Journal of Mathematical Research with Applications
May, 2012, Vol. 32, No. 3, pp. 366–372
DOI:10.3770/j.issn:2095-2651.2012.03.011
Http://jmre.dlut.edu.cn

Robin-Type Boundary Value Problem of Nonlinear Differential Equation with Fractional Order Derivative

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Abstract In this paper, the existence and multiplicity of positive solutions for Robin type boundary value problem of differential equation involving the Riemann-Liouville fractional order derivative are established.

Keywords fractional differential equation; positive solution; fixed point.

MR(2010) Subject Classification 26A33; 34B15

1. Introduction

Due to the development of the theory of fractional calculus and its applications, such as in the fields of physics, Bode's analysis of feedback amplifiers, aerodynamics and polymer archeology etc, many work on fractional calculus, fractional order differential equations have appeared [1–7]. Recently, there have been many papers dealing with the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations (see [8–21] and references along this line). However we notice that most of these papers concerned with the boundary value problems with Dirichlet-type boundary condition. For examples, Su [17] obtained the existence of solution for boundary value problem of a coupled system of nonlinear fractional differential equation

$$D_{0+}^{\alpha}u(t)=f(t,\ v(t),\ D^{p}v(t)),\ D_{0+}^{\beta}v(t)=f(t,\ u(t),\ D^{q}u(t)),\quad 0\le t\le 1,$$

$$u(0)=0=u(1),\ v(0)=0=v(1),$$

where $1<\alpha,\ \beta<2,\ p,q>0,\ \alpha-p\le 1,\ \beta-q\le 1,\ f,g:[0,1]\times R\times R\to R$ and D_{0+}^{α} is the Sturm-Liouville fractional order derivative. By means of lower and upper solution method and fixed-point theorems, some existence results of solutions are established. Bai [18], Jiang [19] considered respectively the existence of positive solution for problem

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\ u(0)=0,\ u(1)=0,\quad 0<\alpha\le 2$$

Received November 12, 2010; Accepted April 25, 2011
Supported by the Anhui Provincial Natural Science Foundation (Grant No. 10040606Q50) and the Natural Science Foundation of Anhui Department of Education (Grant No. KJ2010A285).
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by using the fixed point theorems on cones. As to the problem with complex nonlinearity, there are also some papers dealing with the Dirichlet-type boundary value problem of fractional order differential equations. Based on the Leray-Schauder Continuation Principle, Kosmatov [20] obtained the existence of solution for singular problem

$$D_{0+}^{\alpha}u(t) = f(t,\ u(t),\ u'(t)),\ u(0) = u(1) = 0,\quad 0 < \alpha \leq 2.$$

Agarwal et al. [21] established the existence of positive solution of singular problem with Dirichlet-type boundary condition

$$D_{0+}^{\alpha}u(t) = f(t,\ u(t),\ D^{\mu}u(t)),\ u(0) = u(1) = 0,$$

where $1 < \alpha < 2$, $0 \leq \mu \leq \alpha - 1$ and f satisfies the Caratheodory conditions on $[0,1] \times [0,\ \infty) \times R$ and $f(t,x,y)$ is singular at $x = 0$. The existence results were established by using regularization and sequential techniques.

However, few contributions exist, as far as we know, concerning positive solution to Robin-type boundary value problem of fractional differential equation. The goal of this paper is to fill the gap.

In this paper, we consider the existence of positive solutions for following Robin-type boundary value problem of differential equation involving the Riemann-Liouville fractional order derivative

$$D_{0+}^{\alpha}u(t) = f(t,u(t)),\ t \in (0,\ 1),\ u(0) = 0, u'(1) = 0 \tag{1.1}$$

where $1 < \alpha < 2$ and $f : C([0,1] \times R, R^+)$. We discuss some properties of the associated Green's function for problem (1.1). By using these properties of Green's function and fixed point theorems on cones, we establish the existence and multiplicity of positive solutions for problem (1.1).

2. Preliminary results

Definition 2.1 *The fractional integral of order $\alpha > 0$ of a function $u(t) : (0,\infty) \to R$ is given by*

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s$$

provided the right side is point-wise defined on $(0,\infty)$.

