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The Signless Laplacian Spectral Radius of Tricyclic Graphs with k Pendant Vertices

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Abstract In this paper, we determine the unique graph with the largest signless Laplacian spectral radius among all the tricyclic graphs with n vertices and k pendant vertices.

Keywords signless Laplacian spectral radius; tricyclic graph; pendant vertex.

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1. Introduction

Let $G=(V,E)$ be a simple connected graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. Denote by $d(v_i)$ the degree of the graph G, $N(v_i)$ the set of vertices which are adjacent to vertex v_i. Let $A(G)$ be the adjacency matrix and $Q(G)=D(G)+A(G)$ be the signless Laplacian matrix of the graph G, where $D(G)=\operatorname{diag}(d(v_1),d(v_2),\ldots,d(v_n))$ denotes the diagonal matrix of vertex degrees of G. It is well known that $Q(G)$ is a positive semidefinite matrix. Hence the eigenvalues of $Q(G)$ can be ordered as

$$q_1(G)\ge q_2(G)\ge\cdots\ge q_n(G)\ge 0.$$

The largest eigenvalues of $A(G)$, $L(G)=D(G)-A(G)$ and $Q(G)$ are called the spectral radius, the Laplacian spectral radius and the signless Laplacian spectral radius of G, respectively. The signless Laplacian spectral radius is denoted by $q(G)$ for convenience. It is easy to see that if G is connected, then $Q(G)$ is nonegative irreducible matrix. By the Perron-Frobenius theory, we can see that $q(G)$ has multiplicity one and exists a unique positive unit eigenvector corresponding to $q(G)$. We refer to such an eigenvector as the Perron vector of G.

A tricyclic graph is a connected graph with the number of edges equal to the number of vertices plus two. Denote by T_n^k the set of tricyclic graphs on n vertices and k pendant vertices. Recently, the problem concerning graphs with maximal spectral radius or the Laplacian spectral radius of a given class of graphs has been studied by many authors. Guo [1] determined the graph

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with the largest spectral radius among all the unicyclic and bicyclic graphs with n vertices and k pendant vertices. Guo [2] determined the graph with the largest Laplacian spectral radius among all the unicyclic and bicyclic graphs with n vertices and k pendant vertices. Guo and Wang [3] also determined the graph with the largest Laplacian spectral radius among all the tricyclic graphs with n vertices and k pendant vertices. Geng and Li [4] determined the graph with the largest spectral radius among all the tricyclic graphs with n vertices and k pendant vertices. In this paper, we determine the unique graph with the largest signless Laplacian spectral radius among all the tricyclic and graph with n vertices and k pendant vertices.

Denote by C_n the cycle on n vertices. And a path $P: vv_1v_2\cdots v_k$ is such a graph that v_1 joins v and v_{i+1} joins v_i $(i=1,2,\ldots,k-1)$.

2. Preliminaries

Let $G-x$ or $G-xy$ denote the graph obtained from G by deleting the vertex $x\in V(G)$ or the edge $xy\in E(G)$. Similarly, $G+xy$ is a graph obtained from G by adding an edge xy, where $x,y\in V(G)$ and $xy\notin E(G)$. A pendant vertex of G is a vertex with degree 1. A path $P: v_0v_1v_2\cdots v_k$ in G is called a pendant path, where v_i is adjacent to v_{i+1} $(i=0,1,\ldots,k-1)$ and $d(v_1)=d(v_2)=\cdots=d(v_{k-1})=2, d(v_k)=1$. If $k=1$, then we say vv_1 is a pendant edge of the graph G. k paths $P_{l_1},P_{l_2},\ldots,P_{l_k}$ are said to have almost equal lengths if $l_1,l_2,\ldots,l_k$ satisfy $|l_i-l_j|\le 1$ for $1\le i,j\le k$. We know, by [5], that a tricyclic graph G contains at least 3 cycles and at most 7 cycles, furthermore, there do not exist 5 cycles in G. Then let $T_n^k=T_n^{k,3}\bigcup T_n^{k,,4}\bigcup T_n^{k,6}\bigcup T_n^{k,7}$, where $T_n^{k,i}$ denotes the set of tricyclic graphs in T_n^k with exact i cycles for $i=3,4,6,7$.

