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The Bases and Base Set of Primitive Symmetric Loop-Free Signed Digraphs

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Abstract In this paper, we study the bases and base sets of primitive symmetric loop-free (generalized) signed digraphs on n vertices. We obtain sharp upper bounds of the bases, and show that the base sets of the classes of such digraphs are $\{2, 3, \ldots, 2n-1\}$. We also give a new proof of an important result obtained by Cheng and Liu.

Keywords primitive; signed digraph; symmetric; non-powerful; base.

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1. Introduction

A sign pattern matrix is a matrix each of whose entries is a sign 1, -1 or 0. For a square sign pattern matrix M, notice that in the computations of the entries of the power M^k, the "ambiguous sign" may arise when we add a positive sign 1 to a negative sign -1. So a new symbol "#" was introduced in [1] to denote the ambiguous sign, the set $\Gamma = \{0, 1, -1, \#\}$ is defined as the generalized sign set and the addition and multiplication involving the symbol # are defined as follows:

$$(-1) + 1 = 1 + (-1) = \#;\ a + \# = \# + a = \# \text{ for all } a \in \Gamma \tag{1.1}$$

$$0 \cdot \# = \# \cdot 0 = 0;\ b \cdot \# = \# \cdot b = \# \text{ for all } b \in \Gamma \setminus \{0\}. \tag{1.2}$$

In $[1, 2]$, the matrices with entries in the set Γ are called generalized sign pattern matrices. The addition and multiplication of generalized sign pattern matrices are defined in the usual way, so that the sum and product of the generalized sign pattern matrices are still generalized sign pattern matrices. In this paper, we assume that all the matrix operations considered are operations of the matrices over Γ.

Definition 1.1 ([1]) *A square generalized sign pattern matrix M is called powerful if each power of M contains no # entry.*

Definition 1.2 ([3]) *Let M be a square generalized sign pattern matrix of order n and*

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$M, M^2, M^3, \ldots$ be the sequence of powers of M. Suppose M^b is the first power that is repeated in the sequence. Namely, suppose b is the least positive integer such that there is a positive integer p such that

$$M^b = M^{b+p}. \tag{1.3}$$

Then b is called the generalized base (or simply base) of M, and is denoted by $b(M)$. The least positive integer p such that (1.3) holds for $b = b(M)$ is called the generalized period (or simply period) of M, and is denoted by $p(M)$.

We now introduce some concepts of graph theory.

Let $D = (V, A)$ denote a digraph on n vertices. Loops are permitted, but no multiple arcs. A $u \to v$ walk in D is a sequence of vertices $u, u_1, \ldots, u_k = v$ and a sequence of arcs $e_1 = (u, u_1), e_2 = (u_1, u_2), \ldots, e_k = (u_{k-1}, v)$, where the vertices and the arcs are not necessarily distinct. A closed walk is a $u \to v$ walk where $u = v$. A path is a walk with distinct vertices. A cycle is a closed $u \to v$ walk with distinct vertices except for $u = v$. The length of a walk W is the number of arcs in W, denoted by $l(W)$. A k-cycle is a cycle of length k, denoted by C_k.

A signed digraph S is a digraph where each arc of S is assigned a sign 1 or -1. A generalized signed digraph S is a digraph where each arc of S is assigned a sign 1, -1 or $\#$.

The sign of the walk W in a (generalized) signed digraph, denoted by $\mathrm{sgn}W$, is defined to be $\prod_{i=1}^{k} \mathrm{sgn}(e_i)$, where $e_1, e_2, \ldots, e_k$ is the sequence of arcs of W.

Let $M = (m_{ij})$ be a square (generalized) sign pattern matrix of order n. The associated digraph $D(M) = (V, A)$ of M (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $A = \{(i, j) | m_{ij} \neq 0\}$. The associated (generalized) signed digraph $S(M)$ of M is obtained from $D(M)$ by assigning the sign of m_{ij} to each arc (i, j) in $D(M)$, and we say $D(M)$ is the underlying digraph of $S(M)$.

Let S be a (generalized) signed digraph on n vertices. Then there is a (generalized) sign pattern matrix M of order n whose associated (generalized) signed digraph $S(M)$ is S. We say that S is powerful if M is powerful. Also the base $b(S)$ and period $p(S)$ are defined to be those of M. Namely we define $b(S) = b(M)$ and $p(S) = p(M)$.

A digraph D is said to be strongly connected if there exists a path from u to v for all $u, v \in V$, and D is called primitive if there is a positive integer k such that for each vertex x and each vertex y (not necessarily distinct) in D, there exists a walk of length k from x to y. The least such k is called the primitive exponent (or simply exponent) of D, denoted by $\exp(D)$. It is also well-known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor (simply g.c.d.) of the lengths of all the cycles of D is 1. A (generalized) signed digraph S is called primitive if the underlying digraph D is primitive, and in this case we define $\exp(S) = \exp(D)$.

A digraph D is symmetric if for every arc (u, v) in D, the arc (v, u) is also in D. A (generalized) signed digraph S is called symmetric if the underlying digraph D is symmetric. If a digraph (or a generalized signed digraph) D (or S) is symmetric, then D (or S) can be regarded as an undirected graph (possibly with loops).

A digraph D is loop-free if D has no loops. In this case, if a digraph (or a generalized signed digraph) D (or S) is symmetric and loop-free, then D (or S) can be regarded as a simple graph.

Set $\mathcal{S}_n^{\star\prime} = \{S | S$ is a primitive non-powerful symmetric loop-free signed digraph on n vertices$\}$, $\mathcal{S}_n^{\star} = \{S | S$ is a primitive symmetric loop-free signed digraph on n vertices$\}$ and $\bar{\mathcal{S}}_n^{\star} = \{S | S$ is a primitive symmetric loop-free generalized signed digraph on n vertices$\}$. Clearly, $\mathcal{S}_n^{\star\prime} \subset \mathcal{S}_n^{\star} \subset \bar{\mathcal{S}}_n^{\star}$.

The primitive exponent and exponent set of primitive symmetric digraphs were discussed in [4, 5], and the primitive exponent and exponent set of primitive symmetric loop-free digraphs were discussed in [6, 7]. The following results are useful.

