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Geometrically Continuous Interpolation in Spheres

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Abstract In this paper, a new method for geometrically continuous interpolation in spheres is proposed. The method is entirely based on the spherical Bézier curves defined by the generalized de Casteljau algorithm. Firstly we compute the tangent directions and curvature vectors at the endpoints of a spherical Bézier curve. Then, based on the above results, we design a piecewise spherical Bézier curve with G^1 and G^2 continuity. In order to get the optimal piecewise curve according to two different criteria, we also give a constructive method to determine the shape parameters of the curve. According to the method, any given spherical points can be directly interpolated in the sphere. Experimental results also demonstrate that the method performs well both in uniform speed and magnitude of covariant acceleration.

Keywords interpolation; sphere; geometric continuity; Bézier.

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1. Introduction

The research about interpolation in sphere is very basic and useful for computer graphics (CG), computer aided geometric design (CAGD) and other application fields. For example, in robot kinematics and computer animation, the orientation of a rigid motion can be represented by a curve in the unit quaternion space which is identified with the 3-dimensional unit sphere S^3. Furthermore, for the application in motion design, we need the interplant to have smaller variations in speed and magnitude of acceleration. The above demands can be easily achieved in $(m+1)$-dimensional Euclidean space E^{m+1}. However, when the research is restricted to m-dimensional unit sphere $S^m \subset E^{m+1}$, it becomes much more difficult.

The interpolation curves in S^m are usually constructed in two different ways. One is to interpolate in the ambient space E^{m+1} and then obtain the desired spherical curves by a certain method. Most of the previous results are based on this idea, but it can cause large variations in the speed and magnitude of acceleration of the interpolation curves. Parker and Denham [1] interpolated in the ambient space E^{m+1} and then normalized to produce a curve in S^m. Using the generalized stereographic projection, Dietz et al. [2, 3] constructed the spherical rational curves for interpolation in S^2. However, the method cannot be generalized to S^m with $m \geq 3$. Further work [4] produced C^1 interpolation curves in S^2, but fell down in a similar way. Gfrerrer

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[5] considered rational Lagrange interpolation in S^m. Wang and Qin [6] studied the existence and computation of spherical rational quartic curves for C^1 interpolation. The two methods are both based on algebraic manipulation, instead of stereographic projection. With the idea of functional spline [7], Hartmann [8] developed a method for G^2 interpolation in S^2. The construction needs an appropriate surface/surface intersection algorithm to display the interpolation curve, which is a complex implementation process. A significant improvement was made in [9]. But the interpolation curve is expressed in an implicit form. An even more easily-implemented approach, introduced by [10], involves interpolating in the ambient space E^3 with Bézier spline curve of degree 3 and then projecting the curve onto the smooth surface with ruled surface. The projected curve is of G^1 continuity. Wang et al. [11] presented a new method for up to G^2 interpolation in S^2. The resulting interpolation curve is the intersection of the sphere and the composite conical surface patch. The method provides users with some degree of freedom for interactive control, but still involves surface intersection. Based on normal projection, Wang et al. [12, 13] developed a new method to construct geometrically continuous curves on freeform surface in the form of differential equations. The method overcomes the drawbacks in [8, 11], but the analytical solutions of the equations may not exist.

The other way is to directly construct interpolation curves in S^m. Based on quaternion, [14, 15] constructed spline curves in S^3. However, the methods cannot be generalized to S^m for $m \neq 3$ in an obvious way, since the constructions rely on the group structure of S^3. Shoemake [16] produced the spherical generalizations of de Casteljau's algorithm [17, 18] and then defined spherical Bézier curves. He also gave a method for constructing C^1 spherical Bézier splines for interpolation. Noakes [19] extended the above construction to curves of arbitrary degree in S^m and then introduced two methods for the automatic construction of the control points for generalized Bézier quadratic splines, resulting in spherical splines which are optimal. Working in this amount of generality, Popiel and Noakes [20] solved the problem of interpolation in spheres by constructing C^2 spherical Bézier splines. The method performs well in terms of uniformity of speed and magnitude of (covariant) acceleration. But the construction imposes a strict restriction on the choice of the interpolation points. Sequin et al. [21] presented a blending scheme between circular arcs to produce circle splines that have G^2 continuity. The method is also usable in spheres. But the circle splines have rather complicated transcendental functions.