In order to complete the proof of our main result, we need the following lemmas.

Lemma 1 ([6,7]) *Let G be a connected graph, and u,v be two vertices of G. Suppose that $v_1,v_2,\ldots,v_s\in N(v)\backslash(N(u)\bigcup\{u\})$ $(1\le s\le d(v))$ and $x=(x_1,x_2,\ldots,x_n)$ is the Perron vector of G, where x_i corresponds to the vertex v_i $(1\le i\le n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1\le i\le s)$. If $x_u\ge x_v$, then $q(G)<q(G^*)$.*

Let G be a connected graph, and $uv\in E(G)$. The graph G_{uv} is obtained from G by subdividing the edge uv, i.e., adding a new vertex w and edges wu,wv in $G-uv$.

An internal path of a graph G is a sequence of vertices $v_1,v_2,\ldots,v_m$ with $m\ge 2$ such that:

(1) The vertices in the sequences are distinct (except possibly $v_1=v_m$);

(2) v_i is adjacent to v_{i+1} $(i=1,2,\ldots,m-1)$;

(3) The vertex degrees $d(v_i)$ satisfy $d(v_1)\ge 3$, $d(v_2)=\cdots=d(v_{m-1})=2$ (unless $m=2$) and $d(v_m)\ge 3$.

By similar reasoning to that of Theorem 3.1 of [8] and Lemmas 2 and 7 of [15], we have the following result.

Lemma 2 *Let $P: v_1v_2\cdots v_k$ $(k\ge 2)$ be an internal path of a connected graph G. Let G' be a graph obtained from G by subdividing some edge of P. Then we have $q(G')<q(G)$.*

Let $m_i = \frac{\sum_{v_iv_j\in E} d(v_j)}{d(v_i)}$ be the average of the degrees of the vertices of G adjacent to v_i, which is called average 2-degree of vertex v_i.

From the proof of Theorem 3 of [9] and Theorem 2.10 of [10], we have the following result.

Lemma 3 *If G is a graph, then*

$$q(G) \leq \max\{\frac{d_u(d_u+m_u)+d_v(d_v+m_v)}{d_u+d_v} : uv \in E(G)\}$$

with equality if and only if G is regular or semiregular bipartite.

Let $S(G)$ be a graph obtained by subdividing every edge of G. Then

Lemma 4 ([11, 12]) *Let G be a graph on n vertices and m edges, $P_G(x) = \det(xI - A(G))$, $Q_G(x) = \det(xI - Q(G))$. Then $P_{S(G)} = x^{m-n}Q_G(x^2)$.*

Lemma 5 *Let u be a vertex of a connected graph G and $d(u) \geq 2$. Let $G_{k,l}$ $(k, l \geq 0)$ be the graph obtained from G by attaching two pendant paths of lengths k and l at u, respectively. If $k \geq l \geq 1$, then $q(G_{k,l}) > q(G_{k+1,l-1})$.*

Proof Let $S_1 = S(G_{k,l})$ and $S_2 = S(G_{k+1,l-1})$. It is easy to see that S_1 (S_2) can be obtained from $S(G)$ by attaching pendant paths of lengths $2k-1$ $(2k+1)$ and $2l-1$ $(2l-3)$ at u, respectively. Then applying Theorem 5 ([13]) and Lemma 4, we have

$$\rho(S_1) > \rho(S_2),$$

and consequently $q(G_{k,l}) > q(G_{k+1,l-1})$. $\Box$

Lemma 6 ([6]) *Let G be a simple graph on n vertices which has at least one edge. Then*

$$\triangle(G) + 1 \leq q(G) \leq 2\triangle(G),$$

where $\triangle(G)$ is the largest degree of G. Moreover, if G is connected, then the first equality holds if and only if G is the star $K_{1,n-1}$; and the second equality holds if and only if G is a regular graph.