Theorem 1.A ([7]) *Let D be a primitive symmetric loop-free digraph on n vertices. Then $\exp(D) \leq 2n-4$ and the primitive exponent set of such digraphs is $\{2, 3, \ldots, 2n-4\} \backslash \mathcal{D}$, where $\mathcal{D}$ is the set of odd numbers in $\{n-2, n-1, \ldots, 2n-5\}$.*

In [8], Cheng and Liu studied the bases and base sets of primitive symmetric signed digraphs and generalized signed digraphs.

Theorem 1.B ([8]) *Let S be a primitive symmetric (generalized) signed digraph on n vertices. Then $b(S) \leq 2n$ and the base set of $\mathcal{S}_n(\bar{\mathcal{S}}_n)$ is $\{1, 2, \ldots, 2n\}$, where $\mathcal{S}_n(\bar{\mathcal{S}}_n)$ is the set of the primitive symmetric (generalized) signed digraphs on n vertices.*

A natural question is what are the upper bounds of the bases and the base sets of primitive symmetric loop-free (generalized) signed digraphs on n vertices. As a main result, the sharp upper bounds of the bases are obtained in Section 3, then in Section 4, we show that the base sets of $\mathcal{S}_n^{\star\prime}$, $\mathcal{S}_n^{\star}$ and $\bar{\mathcal{S}}_n^{\star}$ are $\{2, 3, \ldots, 2n-1\}$, and in Section 5, we give a new proof of an important result in [8].

2. Some preliminaries

In this section, we introduce some definitions, theorems and lemmas which we need to use in the presentations and proofs of our main results in this paper. Other definitions and results not in this article can be found in [9–11].

Definition 2.1 ([3]) *Two walks W_1 and W_2 in a signed digraph are called a pair of $SSSD$ walks, if they have the same initial vertex, same terminal vertex and same length, but they have different signs.*

It is easy to see from the above relation between matrices and signed digraphs that a (generalized) sign pattern matrix M is powerful if and only if the associated (generalized) signed digraph $S(M)$ contains no pairs of $SSSD$ walks. Thus for a (generalized) signed digraph S, S is powerful if and only if S contains no pairs of $SSSD$ walks.

In [3], You, Shao and Shan obtained an important characterization for primitive non-powerful signed digraphs from the characterization of powerful irreducible sign pattern matrices [1].

Theorem 2.A ([3]) *If S is a primitive signed digraph, then S is non-powerful if and only if S contains a pair of cycles C' and C'' (say, with lengths p_1 and p_2, respectively) satisfying one of* *the following conditions:*

(A_1) p_1 is odd, p_2 is even and $\mathrm{sgn}C'' = -1$;

(A_2) Both p_1 and p_2 are odd and $\mathrm{sgn}C' = -\mathrm{sgn}C''$.

A pair of cycles C' and C'' satisfying (A1) or (A2) is a "distinguished cycle pair". It is easy to check that if C' and C'' is a distinguished cycle pair with lengths p_1 and p_2, respectively, then the closed walks $W_1 = p_2C'$ (walk around C' by p_2 times) and $W_2 = p_1C''$ have the same length p_1p_2 and different signs:

$$(\mathrm{sgn}C')^{p_2} = -(\mathrm{sgn}C'')^{p_1}. \tag{2.1}$$

The following result can be used to determine the base.

Theorem 2.B ([3]) *Let S be a primitive non-powerful signed digraph. Then*

(1) There is an integer k such that there exists a pair of SSSD walks of length k from each vertex x to each vertex y in S.

(2) If there exists a pair of SSSD walks of length k from each vertex x to each vertex y, then there also exists a pair of SSSD walks of length $k+1$ from each vertex x to each vertex y in S.

(3) The minimal such k (as in (1)) is just $b(S)$-the base of S.

In the rest of the paper, for an undirected walk W of graph G and two vertices x, y on W, let $Q_W(x \to y)$ be the shortest path from x to y on W. Let $Q(x \to y)$ be the shortest path from x to y on G. For a cycle C, if x and y are two (not necessarily distinct) vertices on C and P is a path from x to y along C, then $C\backslash P$ denotes the path or cycle from x to y along C obtained by deleting the edges of P.

The following Lemmas 2.1–2.3 will be used in the proof of Lemma 3.5 and Theorem 3.1.

Lemma 2.1 *Let $S = (V, A)$ be a symmetric signed digraph on $2k$ vertices, where vertex set $V = \{1, 2, \ldots, 2k\}$, arc set $A = \{(i, i+1), (i+1, i) | i = 1, 2, \ldots, 2k-1\} \cup \{(1, 2k), (2k, 1)\}$. If all the signs of 2-cycles are 1 in S, and $\mathrm{sgn}C_{2k} = -1$. Then for any $i \in V(S)$ $(1 \le i \le 2k)$, there exists a vertex $x \in V(S)$, such that there is a pair of $SSSD$ walks with length k from i to x.*

Proof For all $i \in V(1 \le i \le 2k)$, take $x = k + i \,(\mathrm{mod}\, 2k)$. We will show the walks $(i, i+1) + (i+1, i+2) + \cdots + (k+i-1, k+i)$ and $(i, i-1) + (i-1, i-2) + \cdots + (k+i+1, k+i)$ is a pair of $SSSD$ walks with length k from i to x.

Without loss of generality, we may assume $i = 1$. Take $x = k+1 \in V$. Let $W_1 = (1, 2) + (2, 3) + \cdots + (k, k+1)$ and $W_2 = (1, 2k) + (2k, 2k-1) + \cdots + (k+2, k+1)$. Then $\mathrm{sgn}W_1 = \mathrm{sgn}(1, 2) \times \mathrm{sgn}(2, 3) \times \cdots \times \mathrm{sgn}(k, k+1)$, and $\mathrm{sgn}W_2 = \mathrm{sgn}(1, 2k) \times \mathrm{sgn}(2k, 2k-1) \times \cdots \times \mathrm{sgn}(k+2, k+1) = \mathrm{sgn}(2k, 1) \times \mathrm{sgn}(2k-1, 2k) \times \cdots \times \mathrm{sgn}(k+1, k+2)$ because all the signs of 2-cycles are 1.