In this paper, we consider how to construct a spherical interpolation curve with G^2 continuity. In addition, it is important that the resulting interpolation curve exhibits uniform variations in speed and magnitude of acceleration. The idea of our method follows [20], but overcomes the drawbacks of the restriction on the choice of the interpolation points.

The rest of the paper is organized as follows. In Section 2, we review the definition and the properties of spherical Bézier curves in S^m. In Section 3 we compute the initial and final tangent directions and curvature vectors of a spherical Bézier curve of arbitrary degree, and then discuss how to piece the spherical Bézier curve segments together with G^1 and G^2 continuity. In Section 4 we give a method for determining the shape parameters. The method, involving constrained optimization problems, is easy to implement. Some examples are given in Section 5 to demonstrate the advantages of our research. Finally, in Section 6, we make the conclusion.

2. Preliminary knowledge

We use the spherical Bézier curves, which have been introduced in [20], to solve the problem of G^2 interpolation in S^m. For the convenience of the readers, we now review the definition and the main properties of the curves. Most of the notations are adopted from [20].

Denote the Euclidean inner product by $\langle \cdot, \cdot \rangle$. Given two distinct points $p, q \in S^m$ with $d(p,q) < \pi$, where $d(\cdot,\cdot) := \cos^{-1}\langle \cdot, \cdot \rangle$, there exists a unique minimizing geodesic $t \mapsto \varsigma(t,p,q)$, $t \in [0,1]$, joining p to q, and it is well known that

$$\varsigma(t,p,q) = \frac{\sin((1-t)\theta)}{\sin\theta} p + \frac{\sin(t\theta)}{\sin\theta} q, \quad \text{where } \theta := d(p,q). \tag{1}$$

It is easy to verify that

Lemma 1 ([20]) *Given two distinct points $p, q \in S$ with $\theta = d(p,q) < \pi$, then for all $t \in [0,1]$,*

(i) $\langle \varsigma(t,p,q), \dot{\varsigma}(t,p,q) \rangle = 0$;

(ii) $\|\dot{\varsigma}(t,p,q)\| = \theta$;

(iii) $\langle \dot{\varsigma}(0,p,q), q \rangle = \theta \sin\theta$, $\langle \dot{\varsigma}(1,p,q), p \rangle = -\theta \sin\theta$;

(iv) $\langle \dot{\varsigma}(0,p,q), \dot{\varsigma}(1,p,q) \rangle = \theta^2 \cos\theta$.

For any integer $n \geq 1$, denote a sequence of points $x_0, x_1, \ldots, x_n \in S^m$ by x. Set

$$\mathcal{C}^n := \{x : d(x_j, x_{j+1}) < \pi \text{ for all } j = 0, \ldots, n-1\}.$$

Motivated by [16, 22, 23], Popiel and Noakes [20] defined generalized Bézier curves in S^m as follows. Given $x \in \mathcal{C}^n$, a spherical Bézier curve $t \mapsto \beta_n(t;x)$, for $t \in [0,1]$, of degree n with control points $x_0, \ldots, x_n$ takes the form

$$\beta_n(t;x) := \beta_n(t, x_0, \ldots, x_n),$$

where

$$\beta_k(t, x_j, \ldots, x_{j+k}) := \begin{cases} x_j, & \text{if } k = 0, \\ \varsigma(t, \beta_{k-1}(t, x_j, \ldots, x_{j+k-1}), \beta_{k-1}(t, x_{j+1}, \ldots, x_{j+k})), & \text{if } k \geq 1, \end{cases}$$

for all $k = 0, \ldots, n$, all $j = 0, \ldots, n-k$ and all $t \in [0,1]$. The curve $t \mapsto \beta_n(t;x)$ is C^∞. The following properties of the spherical Bézier curves given by [20] are similar to the classical results.

Theorem 1 ([20]) *A spherical Bézier curve $t \mapsto \beta_n(t;x)$ satisfies*

(i) $\beta_n(0;x) = x_0$;

(ii) $\beta_n(1;x) = x_n$;

(iii) $\dot{\beta}_n(0;x) = n\dot{\beta}_1(0, x_0, x_1)$;

(iv) $\dot{\beta}_n(1;x) = n\dot{\beta}_1(1, x_{n-1}, x_n)$.

For $r, m \geq 1$, let $z_0, z_1, \ldots, z_r$ be $r+1$ distinct points on the unit sphere S^m. The general problem of this paper is to construct a geometrically continuous curve in S^m to interpolate the given points $z_0, z_1, \ldots, z_r$.

