{a+c+f} + x^{d+e+f}.$$

Let $x = 1 - \lambda$. Then it follows from [3] that the chromatic polynomial of $K_4(a,b,c,d,e,f)$ is

$$P(K_4(a,b,c,d,e,f)) = (-1)^{n+1}\frac{x}{(x-1)^2}\Big[(x^2+3x+2)+Q(K_4(a,b,c,d,e,f))-x^{n+1})\Big].$$

Hence $P(G) = P(H)$ if and only if $Q(G) = Q(H)$. We solve the equation $Q(G) = Q(H)$ to get all solutions. In the following, we will substitute h.p. for highest power and l.p. for lowest power.

$$\begin{aligned}Q(G) =& -(x+1)(x^2+2x^3+x^d+x^e+x^f)+x^{2+d}+x^{3+f}+x^{3+e}+x^{5+e}+\\ & x^{6+d}+x^{5+f}+x^{d+e+f},\\ Q(H) =& -(x+1)(x+x^2+x^5+x^{d'}+x^{e'}+x^{f'})+x^{1+d'}+x^{2+f'}+x^{3+e'}+x^{5+e'}+\\ & x^{7+d'}+x^{6+f'}+x^{d'+e'+f'}.\end{aligned}$$

Considering the symmetry of the graph $K_4(2,3,3,d,e,f)$, we can assume $e \le f$. From Proposition 2.1, we have that $\min\{d,e,f\} = \min\{d,e\} = 1$, and

$$d+e+f = d'+e'+f'. \tag{1}$$

There are 2 cases to be considered.

Case 1 $\min\{d,e\} = d = 1$. Here we have that $Q(G) = Q(H)$ iff $Q_1(G) = Q_1(H)$, where

$$\begin{aligned}Q_1(G) =& -x^3-2x^4-x^e-x^{e+1}-x^f-x^{f+1}+x^{3+f}+x^{3+e}+x^{5+e}+x^7+x^{5+f},\\ Q_1(H) =& -x^5-x^6-x^{d'}-x^{e'}-x^{e'+1}-x^{f'}-x^{f'+1}+x^{2+f'}+x^{3+e'}+x^{5+e'}+x^{7+d'}+x^{6+f'}.\end{aligned}$$

Since $d+e \ge 5$, we have

$$f \ge e \ge 4. \tag{2}$$

After comparing the powers in $Q_1(G)$ and $Q_1(H)$, we have the h.p. in $Q_1(G)$ is $5+f$. Considering the h.p. in $Q_1(G)$ and $Q_1(H)$, we know there are 3 cases to be considered.

Case 1.1 $\max\{5+e', 7+d', 6+f'\} = 5+e' = 5+f$. Now from the equation $Q_1(G) = Q_1(H)$, we obtain $Q_2(G) = Q_2(H)$ where

$$\begin{aligned}Q_2(G) =& -x^3-2x^4-x^e-x^{e+1}+x^{3+e}+x^{5+e}+x^7,\\ Q_2(H) =& -x^5-x^6-x^{d'}-x^{f'}-x^{f'+1}+x^{2+f'}+x^{7+d'}+x^{6+f'}.\end{aligned}$$

So $d' = 4$, $f' = 3$ and $e = 6$. Thus $K_4(2,3,3,1,6,f) \sim K_4(1,2,5,4,f,3)$.

Case 1.2 $\max\{5+e', 7+d', 6+f'\} = 6+f' = 5+f$. After simplifying $Q_1(G)$ and $Q_1(H)$, we obtain $Q_3(G) = Q_3(H)$ and

$$\begin{aligned}Q_3(G) =& -x^3-2x^4-x^e-x^{e+1}-x^{f+1}+x^{3+f}+x^{3+e}+x^{5+e}+x^7,\\ Q_3(H) =& -x^5-x^6-x^{d'}-x^{e'}-x^{e'+1}-x^{f'}+x^{2+f'}+x^{3+e'}+x^{5+e'}+x^{7+d'}.\end{aligned}$$

Now we can assume $5+e' < 6+f'$ since $5+e' = 5+f$ has been discussed in Case 1.1. As $7+d' \le 6+f'$, the term $x^{2+f'}$ cannot be cancelled by any negative term in $Q_3(H)$, then none of the terms in $Q_3(H)$ is equal to the term $-x^{f+1}$ in $Q_3(G)$ by noting $f+1 = f'+2$. Therefore, $2x^{2+f'} \in Q_3(G)$. Considering (2), we get $3+e = 7 = 2+f'$. Thus $e = 4, f = 6, f' = 5$. Then

$-3x^4 \in Q_3(G)$, but $-3x^4 \notin Q_3(H)$, a contradiction.