Definition 2.2 *The fractional derivative of order $\alpha > 0$ of a continuous function $u(t) : (0,\infty) \to R$ is given by*

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}t})^n\int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s$$

where $n = [\alpha] + 1$, provided that the right side is point-wise defined on $(0,\infty)$.

Lemma 2.1 *Let $\alpha > 0$. If we assume $u \in L(0,1)$, then the fractional differential equation $D_{0+}^{\alpha}u(t) = 0$ has solution*

$$u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N}$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

Lemma 2.2 *Assume that $u \in L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$.

Lemma 2.3 ([22]) *Let E be a Banach space and $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\Omega_2 \setminus \overline{\Omega}_1) \to K$$

be a completely continuous operator such that

$$\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_1, \text{ and } \|Au\| \geq \|u\|, u \in K \cap \partial\Omega_2 \text{ or}$$

$$\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_1, \text{ and } \|Au\| \leq \|u\|, u \in K \cap \partial\Omega_2,$$

then A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

Let $0 < a < b$ be given and let ψ be a nonnegative continuous concave functional on the cone C. Define the convex sets C_r and $C(\psi, a, b)$ by

$$C_r = \{x \in C |\ \|x\| < r\},\ C(\psi, a, b) = \{x \in C |\ a \leq \psi(x),\ \|x\| \leq b\}.$$

Lemma 2.4 ([23]) *Let $T : \overline{C}_r \to \overline{C}_r$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on C such that $\psi(x) \leq \|x\|$ for $x \in \overline{C}_r$. Suppose that there exist $0 < a < b < d \leq c$ such that*

(S_1) $\{x \in C(\psi, b, d) |\ \psi(x) > b\} \neq \emptyset$ and $\psi(Tx) > b$ for $x \in C(\psi, b, d)$;

(S_2) $\|Tx\| < a$ for $\|x\| \leq a$ and

(S_3) $\psi(Tx) > b$ for $x \in C(\psi, b, c)$ with $\|Tx\| \geq d$.

Then T has at least three fixed points x_1, x_2 and x_3 such that

$$\|x_1\| < a, b < \psi(x_2), \|x_3\| > a,\ \psi(x_3) < b.$$

3. Main results

Let $E = C[0, 1]$ be a Banach space endowed with the norm

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)|, \quad u \in E.$$

Define the cone $P \subset E$ by $P = \{u \in E \mid u(t) \geq 0\}$.

Lemma 3.1 *Given $y(t) \in C[0,1]$. Then problem*

$$D_{0+}^{\alpha} u(t) + y(t) = 0,\ u(0) = 0,\ u'(1) = 0, \quad 1 < \alpha < 2, \tag{3.1}$$

is equivalent to

$$u(t) = \int_0^1 G(t,s) y(s) \mathrm{d}s,$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}-(t-s)^{\alpha-1}+t^{\alpha-1}(1-s)^{\alpha-2}, & t\ge s,\\ t^{\alpha-1}(1-s)^{\alpha-2}, & t\le s.\end{cases}$$

Proof By Lemmas 2.1 and 2.2, we see that problem (3.1) is equivalent to

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+C_1t^{\alpha-1}+C_2t^{\alpha-2}.$$

The boundary condition $u(0)=0$ implies that $C_2=0$. In view of the boundary condition $u'(1)=0$, we get

$$C_1=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}y(s)\mathrm{d}s.$$

Thus

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}y(s)\mathrm{d}s=\int_0^1G(t,\ s)y(s)\mathrm{d}s.$$

Lemma 3.2 *The function $G(t,s)$ satisfies the following conditions:*

(1) $0\le G(t,s)\le G(s,s),\quad t,s\in(0,\ 1)$*;*

(2) There exists a positive function $\gamma(s)\in C(0,1)$ such that

$$\min_{1/4\le t\le 3/4}G(t,s)\ge\gamma(s)G(s,s),\quad 0<s<1.$$

Proof (1) It is obvious that $G(t,\ s)\ge0$ for $t\le s$. For $t\ge s$,

$$\Gamma(\alpha)G(t,\ s)=-(t-s)^{\alpha-1}+t^{\alpha-1}(1-s)^{\alpha-2}=t^{\alpha-1}[(1-s)^{\alpha-2}-(1-\frac{s}{t})^{\alpha-1}]>0.$$

For given $s\in(0,\ 1)$, $G(t,\ s)$ is decreasing with respect to t for $s\le t$ and increasing with respect to t for $t\le s$. Thus one can easily check that $G(t,s)\le G(s,s),\ t,s\in(0,\ 1)$.