So $\mathrm{sgn}W_1 \cdot \mathrm{sgn}W_2 = \mathrm{sgn}(1, 2) \times \mathrm{sgn}(2, 3) \times \cdots \times \mathrm{sgn}(2k-1, 2k) \times \mathrm{sgn}(2k, 1) = \mathrm{sgn}C_{2k} = -1$, then $\mathrm{sgn}W_1 = -\mathrm{sgn}W_2$ and $l(W_1) = l(W_2) = k$. Therefore W_1, W_2 is a pair of $SSSD$ walks from 1 to x with length k. □

Lemma 2.2 *Let D be a symmetric digraph on n vertices. If there exist a cycle C and an odd* *cycle C' with lengths of $k(\geq 1)$ and $k'(\geq 1)$ in D such that $C \cap C' = \emptyset$, let P be the shortest path from C to C', and for any $x \in D$, let P_1 (P_2) be the shortest path from x to C (C'). Then we have*

$$l(P_1) + l(P) + l(P_2) \leq 2(n - k - k' + 1) + \max\{[\frac{k}{2}], \frac{k' - 1}{2}\}. \tag{2.2}$$

Proof Suppose P intersects C (C') at v (v').

Case 1 $P_1 \cap C' = \emptyset$ and $P_2 \cap C = \emptyset$.

Subcase 1.1 $(P_1 \cup P_2) \cap P = \emptyset$.

It is easy to see that $l(P_1)+l(P)+l(P_2) \leq 2(n-k-k'+1) \leq 2(n-k-k'+1)+\max\{[\frac{k}{2}], \frac{k'-1}{2}\}$.

Subcase 1.2 $(P_1 \cup P_2) \cap P \neq \emptyset$.

We have $P_1 \cap P \neq \emptyset$ or $P_2 \cap P \neq \emptyset$. Without loss of generality, we may assume $P_1 \cap P \neq \emptyset$. Suppose z is the first vertex on $P_1 \cap P$. Then $l(P_1) + l(P) + l(P_2) \leq l(Q_{P_1}(x \to z)) + l(Q_P(z \to v)) + l(P) + l(Q_{P_1}(x \to z)) + l(Q_P(z \to v')) = 2(l(P) + l(Q_{P_1}(x \to z))) \leq 2(n - k - k' + 1)$.

Case 2 $P_1 \cap C' \neq \emptyset$.

Suppose z is the first vertex on $P_1 \cap C'$. We have $l(P_1) + l(P) + l(P_2) \leq (l(Q_{P_1}(x \to z)) + l(Q_{C'}(z \to v')) + l(P)) + l(P) + l(Q_{P_1}(x \to z)) = 2(l(P) + l(Q_{P_1}(x \to z))) + l(Q_{C'}(z \to v')) \leq 2(n - k - k' + 1) + \frac{k'-1}{2}$.

Case 3 $P_2 \cap C \neq \emptyset$.

Suppose z is the first vertex on $P_2 \cap C$. We have $l(P_1) + l(P) + l(P_2) \leq l(Q_{P_2}(x \to z)) + l(P) + (l(Q_{P_2}(x \to z)) + l(Q_C(z \to v)) + l(P)) = 2(l(P) + l(Q_{P_2}(x \to z))) + l(Q_C(z \to v)) \leq 2(n - k - k' + 1) + [\frac{k}{2}]$.

Combining the above three cases yields (2.2). $\square$

Lemma 2.3 *Let D be a symmetric digraph on n vertices. If there exist a cycle C and an odd cycle C' with lengths of $k(\geq 1)$ and $k'(\geq 1)$ in D such that $C \cap C^{'} = \emptyset$, let P be the shortest path from C to C', $d(x, y)$ be the distance from x to y. Then for any two vertices $x, y \in D$, there exist $x' \in C, y' \in C'$ or $x' \in C', y' \in C$ such that*

$$d(x, x') + l(P) + d(y, y') \leq 2(n - k - k' + 1) + \max\{[\frac{k}{2}], \frac{k' - 1}{2}\}. \tag{2.3}$$

Proof Note that $l(P) \leq n - k - k' + 1$, then we only need to consider the following three cases.

Case 1 $x \in C$ or $y \in C$. Without loss of generality, we may assume $x \in C$.

Take $x' = x$, and for any $y \in D$, there exists $y' \in C'$ such that $d(y, y') \leq [\frac{k}{2}] + n - k - k' + 1$. So $d(x, x') + l(P) + d(y, y') \leq 2(n - k - k' + 1) + [\frac{k}{2}] \leq 2(n - k - k' + 1) + \max\{[\frac{k}{2}], \frac{k'-1}{2}\}$.

Case 2 $x \in C'$ or $y \in C'$. Without loss of generality, we may assume $x \in C'$.

Take $x' = x$. For any $y \in D$, there exists $y' \in C$ such that $d(y, y') \leq \frac{k'-1}{2} + n - k - k' + 1$. So $d(x, x') + l(P) + d(y, y') \leq 2(n - k - k' + 1) + \frac{k'-1}{2} \leq 2(n - k - k' + 1) + \max\{[\frac{k}{2}], \frac{k'-1}{2}\}$.

Case 3 $x \notin C \cup C'$ and $y \notin C \cup C'$.

Let P_1 and P_1' be the shortest paths from x to C and C' respectively, and P_2 and P_2' be the shortest paths from y to C and C' respectively. Assume the result does not hold. Then we have

$$l(P_1)+l(P)+l(P_2')>2(n-k-k'+1)+\max\{[\frac{k}{2}],\frac{k'-1}{2}\},$$

and

$$l(P_1')+l(P)+l(P_2)>2(n-k-k'+1)+\max\{[\frac{k}{2}],\frac{k'-1}{2}\}.$$

Therefore, $l(P_1)+l(P_1')+2l(P)+l(P_2)+l(P_2')>4(n-k-k'+1)+2\max\{[\frac{k}{2}],\frac{k'-1}{2}\}$.

On the other hand, by Lemma 2.2 we have

$$l(P_1)+l(P)+l(P_1')\le 2(n-k-k'+1)+\max\{[\frac{k}{2}],\frac{k'-1}{2}\},$$

and

$$l(P_2)+l(P)+l(P_2')\le 2(n-k-k'+1)+\max\{[\frac{k}{2}],\frac{k'-1}{2}\}.$$

So $l(P_1)+l(P_1')+2l(P)+l(P_2)+l(P_2')\le 4(n-k-k'+1)+2\max\{[\frac{k}{2}],\frac{k'-1}{2}\}$, leading to a contradiction.