Case 1.3 $\max\{5+e', 7+d', 6+f'\} = 7+d' = 5+f$. After discussing the case $5+f = 5+e'$, we can suppose that

$$5+f > 5+e'. \tag{3}$$

Cancelling the same terms of $Q_1(G)$ and $Q_1(H)$, we get

$$Q_4(G) = -x^3 - 2x^4 - x^e - x^{e+1} - x^f - x^{f+1} + x^{3+f} + x^{3+e} + x^{5+e} + x^7,$$
$$Q_4(H) = -x^5 - x^6 - x^{d'} - x^{e'} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{2+f'} + x^{3+e'} + x^{5+e'} + x^{6+f'}.$$

Consider $-x^3$ and $-2x^4$ in $Q_4(G)$. It is due to $Q_4(G) = Q_4(H)$ that there are terms in $Q_4(H)$ which are equal to $-x^3$ and $-2x^4$, respectively. The following cases should be considered.

Case 1.3.1 If $e' = 3$, $f' = 4$, then $e = 4$ from equation (1). After simplification, we obtain

$$Q_5(G) = -x^4 - x^f - x^{f+1} + x^{3+f} + x^9 + 2x^7, \quad Q_5(H) = -x^{d'} - x^5 + x^6 + x^8 + x^{10}.$$

No matter what value d' is, $Q_5(G) \neq Q_5(H)$, which means $Q(G)$ is not equal to $Q(H)$.

Case 1.3.2 If $e' = 4, f' = 3$, here we also have $e = 4$. After cancelling the same terms, we get $Q_6(G) = Q_6(H)$ where

$$Q_6(G) = -x^4 - x^f - x^{f+1} + x^{3+f} + x^7, \quad Q_6(H) = -x^6 - x^{d'} + x^9.$$

It is easy to see that $d' = 4$ and $f = 6$. Thus we obtain the solution where G is isomorphic to $K_4(2,3,3,1,4,6)$ and H is isomorphic to $K_4(1,2,5,4,4,3)$. That is $K_4(2,3,3,1,4,6) \sim K_4(1,2,5,4,4,3)$.

Case 1.3.3 If $d' = e'+1 = 4$, then $f = 6$ and $f' = e$. We obtain $Q_7(G) = Q_7(H)$ after simplifying $Q_4(G)$ and $Q_4(H)$ where

$$Q_7(G) = -x^6 + x^9 + x^{3+e} + x^{5+e}, \quad Q_7(H) = -x^5 + x^{2+f'} + x^8 + x^{6+f'}.$$

As $e \geq 4$ (see(2)), the highest terms of $Q_7(G)$ and $Q_7(H)$ are not equal, a contradiction.

Case 1.3.4 If $d' = f'+1 = 4$, then $f = 6$ and $e' = e$ (noting (1)). By (2) and (3), $e = 4$ or 5. It is easy to see that $Q_4(G) = Q_4(H)$. Thus $K_4(2,3,3,1,4,6) \sim K_4(1,2,5,4,4,3)$, and $K_4(2,3,3,1,5,6) \sim K_4(1,2,5,4,5,3)$.

Case 1.3.5 $d' = 3, e' = f' = 4$. Then $-3x^5 \in Q_4(H)$, but not in $Q_4(G)$, a contradiction.

Case 2 $\min\{d, e\} = e = 1$. Since $d+e \geq 5$, $e+f \geq 6$, we have $d \geq 4$, $f \geq 5$. Cancelling equal terms of $Q(G)$ and $Q(H)$, we know $Q_8(G) = Q_8(H)$ where

$$Q_8(G) = -2x^3 - x^4 - x^d - x^{d+1} - x^f - x^{f+1} + x^{2+d} + x^{3+f} + x^6 + x^{6+d} + x^{5+f},$$
$$Q_8(H) = -x^5 - x^6 - x^{d'} - x^{e'} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{2+f'} + x^{3+e'} + x^{5+e'} + x^{7+d'} + x^{6+f'}.$$

Comparing the l.p. in $Q_8(G)$ and the l.p. in $Q_8(H)$, two of d', e', f' are 3. Since $e'+f' \geq 7$, only two cases need to be considered.

Case 2.1 $d' = e' = 3$. After cancelling the same terms, we get

$$\begin{aligned}Q_9(G) &= -x^d - x^{d+1} - x^f - x^{f+1} + x^{2+d} + x^{3+f} + x^6 + x^{6+d} + x^{5+f},\\ Q_9(H) &= -x^5 - x^{f'} - x^{f'+1} + x^{2+f'} + x^8 + x^{10} + x^{6+f'}.\end{aligned}$$

Noting $f' \geq 4$, so the h.p. in $Q_9(H)$ is $6 + f'$. So $6 + f' = 6 + d$ or $6 + f' = 5 + f$.

If $6 + f' = 6 + d$, we obtain $f = 5$ from (1). Therefore, $Q_9(G) = Q_9(H)$. In fact, H is isomorphic to G in this case.

If $6 + f' = 5 + f$, then $d = 4$ (noting (1)). Cancelling equal terms, we get

$$Q_{10}(G) = -x^4 - x^{f+1} + 2x^6 + x^{3+f}, \quad Q_{10}(H) = -x^{f'} + x^{2+f'} + x^8.$$

When $f' + 1 = f = 5$, $Q_{10}(G) = Q_{10}(H)$. Thus G and H are also isomorphic.