3. Interpolation in spheres

Theorem 1(i) and (ii) imply that the C^∞ spherical Bézier curve interpolates its first and last control points. So the problem proposed at the end of Section 2 can be solved by using a piecewise spherical Bézier curve. The next key point is to stitch two neighboring spherical Bézier curves together with geometric continuity. Set

$$\widetilde{\mathcal{C}}^n := \{x \in \mathcal{C}^n : \theta_j(x) \neq 0 \text{ for } j = 0, n-1\},$$

where $\theta_j(x) := d(x_j, x_{j+1})$, for all $j = 0, \ldots, n-1$. In order to guarantee the curve to be regular nearby the endpoints, from now on, we only concern on the spherical Bézier curve $\beta_n(t; x)$ with control points $x \in \widetilde{\mathcal{C}}^n$.

3.1. G^1 Interpolation in spheres

Two parameterizations meet with G^1 continuity if and only if they have a common tangent direction [24]. We now adopt this fact to construct G^1 piecewise spherical Bézier curves. The following properties are easily verified.

Theorem 2 *Let $\beta_n(t; x)$ be a spherical Bézier curve. Then the initial and final tangent directions of the curve are $\dot\beta_1(0, x_0, x_1)/\theta_0(x)$ and $\dot\beta_1(1, x_{n-1}, x_n)/\theta_{n-1}(x)$, respectively.*

Let i be an integer such that $1 \le i \le r-1$. For $n \ge 2$, take $x^i, x^{i+1} \in \widetilde{\mathcal{C}}^n$ such that

$$x_0^{i+1} = x_n^i. \tag{2}$$

Let $\beta_n(t; x^i)$ and $\beta_n(t; x^{i+1})$ be two spherical Bézier curves with control points x^i and x^{i+1}, respectively. Then by Theorem 1(i) and (ii), we know that these two curves meet with C^0 at x_n^i.

Furthermore, to achieve G^1 continuity, by Theorem 2, we need

$$\frac{\dot\beta_1(1, x_{n-1}^i, x_n^i)}{\theta_{n-1}(x^i)} = \frac{\dot\beta_1(0, x_0^{i+1}, x_1^{i+1})}{\theta_0(x^{i+1})}.$$

Simplification and rearrangement yield

$$x_1^{i+1} = \cos\tau_1^i x_n^i + \frac{\sin\tau_1^i}{\theta_{n-1}(x^i)}\dot\beta_1(1, x_{n-1}^i, x_n^i), \quad \text{where } \tau_1^i \in (0, \pi). \tag{3}$$

In fact, the shape parameter τ_1^i gives the length of tangent vector $\dot\beta_1(0, x_0^{i+1}, x_1^{i+1})$. More geometrically, condition (3) means that x_1^{i+1} lies in the open semicircle with boundary points x_n^i and $-x_n^i$ and inner point $-x_{n-1}^i$. Therefore, we have the following result.

Theorem 3 *Spherical Bézier curves $\beta_n(t; x^i)$ and $\beta_n(t; x^{i+1})$ meet with G^1 continuity at $\beta_n(1; x^i)$ if and only if we choose the control points satisfying (2) and (3).*

In addition, the given points z_{i-1}, z_i and z_{i+1} are interpolated by setting

$$x_0^k := z_{k-1} \text{ and } x_n^k := z_k, \quad \text{where } k = i, i+1. \tag{4}$$

3.2. G^2 Interpolation in spheres

It has been shown that two parameterizations meet with G^2 continuity if and only if they have common tangent direction and curvature vector [24]. We use this fact to piece two spherical Bézier curves together with G^2 continuity. Given $x \in \widetilde{\mathcal{C}}^n$, for brevity, we write

$$\lambda(x) := \langle x_0, \dot{\beta}_1(0, x_1, x_2)\rangle,$$

$$\mu(x) := \langle x_n, \dot{\beta}_1(1, x_{n-2}, x_{n-1})\rangle,$$

$$r_0(x) := \dot{\beta}_1(0, x_1, x_2) + \lambda(x)/(\theta_0(x)\sin\theta_0(x))\dot{\beta}_1(1, x_0, x_1),$$

$$r_n(x) := -\dot{\beta}_1(1, x_{n-2}, x_{n-1}) + \mu(x)/(\theta_{n-1}(x)\sin\theta_{n-1}(x))\dot{\beta}_1(0, x_{n-1}, x_n).$$