(2) Setting

$$g_1(t,\ s)=\frac{-(t-s)^{\alpha-1}+t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)},\ g_2(t,\ s)=\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)},$$

one has

$$\min_{1/4\le t\le 3/4}G(t,s)=\begin{cases}g_1(\frac{3}{4},s), & 0<s\le\frac{1}{4},\\ \min\{g_1(\frac{3}{4},s),\ g_2(\frac{1}{4},s)\}, & \frac{1}{4}\le s\le\frac{3}{4},\\ g_2(\frac{1}{4},s), & \frac{3}{4}\le s<1.\end{cases}$$

Setting $\frac{1}{4}<r<\frac{3}{4}$ to be the unique solution of the equation

$$-(\frac{3}{4}-s)^{\alpha-1}+(\frac{3}{4})^{\alpha-1}(1-s)^{\alpha-2}=(\frac{1}{4})^{\alpha-1}(1-s)^{\alpha-2},$$

we have

$$\min_{1/4\le t\le 3/4}G(t,s)=\begin{cases}\dfrac{-(\frac{3}{4}-s)^{\alpha-1}+(\frac{3}{4})^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0<s\le r,\\ \dfrac{(\frac{1}{4})^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & r\le s<1.\end{cases}$$

Considering the monotonicity of $G(t,s)$, we have

$$\max_{0\le t\le 1} G(t,s) = G(s,s) = \frac{s^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}.$$

Thus, setting

$$\gamma(s) = \begin{cases} \dfrac{-(\frac{3}{4}-s)^{\alpha-1} + (\frac{3}{4})^{\alpha-1}(1-s)^{\alpha-2}}{s^{\alpha-1}(1-s)^{\alpha-2}}, & 0 < s \le r, \\ (\dfrac{1}{4s})^{\alpha-1}, & r \le s < 1 \end{cases}$$

yields

$$\min_{1/4\le t\le 3/4} G(t,s) \ge \gamma(s)G(s,s), \quad 0 < s < 1. \ \square$$

Define the operator $T : P \to E$,

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s.$$

It is obvious that boundary value problem (1.1) is equivalent to the operator equation $u = Tu$. That is, a fixed point of operator T on cone P is the positive solution of problem (1.1).

Theorem 3.1 *The operator $T : P \to P$ is completely continuous.*

Proof It is obvious that the operator $T : P \to P$ is continuous. Let $\Omega \subset P$ be bounded. That is, there exists a positive constant $M_1 > 0$ such that $\|u\| \le M_1$ for all $u \in \Omega$. Then there exists constant $M_2 > 0$ such that $|f(t,\ u)| \le M_2,\ t \in [0,\ 1], u \in \Omega$. Then

$$Tu(t) = \int_0^1 G(t,\ s)f(s,u(s))\mathrm{d}s \le M_2 \int_0^1 G(s,s)\mathrm{d}s.$$

Thus, T is uniformly bounded on the bounded subset of E. On the other hand, for each $u \in \Omega, t_1, t_2 \in [0,1], t_1 < t_2$, one has

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &= \Big| \int_0^1 (G(t_1,s) - G(t_2,s))f(s,\ u(s))\mathrm{d}s \Big| \\ &\le \int_0^1 |G(t_1,s) - G(t_2,s)||f(s,u(s))|\mathrm{d}s \\ &\le M_1 \int_0^1 |G(t_1,s) - G(t_2,s)|\mathrm{d}s \\ &\le M_1 \int_0^1 (1-s)^{\alpha-2}\mathrm{d}s|t_1^{\alpha-1} - t_2^{\alpha-1}|. \end{aligned}$$

This ensures that T is of equi-continuity on the bounded subset of P. Then an application of Ascoli-Arezela Theorem ensures that $T : P \to P$ is completely continuous.