Case 2.2 $d' = f' = 3$. Cancelling equal terms of $Q_8(G)$ and $Q_8(H)$, we know $Q_{11}(G) = Q_{11}(H)$ where

$$\begin{aligned}Q_{11}(G) &= -x^d - x^{d+1} - x^f - x^{f+1} + x^{2+d} + x^{3+f} + x^6 + x^{6+d} + x^{5+f},\\ Q_{11}(H) &= -x^6 - x^{e'} - x^{e'+1} + x^{3+e'} + x^{5+e'} + x^{10} + x^9.\end{aligned}$$

As $e' \geq 4$ (by noting $e' + f' \geq 7$), no positive term in $Q_{11}(H)$ is x^6, thus $-2x^6 \in Q_{11}(G)$. It is easy to see that $d = f = 6$ or $d = f + 1 = 6$ or $d + 1 = f = 6$.

If $d = f = 6$, we get $e' = 7$. We can easily see that $Q_{11}(G) \neq Q_{11}(H)$.

If $d = f + 1 = 6$, we get $e' = 6$. But $Q_{11}(G) \neq Q_{11}(H)$.

If $d + 1 = f = 6$, we get $e' = 6$. But $Q_{11}(G) \neq Q_{11}(H)$.

The proof of the lemma is now completed. □

Lemma 3.2 *If G is in the type of $K_4(2,3,3,d,e,f)$, and H is in the type of $K_4(1,3,4,d',e',f')$, then $G \sim H$ when G is isomorphic to $K_4(2,3,3,1,4,4)$, or $K_4(2,3,3,1,e,e+2)$.*

Proof Let G and H be two graphs such that $G \cong K_4(2,3,3,d,e,f)$ and $H \cong K_4(1,3,4,d',e',f')$. As the above discussion, we know

$$\begin{aligned}Q(G) = &-(x+1)(x^2 + 2x^3 + x^d + x^e + x^f) + (x^{2+d} + x^{3+f} + x^{3+e} + x^{5+e} +\\ &x^{6+d} + x^{5+f} + x^{d+e+f}),\\ Q(H) = &-(x+1)(x + x^3 + x^4 + x^{d'} + x^{e'} + x^{f'}) + (x^{1+d'} + x^{3+f'} +\\ &2x^{4+e'} + x^{5+f'} + x^{7+d'} + x^{d'+e'+f'}).\end{aligned}$$

From Proposition 1 and the symmetry of the graph $K_4(2,3,3,d,e,f)$, we know the equation (1) also holds and $\min\{d,e,f\} = \min\{d,e\} = 1$.

Case 1 $\min\{d,e\} = d = 1$. As $d + e \geq 5$, then

$$f \geq e \geq 4. \tag{4}$$

After simplification, we have $Q_1(G) = Q_1(H)$, where

$$Q_1(G) = -x^3 - x^2 - x^e - x^{e+1} - x^f - x^{f+1} + x^{3+f} + x^{3+e} + x^{5+e} + x^7 + x^{5+f},$$

$$Q_1(H) = -x^5 - x^{d'} - x^{e'} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{3+f'} + 2x^{4+e'} + x^{5+f'} + x^{7+d'}.$$

By considering the h.p. in $Q_1(G)$ and the h.p. in $Q_1(H)$, we have $5+f = \max\{4+e', 7+d', 5+f'\}$.

Case 1.1 $\max\{4+e', 7+d', 5+f'\} = 5+f' = 5+f$. After simplifying $Q_1(G)$ and $Q_1(H)$, we have $Q_2(G) = Q_2(H)$ where

$$Q_2(G) = -x^3 - x^2 - x^e - x^{e+1} + x^{3+e} + x^{5+e} + x^7,$$
$$Q_2(H) = -x^5 - x^{d'} - x^{e'} - x^{e'+1} + 2x^{4+e'} + x^{7+d'}.$$

Consider $-x^2$ in $Q_2(G)$, we know $d' = 2$ or $e' = 2$.

If $d' = 2$, since $d' + e' \geq 5$ and $-x^3$ is in $Q_2(G)$, we get $e' = 3$. Thus $e = 4$ (see (1)), and $G \cong H$.

If $e' = 2$, then $e = d' + 1$ (see (1)). After simplification, we have

$$Q_3(G) = -x^e - x^{e+1} + x^{3+e} + x^{5+e} + x^7, \quad Q_3(H) = -x^5 - x^{d'} + 2x^6 + x^{7+d'}.$$

By considering the h.p. in $Q_3(G)$ and the h.p. in $Q_3(H)$, we know that $Q_3(G) \neq Q_3(H)$, thus $Q(G)$ is not equal to $Q(H)$.