Combining the above three cases gives (2.3). $\Box$

3. The sharp upper bound of $b(S)$

It was shown in [1] that if a primitive signed digraph S is powerful, then $b(S)=\exp(D)$, where D is the underlying digraph of S. This means that the study of the base $b(S)$ for primitive powerful signed digraphs is essentially the study of the base (i.e., primitive exponent) for primitive digraphs. So for a primitive powerful symmetric loop-free signed digraph, Theorem 1.A gives the results. But if S is not powerful, then the situation is totally different. Now we consider primitive non-powerful symmetric loop-free signed digraphs.

Lemma 3.1 *Let S be a primitive non-powerful symmetric loop-free signed digraph on n vertices. If all the signs of 2-cycles are -1 in S. Then $b(S)\le 2n-1$.*

Proof Because S is a primitive loop-free signed digraph, there exists an odd cycle $C_l=v_1v_2\cdots v_lv_1$ of length $l(\ge 3)$ in S. Let x and y be any two (not necessarily distinct) vertices in S.

Consider the directed cycles $C_l=v_1v_2\cdots v_lv_1$ and $C_l'=v_1v_l\cdots v_2v_1$. Since all the signs of 2-cycles are -1 in S, arcs (u,v) and (v,u) have different signs for any two vertices $u,v\in S$. Thus $\mathrm{sgn}C_l\times\mathrm{sgn}C_l'=(-1)^l=-1$ because l is odd and thus $\mathrm{sgn}C_l=-\mathrm{sgn}C_l'$.

Let P_1 be the shortest path from x to C_l and P_1 intersect C_l at x', P_2 be the shortest path from y to C_l and P_2 intersect C_l at y' where $0\le l(P_i)\le n-l$, $i=1,2$. Set

$$W=\begin{cases}P_1+Q_{C_l}(x'\to y')+P_2, & \text{if } l(P_1)+l(Q_{C_l}(x'\to y'))+l(P_2) \text{ is even;}\\ P_1+C_l\setminus Q_{C_l}(x'\to y')+P_2, & \text{otherwise.}\end{cases}$$

Then $l(W)$ is even and $l(W)\le(n-l)+l+(n-l)=2n-l$, thus $W+C_l$, $W+C_l'$ is a pair of $SSSD$ walks from x to y with odd length $\le 2n$. Therefore there exists a pair of $SSSD$ walks of length $2n-1$ from x to y.

Then $b(S) \le 2n-1$ by Theorem 2.B. □

Lemma 3.2 *Let S be a primitive non-powerful symmetric loop-free signed digraph on n vertices and there exists a vertex v in S such that v is contained in a positive 2-cycle C' and a negative 2-cycle C''. Then $b(S) \le 2n-1$.*

Proof Because S is a primitive loop-free signed digraph, there exists an odd cycle $C_l = v_1v_2\cdots v_lv_1$ of length l (≥ 3) in S. Let x and y be any two (not necessarily distinct) vertices in S.

Let P be the shortest path from v to C_l and P intersect C_l at v'. Suppose there are k vertices on P where $k \ge 1$. Then $P \cup C_l$ contains $k+l-1$ vertices.

Let P_1 be the shortest path from x to $P\cup C_l$ and P_1 intersect $P\cup C_l$ at x', P_2 be the shortest path from y to $P\cup C_l$ and P_2 intersect $P\cup C_l$ at y' where $0 \le l(P_i) \le n-k-l+1$, $i=1,2$.

We consider the following three cases.

Case 1 $x' \in P$, $y' \in P$.

Set $a = l(Q_P(x' \to v))$, $b = l(Q_P(v \to y'))$ and

$$W = \begin{cases} P_1 + Q_P(x' \to v) + P + Q_P(v' \to y') + P_2, & \text{if } a \le b; \\ P_1 + Q_P(x' \to v') + P + Q_P(v \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W) \le (n-k-l+1)+(k-1)+(k-1)+(n-k-l+1) = 2n-2l$. If $l(W)$ is odd, we set $W_1 = W$. Otherwise, we set $W_1 = W + C_l$. Therefore, $l(W_1)$ is odd, and $l(W_1) \le 2n-l$, thus W_1+C' and W_1+C'' is a pair of $SSSD$ walks from x to y with odd length $\le 2n-l+2 \le 2n-1$. Therefore, there exists a pair of $SSSD$ walks of length $2n-1$ from x to y.

Case 2 Either x' or y' belongs to P. Without loss of generality, we may assume $x' \in P$ and $y' \notin P$.

Set $w = l(P_1) + l(Q_P(x' \to v)) + l(P) + l(Q_{C_l}(v' \to y')) + l(P_2)$, and

$$W = \begin{cases} P_1 + Q_P(x' \to v) + P + Q_{C_l}(v' \to y') + P_2, & \text{if } w \text{ is odd;} \\ P_1 + Q_P(x' \to v) + P + C_l \setminus Q_{C_l}(v' \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W)$ is odd, and $l(W) \le (n-l-k+1)+(k-1)+(k-1)+(l-1)+(n-l-k+1) = 2n-l-1$, thus the pair $W+C'$ and $W+C''$ is a pair of $SSSD$ walks from x to y with odd length $\le 2n-l+1 < 2n-1$. Therefore, there exists a pair of $SSSD$ walks of length $2n-1$ from x to y.

Case 3 $x' \notin P$, $y' \notin P$.

Subcase 3.1 If $v' \in Q_{C_l}(x' \to y')$, set $w = l(P_1) + l(Q_{C_l}(x' \to y')) + l(P_2)$ and

$$W = \begin{cases} P_1 + Q_{C_l}(x' \to y') + 2P + P_2, & \text{if } w \text{ is odd;} \\ P_1 + Q_{C_l}(x' \to y') + 2P + C_l + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W)$ is odd, and $l(W) \le (n-l-k+1) + \frac{(l-1)}{2} + 2(k-1) + l + (n-l-k+1) = 2n - \frac{l}{2} - \frac{1}{2}$, thus the pair $W+C'$ and $W+C''$ is a pair of $SSSD$ walks from x to y with odd length $\le 2n-\frac{l}{2}+\frac{3}{2}\le 2n$ $(l\ge 3)$. Since $l(W)$ is odd and $2n$ is even, $b(S)\le 2n-1$. Therefore, there exists a pair of $SSSD$ walks of length $2n-1$ from x to y.