Lemma 7 ([14]) *Let e be an edge of the graph G. Then*

$$q_1(G) \geq q_1(G-e) \geq q_2(G) \geq q_2(G-e) \geq \cdots \geq q_n(G) \geq q_n(G-e) \geq 0.$$

Let $B_3(1)$ be a tricyclic graph in T_n^k obtained from the graph G_1 in Figure 1 by attaching k paths with almost equal lengths to the vertex with degree 6.

Let $B_4(1)$ be a tyicyclic graph in T_n^k obtained from the graph G_2 in Figure 1 by attaching k paths with almost equal lengths to the vertex with degree 5.

Let $B_6(1)$ be a tyicyclic graph in T_n^k obtained from the graph G_3 in Figure 1 by attaching k paths with almost equal lengths to some vertex with degree 4.

Let $B_7(1)$ be a tyicyclic graph in T_n^k obtained from K_4 by attaching k paths with almost equal lengths to a vertex of K_4.

If $G \in T_n^{k,3}$, then G is obtained by attaching some trees to some vertices of graph G', where $G' \in \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$ (see Figure 1).

If $G \in T_n^{k,4}$, then G is obtained by attaching some trees to some vertices of graph G', where $G' \in \{T_8, T_9, T_{10}, T_{11}\}$ (see Figure 1).

If $G \in T_n^{k,6}$, then G is obtained by attaching some trees to some vertices of graph G', where $G' \in \{T_{12}, T_{13}, T_{14}\}$ (see Figure 1).

If $G \in T_n^{k,7}$, then G is obtained by attaching some trees to some vertices of graph T_{15} (see Figure 1).

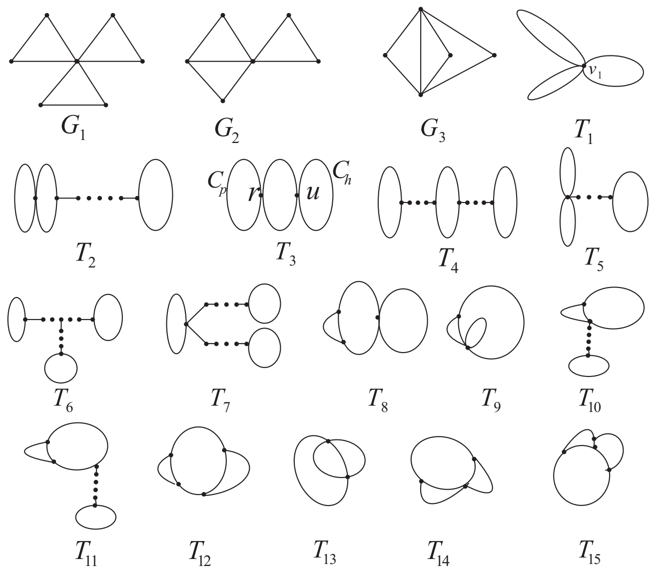

Figure 1 Graphs $G_1 - G_3$ $T_1 - T_{15}$

3. Main results

Lemma 8 *If both* $B_3(1)$ *and* $B_4(1)$ *exist, then* $q(B_4(1)) < q(B_3(1))$.