Case 1.2 $\max\{4+e', 7+d', 5+f'\} = 7+d' = 5+f$. We have discussed the case $5+f = 5+f'$, so we can assume that $5+f > 5+f'$. Cancelling the same terms in $Q_1(G)$ and $Q_1(H)$, we have $Q_4(G) = Q_4(H)$ where

$$Q_4(G) = -x^3 - x^2 - x^e - x^{e+1} - x^f - x^{f+1} + x^{3+f} + x^{3+e} + x^{5+e} + x^7,$$
$$Q_4(H) = -x^5 - x^{d'} - x^{e'} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{3+f'} + 2x^{4+e'} + x^{5+f'}.$$

Comparing the lowest power of $Q_4(G)$ to the lowest power of $Q_4(H)$, we get $\min\{d', e', f'\} = 2$.

If $d' = 2$, then we get $e = f = d' + 2 = 4$ (by (4)) and $e' + f' = 7$ (see (1)). Cancelling equal terms, we obtain $Q_5(G) = Q_5(H)$ where

$$Q_5(G) = -x^3 - x^5 - 2x^4 + 3x^7 + x^9,$$
$$Q_5(H) = -x^{e'} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{3+f'} + 2x^{4+e'} + x^{5+f'}.$$

Consider $3x^7$ in $Q_5(G)$, we know that $e' = 3$, $f' = 4$ and then $Q_5(G) = Q_5(H)$. In fact, $K_4(2,3,3,1,4,4) \cong K_4(1,3,4,2,3,4)$.

If $e' = 2$, we know $f' = e + 1$ from (1). Cancelling equal terms in $Q_4(G)$ and $Q_4(H)$, we get $Q_6(G) = Q_6(H)$ where

$$Q_6(G) = -x^e - x^f - x^{f+1} + x^{3+f} + x^{3+e} + x^{5+e} + x^7,$$
$$Q_6(H) = -x^5 - x^{d'} - x^{f'+1} + x^{3+f'} + 2x^6 + x^{5+f'}.$$

As $5 + f' = 6 + e > 7$, and the h.p. in $Q_6(H)$ is $5 + f'$, then we get $3 + f = 5 + f'$, that is $f = 2 + d' = 2 + f' = 3 + e$. After simplification, we have $Q_7(G) = Q_7(H)$, where

$$Q_7(G) = -x^e - x^{f+1} + x^{5+e} + x^7, \quad Q_7(H) = -x^5 - x^{d'} - x^{f'+1} + x^{3+f'} + 2x^6.$$

Since $f'+3=f+1$, and no negative terms can cancel the term $x^{f'+3}$ (noting $e'+f'\geq 7$), $2x^{f+1}$ should be in $Q_7(G)$, which is impossible.

If $f'=2$, then $e'=e+1$(by(1)). After simplifying $Q_4(G)$ and $Q_4(H)$, we obtain $Q_8(G)=Q_8(H)$ where

$$Q_8(G)=-x^e-x^f-x^{f+1}+x^{3+f}+x^{3+e},\quad Q_8(H)=-x^{d'}-x^{e'+1}+x^{4+e'}.$$

We can check that $d'=e$ is a solution of $Q_8(G)=Q_8(H)$. Thus we obtain the solution of $Q(G)=Q(H)$ where G is isomorphic to $K_4(2,3,3,1,e,e+2)$ and H is isomorphic to $K_4(1,3,4,e,e+1,2)$.

Case 1.3 $\max\{4+e',7+d',5+f'\}=4+e'=5+f$. As the coefficient of $x^{4+e'}$ is 2, we know $5+e$ should also be equal to $4+e'$. After simplifying $Q_1(G)$ and $Q_1(H)$, we have $Q_9(G)=Q_9(H)$, where

$$\begin{aligned}Q_9(G)=&-x^3-x^2-2x^e-x^{e+1}+2x^{3+e}+x^7,\\ Q_9(H)=&-x^5-x^{d'}-x^{e'+1}-x^{f'}-x^{f'+1}+x^{3+f'}+x^{5+f'}+x^{7+d'}.\end{aligned}$$

Since the lowest term in $Q_9(G)$ is $-x^2$, we have $d'=2$ or $f'=2$.

If $d'=2$, noting that $-x^3\in Q_3(G)$, we have $f'=3$. Since $d+e+f=d'+e'+f'$, we get $e=f=5$, $e'=6$. It is easy to check $Q_9(G)$ is not equal to $Q_9(H)$, a contradiction.

If $f'=2$, cancelling equal terms of $Q_9(G)$ and $Q_9(H)$, we get $Q_{10}(G)=Q_{10}(H)$, where

$$Q_{10}(G)=-2x^e-x^{e+1}+2x^{3+e},\quad Q_{10}(H)=-x^{d'}-x^{e'+1}+x^{7+d'}.$$

Because there are five terms in $Q_{10}(G)$, and no positive terms can cancel negative terms, but there are only three terms in $Q_{10}(H)$, $Q_{10}(G)\neq Q_{10}(H)$.