Subcase 3.2 If $v'\notin Q_{C_l}(x'\to y')$, set $w=l(P_1)+l(Q_{C_l}(x'\to y'))+l(P_2)$. If w is even, then set $W=P_1+Q_{C_l}(x'\to y')+C_l+2P+P_2$. If w is odd, then set $a=l(Q_{C_l}(x'\to v'))$, $b=l(Q_{C_l}(y'\to v'))$ and

$$W=\begin{cases} P_1+2Q_{C_l}(x'\to v')+Q_{C_l}(x'\to y')+2P+P_2, & \text{if } a\le b;\\ P_1+Q_{C_l}(x'\to y')+2Q_{C_l}(y'\to v')+2P+P_2, & \text{otherwise.}\end{cases}$$

Then $l(W)$ is odd, and $l(W)\le (n-l-k+1)+\frac{(l-1)}{2}+l+2(k-1)+(n-l-k+1)=2n-\frac{l}{2}-\frac{1}{2}$. Thus the pair $W+C'$ and $W+C''$ is a pair of $SSSD$ walks from x to y with odd length $\le 2n-\frac{l}{2}+\frac{3}{2}\le 2n$ $(l\ge 3)$. Since $l(W)$ is odd and $2n$ is even, $b(S)\le 2n-1$. Therefore there exists a pair of $SSSD$ walks of length $2n-1$ from x to y.

Then $b(S)\le 2n-1$ by Theorem 2.B. $\Box$

In the above lemma, we actually have a pair of $SSSD$ walks of length 2. Using the same way, we can prove the following result.

Lemma 3.3 *Let S be a primitive symmetric loop-free generalized signed digraph on n vertices and there exists a closed walk W of length 2 in S with the property that* $\mathrm{sgn}W=\#$. *Then* $b(S)\le 2n-1$.

Lemma 3.4 *Let S be a primitive non-powerful symmetric loop-free signed digraph on n vertices and there exists a negative 2-cycle. Then $b(S)\le 2n-1$.*

Proof If the sign of every 2-cycle is -1, then the result follows from Lemma 3.1.

If there exists a positive 2-cycle, then there exist at least a positive 2-cycle and a negative 2-cycle. By the fact that the underlying digraph D is strongly connected, there exists a vertex v such that v is contained in a positive 2-cycle and a negative 2-cycle. By Lemma 3.2, we have $b(S)\le 2n-1$. $\Box$

Lemma 3.5 *Let S be a primitive non-powerful symmetric loop-free signed digraph on n vertices and there exists a negative even cycle in S. Then $b(S)\le 2n-1$.*

Proof Let C be the shortest even cycle in S with the property that $\mathrm{sgn}C=-1$ and let $l(C)=k$.

If $k=2$, then the result follows from Lemma 3.4.

If $k\ge 4$, then the sign of any 2-cycle is 1. Since S is primitive, there exists an odd cycle C' in S, say length k' (≥ 3). Let x and y be any two (not necessarily distinct) vertices in S.

Case 1 $C\cap C'=\emptyset$.

Let P be the shortest path from C to C'. Assume P intersects C at v, P intersects C' at v'. By Lemma 2.3, there exist $x'\in C, y'\in C'$ or $x'\in C', y'\in C$ such that (2.3) holds. Without loss of generality, suppose there exist $x'\in C, y'\in C'$ such that (2.3) holds. For convenience, let P_1

be the shortest path from x to x' and P_2 be the shortest path from y to y'.

Set $w = l(P_1) + l(Q_C(x' \to v)) + l(P) + l(Q_{C'}(v' \to y')) + l(P_2)$, and

$$W = \begin{cases} P_1 + Q_C(x' \to v) + P + Q_{C'}(v' \to y') + P_2, & \text{if } w \text{ is odd;} \\ P_1 + Q_C(x' \to v) + P + C' \setminus Q_{C'}(v' \to y') + P_2, & \text{otherwise.} \end{cases}$$

Note that C and $\frac{k}{2}C_2$ (walk around a 2-cycle $\frac{k}{2}$ times) have different signs, so $W_1 = W + C$ and $W_2 = W + \frac{k}{2}C_2$ is a pair of $SSSD$ walks from x to y with odd length $\leq 2(n-k-k'+1) + \max\{[\frac{k}{2}] + \frac{k'-1}{2}\} + \frac{k}{2} + k' + k = 2n - k' - \frac{1}{2}k + 2 + \max\{[\frac{k}{2}], \frac{k'-1}{2}\}$ by (2.3).

If $[\frac{k}{2}] \geq \frac{k'-1}{2}$, then $l(W_1) \leq 2n - k' - \frac{1}{2}k + 2 + [\frac{k}{2}] \leq 2n - k' + 2 \leq 2n - 1$.

If $[\frac{k}{2}] < \frac{k'-1}{2}$, then $l(W_1) \leq 2n - k' - \frac{1}{2}k + 2 + \frac{k'-1}{2} = 2n - \frac{k+k'}{2} + \frac{3}{2} < 2n - 1$.

Therefore there exists a pair of $SSSD$ walks of length $2n-1$ from x to y.

Case 2 $C \cap C' \neq \emptyset$.

Let P_1 be the shortest path from x to C and P_1 intersect C at x' where $0 \leq l(P_1) \leq n-k$, P_2 be the shortest path from y to C' and P_2 intersect C' at y' where $0 \leq l(P_2) \leq n-k'$.

By Lemma 2.1, there exists $x'' \in C$ such that there exists a pair of $SSSD$ walks from x' to x'' with length $\frac{k}{2}$, denoted by W_1 and W_2.

Because $C \cap C' \neq \emptyset$, there exists $z \in C \cap C'$. So we set $w = l(P_1) + \frac{k}{2} + l(Q_C(x'' \to z)) + l(Q_{C'}(z \to y')) + l(P_2)$, and for $i = 1, 2$, set

$$W_i' = \begin{cases} P_1 + W_i + Q_C(x'' \to z) + Q_{C'}(z \to y') + P_2, & \text{if } w \text{ is odd;} \\ P_1 + W_i + Q_C(x'' \to z) + C' \setminus Q_{C'}(z \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then W_1', W_2' is a pair of $SSSD$ walks from x to y with odd length $\leq n-k+\frac{k}{2}+\frac{k}{2}+k'+n-k' = 2n$. Therefore, there exists a pair of $SSSD$ walks of length $2n-1$ from x to y.