Proof Let

$$t_1 = \frac{(k+5)(k+5+\frac{2k+2+3+2+2+2}{k+5}) + 3(3+\frac{2+k+5+2}{3})}{k+5+3},$$
$$t_2 = \frac{(k+5)(k+5+\frac{2k+2+3+2+2+2}{k+5}) + 2(2+\frac{3+k+5}{2})}{k+5+2},$$
$$t_3 = \frac{(k+5)(k+5+\frac{2k+2+3+2+2+2}{k+5}) + 2(2+\frac{2+k+5}{2})}{k+5+2},$$
$$t_4 = \frac{(k+5)(k+5+\frac{2k+2+3+2+2+2}{k+5}) + 2(2+\frac{k+5+2}{2})}{k+5+2},$$
$$t_5 = \frac{3(3+\frac{2+2+k+5}{3}) + 2(2+\frac{3+k+5}{2})}{3+2},$$

$$t_6 = \frac{2(2+\frac{k+5+2}{2}) + 2(2+\frac{k+5+2}{2})}{2+2},$$
$$t_7 = \frac{2(2+\frac{k+5+2}{2}) + 2(2+\frac{2+2}{2})}{2+2},$$
$$t_8 = \frac{2(2+\frac{2+2}{2}) + 2(2+\frac{2+2}{2})}{2+2},$$
$$t_9 = \frac{(k+5)(k+5+\frac{2k+2+3+2+2+2}{k+5}) + 1(1+\frac{k+5}{1})}{k+5+1}.$$

By Lemmas 3 and 6, we can get

$$q(B_4(1)) \le \max\{t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8,t_9\} \le k+7 = \triangle(B_3(1))+1 < q(B_3(1)).$$

By similar reasoning to that of Lemma 8, we can get the following lemma.

Lemma 9 *If $B_3(1)$, $B_6(1)$ and $B_7(1)$ exist, then $q(B_6(1)) < q(B_3(1))$, $q(B_7(1)) < q(B_3(1))$.*

Theorem 1 *Let $G \in T_n^{k,3}$. Then $q(G) \le q(B_3(1))$; the equality holds if and only if $G \cong B_3(1)$.*

Proof Choose $G \in T_n^{k,3}$ such that $q(G)$ is as large as possible. Denote by C_p, C_q, C_h the three cycles of G, respectively.

We first prove that G must be obtained by attaching some trees to some vertices of T_1 in Figure 1.

Denote the vertex set of G by $\{v_1,v_2,\dots,v_n\}$ and the Perron vector of G by $x=(x_1,x_2,\dots,x_n)$, where x_i corresponds to v_i.

Assume G is obtained by attaching some trees to some vertices of graph T_3 in Figure 1. If $x_r \ge x_u$, then let $G^* = G - uv_{i+1} - uv_{i-1} - uf_1 - \cdots - uf_z + rv_{i+1} + rv_{i-1} + rf_1 + \cdots + rf_z$, where $uv_{i+1}, uv_{i-1} \in E(C_h)$, and $f_1,\dots,f_z$ are all the neighbors of u in those trees (if exist) attaching to u. If $x_r < x_u$, then let $G^* = G - rv_{j+1} - rv_{j-1} - rq_1 - \cdots - rq_s + uv_{j+1} + uv_{j-1} + uq_1 + \cdots + uq_s$, where $rv_{j+1}, rv_{j-1} \in E(C_p)$, and $q_1,\dots,q_s$ are all the neighbors of r in those trees (if exist) attaching to r. Combining two cases above, by Lemma 1, we can see that $q(G^*) > q(G)$ and $G^* \in T_n^{k,3}$, a contradiction. Hence G cannot be obtained by attaching some trees to some vertices of graph T_3.

By similar reasoning, it is easy to prove that G cannot be obtained by attaching trees to some vertices of graph T_2, T_4, T_5, T_6, T_7. Hence G must be obtained by attaching some trees to vertices of T_1.

Next, we will prove that G must be obtained by attaching exactly one tree to some vertex of T_1.