Case 2 $e=1$. Cancelling equal terms of $Q(G)$ and $Q(H)$, we have $Q_{11}(G)=Q_{11}(H)$, where

$$\begin{aligned}Q_{11}(G)=&-2x^3-x^2-x^d-x^{d+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^4+x^6+x^{6+d}+x^{5+f},\\ Q_{11}(H)=&-x^5-x^{d'}-x^{e'}-x^{e'+1}-x^{f'}-x^{f'+1}+x^{3+f'}+2x^{4+e'}+x^{5+f'}+x^{7+d'}.\end{aligned}$$

Consider the l.p. in $Q_{11}(G)$ and the l.p. in $Q_{11}(H)$, we have $\min\{d',e',f'\}=2$.

Case 2.1 $d'=2$. After simplifying, we have $Q_{12}(G)=Q_{12}(H)$, where

$$\begin{aligned}Q_{12}(G)=&-2x^3-x^d-x^{d+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^4+x^6+x^{6+d}+x^{5+f},\\ Q_{12}(H)=&-x^5-x^{e'}-x^{e'+1}-x^{f'}-x^{f'+1}+x^{3+f'}+2x^{4+e'}+x^{5+f'}+x^9.\end{aligned}$$

Since $-2x^3\in Q_{12}(G)$, and the power of each positive term is not equal to 3, $-2x^3\in Q_{12}(H)$. Then $e'+f'\leq 6$. It means length of a cycle of graph H is less than 8, a contradiction.

Case 2.2 $e'=2$. As $e'+f'\geq 7$, we have $f'\geq 5$. After simplifying $Q_{11}(G)$ and $Q_{11}(H)$, we have $Q_{13}(G)=Q_{13}(H)$ where

$$\begin{aligned}Q_{13}(G)=&-x^3-x^d-x^{d+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^4+x^{6+d}+x^{5+f},\\ Q_{13}(H)=&-x^5-x^{d'}-x^{f'}-x^{f'+1}+x^{3+f'}+x^6+x^{5+f'}+x^{7+d'}.\end{aligned}$$

Since $-x^3$ is in $Q_{13}(G)$, $d'=3$ (noting $f'\ge 5$). Cancelling equal terms, we get $Q_{14}(G)=Q_{14}(H)$ where

$$Q_{14}(G)=-x^d-x^{d+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^4+x^{6+d}+x^{5+f},$$
$$Q_{14}(H)=-x^5-x^{f'}-x^{f'+1}+x^{3+f'}+x^6+x^{5+f'}+x^{10}.$$

By considering the h.p. in $Q_{14}(G)$ and the h.p. in $Q_{14}(H)$, we have $5+f'=\max\{6+d,5+f\}$.

If $5+f'=5+f$, we have $d=4$. Thus $G\cong H$ in this case.

If $5+f'=6+d$, we know $f=5$ from equation $d+e+f=d'+e'+f'$. We now suppose $f'>f=5$, as $5+f=5+f'$ has been discussed just now. For $d=f'-1>4$, then we note x^4 is in $Q_{14}(G)$, but not in $Q_{14}(H)$, a contradiction.

Case 2.3 $f'=2$. As $e'+f'\ge 7$, we have $e'\ge 5$. By $Q_{11}(G)=Q_{11}(H)$, after simplification, we have $Q_{15}(G)=Q_{15}(H)$ where

$$Q_{15}(G)=-x^3-x^d-x^{d+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^4+x^6+x^{6+d}+x^{5+f},$$
$$Q_{15}(H)=-x^{d'}-x^{e'}-x^{e'+1}+2x^{4+e'}+x^7+x^{7+d'}.$$

Consider $-x^3$ in $Q_{15}(G)$. It is due to $Q_{15}(G)=Q_{15}(H)$ that there is one term in $Q_{15}(H)$ which is equal to $-x^3$. So we have $d'=3$ and then we get

$$Q_{16}(G)=-x^d-x^{d+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^4+x^6+x^{6+d}+x^{5+f},$$
$$Q_{16}(H)=-x^{e'}-x^{e'+1}+2x^{4+e'}+x^7+x^{10}.$$

Then we note $x^4\in Q_{16}(G)$, but the lowest power in $Q_{16}(H)$ is greater than 5. So one of the negative terms should be $-x^4$ in $Q_{16}(G)$. Noting $e=1$ and $f+e\ge 6$, we get $d=4$. From (1), $f=e'$. It is easy to say that $Q_{16}(G)\ne Q_{16}(H)$, which is a contradiction. So this lemma holds. □

Lemma 3.3 *If G is in the type of $K_4(2,3,3,d,e,f)$, and H is in the type of $K_4(1,2,c',2,e',3)$, then there is no graph G satisfying $G\sim H$.*

Proof Let G and H be two graphs such that $G\cong K_4(2,3,3,d,e,f)$ and $H\cong K_4(1,2,c',2,e',3)$. Then

$$Q(G)=-(x+1)(x^2+2x^3+x^d+x^e+x^f)+(x^{2+d}+x^{3+f}+x^{3+e}+x^{5+e}+ x^{6+d}+x^{5+f}+x^{d+e+f}),$$
$$Q(H)=-(x+1)(x+2x^2+x^3+x^{c'}+x^{e'})+(x^3+x^5+x^{3+e'}+2x^{4+c'}+ x^{5+e'}+x^{c'+e'}).$$