Then $b(S) \leq 2n-1$ by Theorem 2.B. □

Theorem 3.1 *Let S be a primitive non-powerful symmetric loop-free signed digraph on n vertices. Then $b(S) \leq 2n-1$.*

Proof Because S is primitive and non-powerful, it follows that Theorem 2.A holds.

If (A_1) of Theorem 2.A holds, then there exists an even cycle C with the property $\operatorname{sgn} C = -1$. By Lemma 3.5, we have $b(S) \leq 2n-1$.

If (A_1) of Theorem 2.A does not hold, then the sign of every even cycle is 1, and (A_2) of Theorem 2.A holds. Therefore there exist odd cycles C and C' such that $\text{sgn}C = -\text{sgn}C'$. We can assume that $l(C) \geq l(C')$. Let $l = l(C)$ (≥ 3) and $l' = l(C')$ (≥ 3). Suppose $W_1 = C$ and $W_2 = C' + \frac{l-l'}{2}C_2$, then W_1 and W_2 have the same length l and different signs. Two cases need to be considered.

Case 1 $C \cap C' = \emptyset$.

The proof is similar to the case 1 of Lemma 3.5.

Case 2 $C \cap C' \neq \emptyset$.

Let x and y be any two (not necessarily distinct) vertices in S. Let P_1 be the shortest path from x to C and P_1 intersect C at x', P_2 be the shortest path from y to C and P_2 intersect C at y' where $0 \le l(P_i) \le n-l$ $(i=1,2)$.

Set $W_0 = Q_C(x' \to y')$, then W_0 or $C \setminus W_0$ must contain at least one common vertex of C and C'. Assume W_0 contains at least one common vertex of C and C', denoted by z.

Set $w = l(P_1)+l(W_0)+l(P_2)$, $a = l(Q_{W_0}(x' \to z)$, $b = l(Q_{W_0}(y' \to z)$ and

$$W = \begin{cases} P_1 + W_0 + P_2, & \text{if } w \text{ is even;} \\ P_1 + C \setminus W_0 + 2Q_{W_0}(x' \to z) + P_2, & \text{if } w \text{ is odd and } a \le b; \\ P_1 + C \setminus W_0 + 2Q_{W_0}(y' \to z) + P_2, & \text{if } w \text{ is odd and } a > b. \end{cases}$$

Set $W_1 = W + C$, $W_2 = W + C' + \frac{l-l'}{2}C_2$, then W_1, W_2 is a pair of $SSSD$ walks from x to y with odd length $\le 2(n-l)+l+l \le 2n$. Therefore, there exists a pair of $SSSD$ walks of length $2n-1$ from x to y.

Then $b(S) \le 2n-1$ by Theorem 2.B. $\square$

Theorem 3.2 *Let S be a primitive symmetric loop-free signed digraph on n vertices. Then $b(S) \le 2n-1$.*

Proof We consider the following two cases.

Case 1 If S is powerful.

From Theorem 1.A, we have $b(S) \le 2n-4$.

Case 2 If S is not powerful.

From Theorem 3.1, we have $b(S) \le 2n-1$. $\square$

Theorem 3.3 *Let S be a primitive symmetric loop-free generalized signed digraph on n vertices. Then $b(S) \le 2n-1$.*

Proof If S is primitive symmetric signed digraph, then $b(S) \le 2n-1$ by Theorem 3.2.

Otherwise, there exists an arc (u,v) where $u, v \in V(S)$ such that the arc is assigned a sign $\#$. Set $W = (u,v)+(v,u)$, thus W is a closed walk of length 2 and $\mathrm{sgn} W = \#$. Therefore $b(S) \le 2n-1$ by Lemma 3.3. $\square$

4. Base sets of $\mathcal{S}_n^{\star\prime}$, $\mathcal{S}_n^{\star}$ and $\bar{\mathcal{S}}_n^{\star}$

Let $\mathcal{S}_n^{\star\prime} = \{S | S$ is a primitive non-powerful symmetric loop-free signed digraph on n vertices$\}$, $\mathcal{S}_n^{\star} = \{S|S$ is a primitive symmetric loop-free signed digraph on n vertices$\}$, and $\bar{\mathcal{S}}_n^{\star} = \{S|S$ is a primitive symmetric loop-free generalized signed digraph on n vertices$\}$. Clearly, $\mathcal{S}_n^{\star\prime} \subset \mathcal{S}_n^{\star} \subset \bar{\mathcal{S}}_n^{\star}$. In this section, we show the base sets of $\mathcal{S}_n^{\star\prime}$, $\mathcal{S}_n^{\star}$ and $\bar{\mathcal{S}}_n^{\star}$ are $\{2,3,\ldots,2n-1\}$.

In [8], Cheng and Liu defined connected graph $(v_1, v_2; k, l, m)$-lollipop and obtained the following result.

Lemma 4.1 ([8]) *Suppose S is a signed digraph on n vertices and the underlying digraph D is* *a $(v_1, v_n; k, l, m)$-lollipop, where $n = k + l + m - 1$, $l > 1$ and l is odd. If there exist no positive 2-cycles in S, then*

(1) $b(S) = 2l + 2k - 3$ if $k \geq 2$;

(2) $b(S) = 2l + 1$ if $k = 1$ and $m > 0$;

(3) $b(S) = 2l - 1$ if $k = 1$ and $m = 0$.

It is obvious that S is primitive non-powerful symmetric loop-free signed digraph on n (≥ 4) vertices if S satisfies the conditions of Lemma 4.1, then by the result of Lemma 4.1, we have

Corollary 4.1 *Let $n \geq 4$, $E_{n,0} = \{b(S) \mid S \in \mathcal{S}_n^{\star\prime}\}$, then $\{7, 9, \ldots, 2n-1\} \subseteq E_{n,0}$.*

Lemma 4.2 *Let $1 \leq k \leq n-3$ and $S = (V, A)$ be a signed digraph, where $V = \{1, 2, 3, \ldots, n\}$, $A = \{(i, i+1), (i+1, i) | 1 \leq i \leq k+1\} \cup \{(k+2, j), (j, k+2) | k+3 \leq j \leq n\} \cup \{(1, 3), (3, 1)\}$, $\operatorname{sgn}(1, 3) = -1$, and the signs of the other arcs of S are 1. Then S is a primitive non-powerful symmetric loop-free signed digraph with $b(S) = 2k + 2$.*

Proof It is obvious that S is a primitive non-powerful symmetric loop-free signed digraph by Theorem 2.A. Now we show $b(S) = 2k + 2$.