Assume there exist two trees, say T_1', T_2' are attached to vertices w_1, w_2 of T_1, respectively. If $x_{w_1} \le x_{w_2}$, then let $G^* = G - w_1u_1 - w_1u_2 - \cdots - w_1u_g + w_2u_1 + \cdots + w_2u_g$, where $u_1,\dots,u_g$ are all the neighbors of w_1 in T_1'. If $x_{w_1} > x_{w_2}$, then let $G^* = G - w_2u_1' - w_2u_2' - \cdots - w_2u_l' + w_1u_1' + \cdots + w_1u_l'$, where $u_1',\dots,u_l'$ are all the neighbors of w_2 in T_2'. By Lemma 1, we can see that $q(G^*) > q(G)$ and $G^* \in T_n^{k,3}$, a contradiction. Hence G has only one tree, say T^*, attached to some vertex, say v, of T_1.

Thirdly, we prove that $d(u) \leq 2$, for any $u \in V(T^*)$, $u \notin V(T_1)$, where T^* is a tree which attaches to some vertex of T_1. If $d(u) > 2$, denote $N(u) = \{z_1, z_2, \ldots, z_s\}$ and $N(v) = \{w_1, w_2, \ldots, w_t\}$, $t \geq 3$. Let z_1, w_3 belong to the path joining v and u, and w_1 belong to one cycle in G. If $x_v \geq x_u$, let $G^* = G - uz_3 - \cdots - uz_s + vz_3 + \cdots + vz_s$. If $x_v < x_u$, let $G^* = G - vw_1 + uw_1$. It is easy to see that $G^* \in T_n^{k,3}$. By Lemma 1, we can get that $q(G^*) > q(G)$, a contradiction. Hence, G is a graph obtained from T_1 by attaching k paths.

By Lemma 5, it is easy to get that the k paths attached to v of T_1 have almost equal lengths.

Let v_1 be the common vertex of the three cycles of T_1. Finally, we prove that $v = v_1$.

Assume that $v \neq v_1$. Without loss of generality, suppose that $v \in C_p$, where C_p is some cycle of T_1. Let $P_1, P_2, \ldots, P_k$ be the k paths attached to v, and $vw_{i1} \in P_i$ $(i = 1, 2, \ldots, k)$. Denote $v_1v'_{m-1}, v_1v'_{m+1} \in C_q, v_1v'_{j-1}, v_1v'_{j+1} \in C_h$, where C_q and C_h are the two cycles except C_p of T_1.

If $x_v \geq x_{v_1}$, then let $G^* = G - v_1v'_{i-1} - v_1v'_{i+1} - v_1v'_{j-1} - v_1v'_{j+1} + vv'_{i-1} + vv'_{i+1} + vv'_{j-1} + vv'_{j+1}$. If $x_v < x_{v_1}$, then let $G^* = G - vw_{11} - vw_{21} - \cdots - vw_{k1} + v_1w_{11} + v_1w_{21} + \cdots + v_1w_{k1}$. Obviously, $G^* \in T_n^{k,3}$, and by Lemma 1, we get $q(G^*) > q(G)$, a contradiction. Hence $v = v_1$.

By Lemmas 2 and 7, it is easy to prove that all the cycles in G have length 3. Then $G \cong B_3(1)$. $\Box$

By similar reasoning to that of Theorem 1, it is not difficult to prove the following theorems.

Theorem 2 *Let $G \in T_n^{k,4}$. Then $q(G) \leq q(B_4(1))$, and the equality holds if and only if $G \cong B_4(1)$.*

Theorem 3 *Let $G \in T_n^{k,6}$. Then $q(G) \leq q(B_6(1))$, and the equality holds if and only if $G \cong B_6(1)$.*

Theorem 4 *Let $G \in T_n^{k,7}$. Then $q(G) \leq q(B_7(1))$, and the equality holds if and only if $G \cong B_7(1)$.*

From Lemmas 8, 9 and Theorems 1–4, we get the main result.

Theorem 5 *Let $G \in T_n^k$, $k \geq 1$. Then $q(G) \leq q(B_3(1))$, and the equality holds if and only if $G \cong B_3(1)$.*

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