From Proposition 1, we know that $\min\{d,e,f\}=\min\{d,e\}=1$ and

$$d+e+f=c'+e'. \tag{5}$$

Cancelling equal terms, we have $Q_1(G)=Q_1(H)$ where

$$Q_1(G)=-x^3-x^4-x^d-x^{d+1}-x^e-x^{e+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^{3+e}+$$

$$x^{5+e}+x^{6+d}+x^{5+f},$$
$$Q_1(H)=-x-2x^2-x^{c'}-x^{c'+1}-x^{e'}-x^{e'+1}+x^5+x^{3+e'}+2x^{4+c'}+x^{5+e'}.$$

Consider $-x$ and $-2x^2$ in $Q_1(H)$. It is due to $Q_1(G)=Q_1(H)$ that there are terms in $Q_1(G)$ which are equal to $-x$ and $-2x^2$, so one of d,e,f is 1, and one of the left two is 2. However, the girth of G and H is 8, which needs $d+e\geq 5$ and $e+f\geq 6$. Hence we know there is no solution to $Q(G)=Q(H)$. □

Lemma 3.4 *If G is in the type of $K_4(2,3,3,d,e,f)$, and H is in the type of $K_4(1,2,c',3,e',2)$, then there is no graph G satisfying $G\sim H$.*

Proof Let G and H be two graphs such that $G\cong K_4(2,3,3,d,e,f)$ and $H\cong K_4(1,2,c',3,e',2)$. Then

$$Q(G)=-(x+1)(x^2+2x^3+x^d+x^e+x^f)+(x^{2+d}+x^{3+f}+x^{3+e}+x^{5+e}+x^{6+d}+x^{5+f}+x^{d+e+f}),$$
$$Q(H)=-(x+1)(x+2x^2+x^3+x^{c'}+x^{e'})+(2x^4+x^{3+e'}+x^{3+c'}+x^{5+c'}+x^{5+e'}+x^{c'+e'}).$$

From Proposition 1, the equation (5) also holds. After simplifying $Q(G)$ and $Q(H)$, we have $Q_1(G)=Q_1(H)$, where

$$Q_1(G)=-x^4-x^d-x^{d+1}-x^e-x^{e+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^{3+e}+x^{5+e}+x^{6+d}+x^{5+f},$$
$$Q_1(H)=-x-2x^2-x^{c'}-x^{c'+1}-x^{e'}-x^{e'+1}+2x^4+x^{3+e'}+x^{3+c'}+x^{5+c'}+x^{5+e'}.$$

It is easy to handle these cases in the same way as the proof of Lemma 3.3. □

Lemma 3.5 *If G is in the type of $K_4(2,3,3,d,e,f)$, and H is in the type of $K_4(2,2,4,d',e',f')$, then there is no graph G satisfying $G\sim H$ unless $G\cong H$.*

Proof Let G and H be two graphs such that $G\cong K_4(2,3,3,d,e,f)$ and $H\cong K_4(2,2,4,d',e',f')$. Then

$$Q(G)=-(x+1)(x^2+2x^3+x^d+x^e+x^f)+(x^{2+d}+x^{3+f}+x^{3+e}+x^{5+e}+x^{6+d}+x^{5+f}+x^{d+e+f}),$$
$$Q(H)=-(x+1)(2x^2+x^4+x^{d'}+x^{e'}+x^{f'})+(x^{2+d'}+x^{2+f'}+2x^{4+e'}+x^{6+d'}+x^{6+f'}+x^{d'+e'+f'}).$$

Now both $K_4(2,3,3,d,e,f)$ and $K_4(2,2,4,d',e',f')$ have the property of symmetry, thus we can assume $e\leq f$ and $e'\leq f'$. As Proposition 1 shows, equation (1) also holds. Cancelling equal terms, we have $Q_1(G)=Q_1(H)$ where

$$Q_1(G)=-x^3-x^4-x^d-x^{d+1}-x^e-x^{e+1}-x^f-x^{f+1}+x^{2+d}+x^{3+f}+x^{3+e}+x^{5+e}+x^{6+d}+x^{5+f},$$

$$Q_1(H) = -x^2 - x^5 - x^{d'} - x^{d'+1} - x^{e'} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{2+d'} + x^{2+f'} + 2x^{4+e'} + x^{6+d'} + x^{6+f'}.$$

Case 1 $\min\{d,e,f\} = \min\{d,e\} = 1$. From Proposition 1, $\min\{d',e',f'\} = \min\{d',e'\} = 1$.