Theorem 3.2 *Assume that there exist two positive constants $r_2 > r_1 > 0$ such that*

(A1) $f(t,\ u) \le Mr_2,\ (t,u) \in [0,1] \times [0,r_2]$;

(A2) $f(t,\ u) \geq Nr_1,\ (t,u) \in [0,1] \times [0,r_1]$

where

$$M = \Big(\int_0^1 G(s,s)\mathrm{d}s \Big)^{-1},\ N = \Big(\int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s)G(s,s)\mathrm{d}s \Big)^{-1}.$$

Then problem (1.1) has at least one positive solution u such that $r_1 \leq \|u\| \leq r_2$.

Proof Let $\Omega_1 = \{u \in P | \|u\| \leq r_1\}$. For $u \in \partial\Omega_1$ and assumption (A2), we have

$$0 \leq u(t) \leq r_1, \text{ and } f(t,u) \geq Nr_1, \quad t \in [0,1].$$

Then for $t \in [\frac{1}{4}, \frac{3}{4}]$, we have that

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \geq \int_{1/4}^{3/4} \gamma(s)G(s,s)f(s,u(s))\mathrm{d}s \geq r_1.$$

Thus $\|Tu\| \geq \|u\|$, $u \in \partial\Omega_1$.

Let $\Omega_2 = \{u \in P | \|u\| \leq r_2\}$. For $u \in \partial\Omega_2$ and assumption (A1), we get that

$$0 \leq u(t) \leq r_2, \text{ and } f(t,\ u) \leq Mr_2, \quad t \in [0,1].$$

Then for $t \in [0,1]$,

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \leq Mr_2 \int_0^1 G(s,s)\mathrm{d}s \leq r_2.$$

Thus $\|Tu\| \leq \|u\|$, $u \in \partial\Omega_2$. An application of Lemma 2.3 ensures the existence of at least one positive solution $u(t)$ of problem (1.1).

Theorem 3.3 *Suppose $f(t,u)$ is continuous and there exist constants $0 < a < b < c$ such that*

(A3) $f(t,u) < Ma$, *for* $(t,u) \in [0,1] \times [0,a]$*;*

(A4) $f(t,u) \geq Nb$, *for* $(t,u) \in [1/4,3/4] \times [b,c]$*;*

(A5) $f(t,u) \leq Mc$, *for* $(t,u) \in [0,1] \times [0,c]$*,*

then problem (1.1) has at least three positive solutions u_1, u_2, u_3 with

$$\max_{0\leq t\leq 1} |u_1| \leq a,\ b < \min_{1/4\leq t\leq 3/4} |u_2| < \max_{0\leq t\leq 1} |u_2| \leq c,\ a < \max_{0\leq t\leq 1} |u_3| \leq c,\ \min_{1/4\leq t\leq 3/4} |u_3| < b.$$

Proof Define the nonnegative continuous concave functional θ on the cone P by

$$\theta(u) = \min_{1/4\leq t\leq 3/4} |u(t)|.$$

If $u \in \overline{P}_c$, then $\|u\| \leq c$. Then by condition (A5), we have

$$|T(u)(t)| = |\int_0^1 G(t,s)f(s,\ u(s))\mathrm{d}s| \leq Mc \int_0^1 G(s,s)\mathrm{d}s = c,$$

which yields that $T : \overline{P}_c \to \overline{P}_c$. In the same way, we get that

$$\|Tu\| < a, \text{ for } u \leq a.$$

The fact that constant function $u \in \{u \in P(\theta,b,c) | \theta(u) > b\}$ ensures that $\{u \in P(\theta,b,c) | \theta(u) > b\} \neq \emptyset$.

And for $u \in P(\theta,b,c)$, we have

$$f(t,\ u(t)) \geq Nb, \quad t \in [1/4,3/4].$$

Then

$$\theta(Tu)=\min_{1/4\le t\le 3/4}|\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s|>Nb\int_{1/4}^{3/4}\gamma(s)G(s,s)\mathrm{d}s=b,$$

which yields that $\theta(Tu)>b$, for $u\in P(\theta,b,c)$. Then an application of Lemma 2.4 ensures that problem (1.1) has at least three positive solutions with

$$\max_{0\le t\le 1}|u_1|\le a,\ b<\min_{1/4\le t\le 3/4}|u_2|<\max_{0\le t\le 1}|u_2|\le c, a<\max_{0\le t\le 1}|u_3|\le c,\ \min_{1/4\le t\le 3/4}|u_3|<b.$$

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