It is easy to see that there exist no pairs of $SSSD$ walks with length $2k + 1$ from n to n, so $b(S) \geq 2k + 2$.

On the other hand, the vertex 3 is contained in a positive 2-cycle C_2 and a negative 2-cycle C_2', so there is a pair of $SSSD$ walks from 3 to 3 with length 2. At the same time, the vertex 3 is also contained in a positive 3-cycle C_3 and a negative 3-cycle C_3', so there is a pair of $SSSD$ walks from 3 to 3 with length 3.

For any two vertices $x, y \in V$, let P_1 be the shortest path from x to 3, P_2 be the shortest path from 3 to y where $0 \leq l(P_i) \leq k, i = 1, 2$. Set $w = l(P_1) + l(P_2)$, then $w \leq 2k$.

If w is even, we set $W_1 = P_1 + C_2 + P_2$, $W_2 = P_1 + C_2' + P_2$. Then W_1, W_2 is a pair of $SSSD$ walks from x to y with even length $\leq 2k + 2$.

If w is odd, then $w \leq 2k - 1$, and we set $W_1 = P_1 + C_3 + P_2$, $W_2 = P_1 + C_3' + P_2$. Then W_1, W_2 is a pair of $SSSD$ walks from x to y with even length $\leq 2k + 2$.

Therefore there exists a pair of $SSSD$ walks of length $2k + 2$ from x to y and $b(S) \leq 2k + 2$ by Theorem 2.B.

Combining the above two inequalities, we have $b(S) = 2k + 2$. □

By Lemma 4.2 and the fact $1 \leq k \leq n - 3$, we have

Corollary 4.2 *Let $n \geq 4$, then $\{4, 6, 8, \ldots, 2n-4\} \subseteq E_{n,0}$.*

Lemma 4.3 *Let $n \geq 4$ and $S = (V, A)$ be a signed digraph, where $V = \{1, 2, \ldots, n\}$, $A = \{(i, i+1) \cup (i+1, i) | 1 \leq i \leq n-3\} \cup \{(1, n-1), (n-1, 1), (1, n), (n, 1), (n-1, n), (n, n-1)\}$, $\operatorname{sgn}(1, n-1) = \operatorname{sgn}(n, 1) = -1$ and the signs of other arcs of S are 1. Then S is a primitive non-powerful symmetric loop-free signed digraph and $b(S) = 2n - 2$.*

Proof It is obvious that S is a primitive non-powerful symmetric loop-free signed digraph by

Theorem 2.A. Now we show $b(S) = 2n-2$.

It is easy to see that there exist no pairs of $SSSD$ walks with length $2n-3$ from $n-2$ to $n-2$, so $b(S) \geq 2n-2$.

On the other hand, the vertex 1 is contained in a positive 2-cycle C_2 and a negative 2-cycle C_2', so there is a pair of $SSSD$ walks from 1 to 1 with length 2.

For any two vertices $x, y \in V$, let P_1 be the shortest path from x to 1, P_2 be the shortest path from 1 to y where $0 \leq l(P_i) \leq n-3$ $(i = 1, 2)$, and C_3 be 3-cycle in S. Set $w = l(P_1) + l(P_2)$, and

$$W = \begin{cases} P_1 + P_2, & \text{if w is even;} \\ P_1 + C_3 + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W) \leq 2(n-3)+3 = 2n-3$ and $l(W)$ is even, and thus $W_1 = W + C_2$ and $W_2 = W + C_2'$ are a pair of $SSSD$ walks from x to y with even length $\leq 2n-1$. Therefore there exists a pair of $SSSD$ walks of length $2n-2$ from x to y and $b(S) \leq 2n-2$ by Theorem 2.B.

Combining the above arguments, we have $b(S) = 2n-2$. □

The proofs of the following lemmas are easy, so we omit them.

Lemma 4.4 *Let $S = (V, A)$ be a signed digraph of order $n = 2m+1 (\geq 5)$, where $V = \{1, 2, \ldots, 2m, 2m+1\}$, $A = \{(1, i), (i, 1) \mid 2 \leq i \leq 2m+1\} \cup \{(2, i), (i, 2) \mid 3 \leq i \leq 2m+1\} \cup \{(2i, 2i+1), (2i+1, 2i) \mid 2 \leq i \leq m\} \cup \{(3, 4), (4, 3)\}$, for all $1 \leq i \leq m$, $\mathrm{sgn}(1, 2i) = -1$, and for all $1 \leq i \leq m+1$, $\mathrm{sgn}(2, 2i-1) = -1$, and the signs of the other arcs of S are 1. Then S is a primitive non-powerful symmetric loop-free signed digraph with $b(S) = 2$.*

When n is even, Wang, You and Ma [12] gave the following signed digraph $S = (V, A)$ where $V = \{1, 2, \ldots, 2m-1, 2m\}$, $A = \{(1, i), (i, 1) | 2 \leq i \leq 2m\} \cup \{(2, i), (i, 2) | 3 \leq i \leq 2m\} \cup \{(2i-1, 2i), (2i, 2i-1) | 2 \leq i \leq m\}$, for all $1 \leq i \leq m$, $\mathrm{sgn}(1, 2i) = -1$, $\mathrm{sgn}(2, 2i-1) = -1$, and the signs of the other arcs of S are 1. They showed S is a primitive non-powerful symmetric loop-free signed digraph on $n = 2m$ (≥ 4) vertices with $b(S) = 2$.