If $d = e' = 1$. As $e + d \geq 5$ and $d' + e' \geq 6$, we have $f \geq e \geq 4$, and $f' \geq d' \geq 5$. After simplifying $Q_1(G)$ and $Q_1(H)$, we have $Q_2(G) = Q_2(H)$ where

$$Q_2(G) = -x^4 - x^e - x^{e+1} - x^f - x^{f+1} + x^{3+f} + x^{3+e} + x^{5+e} + x^7 + x^{5+f},$$
$$Q_2(H) = -x^2 - x^{d'} - x^{d'+1} - x^{f'} - x^{f'+1} + x^{2+d'} + x^{2+f'} + x^5 + x^{6+d'} + x^{6+f'}.$$

Comparing the l.p. in $Q_2(G)$ with the l.p. in $Q_2(H)$, we know that $Q_2(G) \neq Q_2(H)$.

It is easy to handle other left cases in the same fashion as Case 1, and we obtain that $Q(G) \neq Q(H)$ if one of the three parameters is 1. In the following, we can suppose that $\min\{d,e,f\} \geq 2$.

Case 2 $\min\{d,e,f\} = \min\{d,e\} = 2$.

Case 2.1 $d = 2$. From $d + e \geq 5$, we obtain

$$f \geq e \geq 3. \tag{6}$$

After simplifying $Q_1(G)$ and $Q_1(H)$, we have $Q_3(G) = Q_3(H)$ where

$$Q_3(G) = -2x^3 - x^e - x^{e+1} - x^f - x^{f+1} + x^{3+f} + x^{3+e} + x^{5+e} + x^8 + x^{5+f},$$
$$Q_3(H) = -x^5 - x^{d'} - x^{d'+1} - x^{e'} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{2+d'} + x^{2+f'} + 2x^{4+e'} + x^{6+d'} + x^{6+f'}.$$

Comparing the h.p. in $Q_3(G)$ with the h.p. in $Q_3(H)$, we have $5 + f = \max\{4 + e', 6 + f'\}$.

Case 2.1.1 $\max\{4 + e', 6 + f'\} = 4 + e' = 5 + f$. Note the coefficient of $x^{4+e'}$ is 2, we know $5 + e$ must also be equal to $4 + e'$. Cancelling equal terms of $Q_3(G)$ and $Q_3(H)$, we have $Q_4(G) = Q_4(H)$ where

$$Q_4(G) = -2x^3 - 2x^e - x^{e+1} + 2x^{3+e} + x^8,$$
$$Q_4(H) = -x^5 - x^{d'} - x^{d'+1} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{2+d'} + x^{2+f'} + x^{6+d'} + x^{6+f'}.$$

The lowest power of $Q_4(G)$ is 3 (see (6)) and since $Q_4(G) = Q_4(H)$, there are two terms in $Q_4(H)$ which are equal to $-x^3$. Therefore, $d' = f' = 3$. From $e = f = e' - 1$ and $d + e + f = d' + e' + f'$, we know $e = f = 5$. Thus $Q_4(G) \neq Q_4(H)$.

Case 2.1.2 $\max\{4 + e', 6 + f'\} = 6 + f' = 5 + f$. After simplifying $Q_3(G)$ and $Q_3(H)$, we have $Q_5(G) = Q_5(H)$ where

$$Q_5(G) = -2x^3 - x^e - x^{e+1} - x^{f+1} + x^{3+f} + x^{3+e} + x^{5+e} + x^8,$$
$$Q_5(H) = -x^5 - x^{d'} - x^{d'+1} - x^{e'} - x^{e'+1} - x^{f'} + x^{2+d'} + x^{2+f'} + 2x^{4+e'} + x^{6+d'}.$$

For the same reason as above discussion given, 3 is the l.p. in $Q_5(G)$, and for $Q_5(G) = Q_5(H)$, $-2x^3 \in Q_5(H)$, we know $d' = e' = 3$ or $d' = f' = 3$.

If $d' = e' = 3$, noting equations $f = f' + 1$ and $d + e + f = d' + e' + f'$, we know $e = 3$. Now after simplifying, we get

$$Q_6(G) = -x^3 - x^f - x^{f+1} + x^{3+f} + x^6 + 2x^8,$$
$$Q_6(H) = -x^4 - x^{f'} - x^{f'+1} + x^{2+f'} + 2x^7 + x^9.$$

It is easy to see $f' = 3$, then $Q_6(G) \neq Q_6(H)$, which means $Q(G) \neq Q(H)$.

If $d' = f' = 3$, then $f = 4$ and $e = e'$. Simplifying $Q_5(G)$ and $Q_5(H)$, we obtain

$$Q_7(G) = -x^5 + x^7 + x^{3+e} + x^{5+e} + x^8, \quad Q_7(H) = -x^4 + x^5 + 2x^{4+e'} + x^9.$$

Consider term x^5. It is due to $Q_7(G) = Q_7(H)$ that $2x^5$ must be in $Q_7(G)$, which is impossible.

Case 2.2 $e = 2$. After cancelling equal terms in $Q_1(G)$ and $Q_1(H)$, we have $Q_8(G) = Q_8(H)$ where

$$Q_8(G) = -2x^3 - x^4 - x^d - x^{d+1} - x^f - x^{f+1} + x^{2+d} + x^{3+f} + x^5 + x^7 + x^{6+d} + x^{5+f},$$
$$Q_8(H) = -x^5 - x^{d'} - x^{d'+1} - x^{e'} - x^{e'+1} - x^{f'} - x^{f'+1} + x^{2+d'} + x^{2+f'} + 2x^{4+e'} + x^{6+d'} + x^{6+f'}.$$

Consider $-2x^3$ in $Q_8(G)$. Because

$$d + e \geq 5, \quad f + e \geq 6, \tag{7}$$

3 is l.p. in $Q_8(G)$. So two cases need to be considered.

Case 2.2.1 $d' = e' = 3$. After simplifying, we obtain $Q_9(G) = Q_9(H)$ where

$$Q_9(G) = -x^d - x^{d+1} - x^f - x^{f+1} + x^{2+d} + x^{3+f} + x^5 + x^{6+d} + x^{5+f},$$
$$Q_9(H) = -x^4 - x^{f'} - x^{f'+1} + x^{2+f'} + x^7 + x^9 + x^{6+f'}.$$

Comparing the h.p. of $Q_9(G)$ with the h.p. of $Q_9(H)$, we obtain $6 + f' = \max\{6 + d, 5 + f\}$.

If $6 + f' = 6 + d$, then we know $f = 4$ for $d + e + f = d' + e' + f'$. Thus $G \cong H$.

If $6 + f' = 5 + f$, then $d = 3$. It is easy to get $f = f' + 1 = 4$, so $f' = d = 3$. We can see this is just a special case of $6 + f' = 6 + d$.

Case 2.2.2 $d' = f' = 3$. By $Q_8(G) = Q_8(H)$, and after simplifying, we obtain $Q_{10}(G) = Q_{10}(H)$ where

$$Q_{10}(G) = -x^d - x^{d+1} - x^f - x^{f+1} + x^{2+d} + x^{3+f} + x^7 + x^{6+d} + x^{5+f},$$
$$Q_{10}(H) = -x^4 - x^{e'} - x^{e'+1} + 2x^{4+e'} + 2x^9.$$

The highest power of $Q_{10}(H)$ is $\max\{4 + e', 9\}$, and the coefficient of highest term is at least 2. As $d \geq 3, f \geq 4$ (see (7)), $6 + d$ must be equal to $5 + f$.

If $6 + d = 5 + f = 4 + e'$, we get $d + 1 = f = 6$, $e' = 7$, since $d + e + f = d' + e' + f'$. Thus $Q_{10}(G) \neq Q_{10}(H)$.

If $6 + d = 5 + f = 9$, then $d + 1 = f = 4$, and $e' = 3$. Thus $G \cong H$. So this lemma holds. □

Lemma 3.6 *If G is in the type of $K_4(2,3,3,d,e,f)$, and H is in the type of $K_4(2,2,c',2,e',2)$, then there is no graph G satisfying $G \sim H$.*

Proof From Proposition 2, we know that $K_4(2,2,c',2,e',2)$ is chromatically unique. □

Theorem 3.7 *K_4-homeomorphs $K_4(2,3,3,d,e,f)$ with girth 8 is not χ-unique if and only if it is isomorphic to $K_4(2,3,3,1,6,\alpha)$ $(\alpha \geq 6)$, $K_4(2,3,3,1,\beta,\beta+2)$ $(\beta \geq 4)$, or $K_4(2,3,3,1,5,6)$.*

Proof Let G and H be two graphs such that $G \cong K_4(2,3,3,d,e,f)$ and $H \sim G$. Since the girth of G is 8, there is at most one 1 among d,e and f. Moreover, from (ii) and (iii) of Proposition 2.1, it follows that H is a K_4-homeomorph with girth 8. So H must be one of the following 7 types.

Type 1. $K_4(1,2,5,d',e',f')$, where $d'+e' \geq 6$, $d'+f' \geq 5$, $e'+f' \geq 7$.
Type 2. $K_4(1,3,4,d',e',f')$, where $d'+e' \geq 5$, $d'+f' \geq 4$, $e'+f' \geq 7$.
Type 3. $K_4(1,2,c',2,e',3)$, where $c' \geq 5$, $e' \geq 4$.
Type 4. $K_4(1,2,c',3,e',2)$, where $e' \geq c \geq 5$.
Type 5. $K_4(2,3,3,d',e',f')$, where $d'+e' \geq 5$, $e'+f' \geq 6$, $f' \geq e' \geq 1$.
Type 6. $K_4(2,2,4,d',e',f')$, where $d'+e' \geq 6$, $d'+f' \geq 4$, $f' \geq d' \geq 1$.
Type 7. $K_4(2,2,c',2,e',2)$, where $e' \geq c' \geq 4$.

From Lemma 1 and the lemmas in this section, we get the conclusion. □

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