Lemma 4.5 *Let $n \geq 3$ and $S = (V, A)$ be a signed digraph, where $V = \{1, 2, 3 \ldots, n\}$, $A = \{(1, 2), (2, 1)\} \cup \{(1, i), (i, 1), (2, i), (i, 2) | 3 \leq i \leq n\}$, $\mathrm{sgn}(1, 2) = -1$ and the signs of the other arcs of S are 1. Then S is a primitive non-powerful symmetric loop-free signed digraph with $b(S) = 3$.*

Lemma 4.6 *Let $n \geq 5$ and $S = (V, A)$ be a signed digraph, where $V = \{1, 2, 3, \ldots, n\}$, $A = \{(i, i+1), (i+1, i) | 1 \leq i \leq 3\} \cup \{(1, j), (j, 1), (4, j), (j, 4) | 5 \leq j \leq n\}$, and for all $j \in \{5, \ldots, n\}$, $\mathrm{sgn}(j, 1) = -1$, $\mathrm{sgn}(1, 2) = \mathrm{sgn}(3, 4) = -1$, the signs of the other arcs of S are 1. Then S is a primitive non-powerful symmetric loop-free signed digraph with $b(S) = 5$.*

Because S is loop-free, $1 \notin E_{n,0}$. By Lemmas 4.3–4.6, we have $2, 3, 5, 2n-2 \in E_{n,0}$. So by Theorems 3.1–3.3, Corollaries 4.1 and 4.2, and $\mathcal{S}_n^{\star\prime} \subset \mathcal{S}_n^{\star} \subset \bar{\mathcal{S}}_n^{\star}$, we have

Theorem 4.1 *(1) Let $n \geq 5$, then $E_{n,0} = \{2, 3, 4, \ldots, 2n-1\}$.*

(2) The base set of $\mathcal{S}_n^{\star}$ $(\bar{\mathcal{S}}_n^{\star})$ is $\{2, 3, 4, \ldots, 2n-1\}$ for $n \geq 5$.

Let ${\mathcal{S}_n}' = \{S|S$ is a primitive non-powerful symmetric signed digraph on n vertices $\}$. Then ${\mathcal{S}_n^{\star}}' \subseteq {\mathcal{S}_n}'$. It is easy to see that there exist primitive non-powerful symmetric signed digraphs on n vertices such that the bases are 1 or $2n$. Then by Theorem 4.1, we have

Corollary 4.3 *The base set of* ${\mathcal{S}_n}'$ *is* $\{1, 2, \ldots, 2n-1, 2n\}$.

5. A new proof of an important result in [8]

In this section, we give a new proof of Lemma 4.5 [8, pp. 726–729]. The following Lemma 5.1 in terms of graph theoretical version is useful.

Lemma 5.1 ([8]) *Suppose S is a primitive non-powerful symmetric signed digraph on n vertices and there exists a negative 2-cycle. Then $b(S) \leq 2n$.*

Theorem 5.1 ([8]) *Suppose S is a primitive non-powerful symmetric signed digraph on n vertices and there exists a negative even cycle. Then $b(S) \leq 2n$.*

Proof Let C be the shortest even cycle in S with the property that $\mathrm{sgn} C = -1$ and let $l(C) = k$ (k is even).

If $k = 2$, then the result follows from Lemma 5.1.

If $k \geq 4$, then the sign of any 2-cycle is 1. Since S is primitive, there exists an odd cycle C' in S with length k' (≥ 1). Let x and y be any two (not necessarily distinct) vertices in S.

Case 1 $C \cap C' = \emptyset$.

Let P be the shortest path from C to C'. Assume P intersects C at v and P intersects C' at v'. By Lemma 2.3, there exist $x' \in C, y' \in C'$ or $x' \in C', y' \in C$ such that (2.3) holds. Without loss of generality, we assume that there exist $x' \in C, y' \in C'$ such that (2.3) holds. For convenience, let P_1 be the shortest path from x to x' and P_2 be the shortest path from y to y' with property $0 \leq l(P_i) \leq n - k - k' + 1$ ($i = 1, 2$).

Set $w = l(P_1) + l(Q_C(x' \to v)) + l(P) + l(Q_{C'}(v' \to y')) + l(P_2)$, and

$$W = \begin{cases} P_1 + Q_C(x' \to v) + P + Q_{C'}(v' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + Q_C(x' \to v) + P + C' \setminus Q_{C'}(v' \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $W_1 = W + C$ and $W_2 = W + \frac{k}{2}C_2$ is a pair of $SSSD$ walks from x to y with even length $\leq 2(n-k-k'+1) + \max\{[\frac{k}{2}] + \frac{k'-1}{2}\} + \frac{k}{2} + k' + k = 2n - k' - \frac{1}{2}k + 2 + \max\{[\frac{k}{2}], \frac{k'-1}{2}\}$ by (2.3).

If $[\frac{k}{2}] > \frac{k'-1}{2}$, then $l(W_1) \leq 2n - k' - \frac{1}{2}k + 2 + [\frac{k}{2}] = 2n - k' + 2 \leq 2n+1$, and thus $l(W_1) \leq 2n$ since $l(W_1)$ is even.

If $[\frac{k}{2}] \leq \frac{k'-1}{2}$, then $l(W_1) \leq 2n - k' - \frac{1}{2}k + 2 + \frac{k'-1}{2} = 2n - \frac{k+k'}{2} + \frac{3}{2} \leq 2n$.

Therefore there exists a pair of $SSSD$ walks of length $2n$ from x to y.

Case 2 $C \cap C' \neq \emptyset$.

Let P_1 be the shortest path from x to C and P_1 intersect C at x' with property $0 \leq l(P_1) \leq n - k$, P_2 be the shortest path from y to C' and P_2 intersect C' at y' with property $0 \leq l(P_2) \leq n - k'$.

By Lemma 2.1, there exists $x'' \in C$ such that there exists a pair of $SSSD$ walks from x' to x'' with length $\frac{k}{2}$, denoted by W_1 and W_2.

Because $C \cap C' \neq \emptyset$, there exists $z \in C \cap C'$. So we set $w = l(P_1) + \frac{k}{2} + l(Q_C(x'' \to z)) + l(Q_{C'}(z \to y')) + l(P_2)$, and for $i = 1, 2$, set

$$W_i' = \begin{cases} P_1 + W_i + Q_C(x'' \to z) + Q_{C'}(z \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + W_i + Q_C(x'' \to z) + C' \setminus Q_{C'}(z \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then W_1', W_2' is a pair of $SSSD$ walks from x to y with even length $\leq n-k+\frac{k}{2}+\frac{k}{2}+k'+n-k' = 2n$. Therefore there exists a pair of $SSSD$ walks of length $2n$ from x to y.

Thus $b(S) \leq 2n$ by Theorem 2.B. $\Box$

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