Then by Lemma 1, we have

$$\langle r_n(x), x_{n-1}\rangle = 0, \quad \langle r_n(x), x_n\rangle = 0, \tag{5}$$
$$\|r_n(x)\|^2 = \theta_{n-2}(x)^2 - \frac{\mu^2(x)}{\sin^2\theta_{n-1}(x)}.$$

Define $s : [0,1] \times \widetilde{\mathcal{C}}^n \to \mathbb{R}$ by

$$s(t;x) = \int_0^t \|\dot{\beta}_n(u;x)\|\mathrm{d}u. \tag{6}$$

Then $s(t;x)$ is the arc length of $\beta_n(u;x)$ from 0 to t.

Lemma 2 *Given $x \in \widetilde{\mathcal{C}}^n$, then*

(i) $\dot{s}(0;x) = n\theta_0(x)$;

(ii) $\dot{s}(1;x) = n\theta_{n-1}(x)$;

(iii) $\ddot{s}(0;x) = -n(n-1)(\theta_0(x) + \lambda(x)/\sin\theta_0(x))$;

(iv) $\ddot{s}(1;x) = n(n-1)(\theta_{n-1}(x) - \mu(x)/\sin\theta_{n-1}(x))$.

Proof Part (i) and (ii) can be easily verified by Theorem 1(iii), (iv) and Lemma 1(ii). Differentiating (6) twice with respect to t gives

$$\ddot{s}(t;x) = \langle\dot{\beta}_n(t;x), \ddot{\beta}_n(t;x)\rangle/\|\dot{\beta}_n(t;x)\|.$$

To prove (iii), we express $\dot{\beta}_n(0;x)$ and $\ddot{\beta}_n(0;x)$ by Theorem 1 (iii) and Theorem 2(i) of [20] respectively, and then compute $\langle\dot{\beta}_n(0;x), \ddot{\beta}_n(0;x)\rangle$. The resulting expression for $\ddot{s}(0;x)$ can be simplified to the stated form by Lemma 1. We omit the details. The proof of (iv) is completed similarly. $\square$

We now give the endpoints curvature vectors of a spherical Bézier curve.

Theorem 4 *Let $\beta_n(t;x)$ be a spherical Bézier curve. Then the initial and final curvature vectors of the curve are*

$$(n-1)/(n\theta_0(x)\sin\theta_0(x))r_0(x) - x_0$$

and

$$(n-1)/(n\theta_{n-1}(x)\sin\theta_{n-1}(x))r_n(x) - x_n,$$

respectively.

Proof The second derivative of $\beta_n(t;x)$ with respect to arc length is given by

$$\ddot{\gamma}_n(s;x) = \frac{\ddot{\beta}_n(t;x) - \dot{\gamma}_n(s;x)\ddot{s}(t;x)}{\dot{s}(t;x)^2},$$

where $\gamma_n(s;x)$ is the arc-length parametrization of $\beta_n(t;x)$. Then by Theorem 2 and Lemma 2(i), (iii),

$$\ddot{\gamma}_n(0;x) = (\ddot{\beta}_n(0;x) - \dot{\beta}_1(0,x_0,x_1)/\theta_0(x)n(1-n)(\theta_0(x) + \lambda(x)/\sin\theta_0(x)))/(n\theta_0(x))^2.$$

Since $\beta_n(t;x)$ is a curve in S^m, $\langle \beta_n, \beta_n\rangle = 1$. So $\langle \ddot{\beta}_n, \beta_n\rangle = -\langle \dot{\beta}_n, \dot{\beta}_n\rangle$, and then

$$\ddot{\beta}_n(t;x) = D/du|_{u=t}\dot{\beta}_n(u;x) - \langle \dot{\beta}_n(t;x), \dot{\beta}_n(t;x)\rangle \beta_n(t;x),$$

where $D/du|_{u=t}\dot{\beta}_n(u;x)$ is the covariant acceleration of β_n at $u=t$. So by Corollary 1(i) of [20], Theorem 1(iii) and Lemma 1(ii), the expression of $\ddot{\gamma}_n(0;x)$ can be simplified to the following form:

$$\ddot{\gamma}_n(0;x) = (n-1)/(n\theta_0(x)\sin\theta_0(x))r_0(x) - x_0.$$

The second derivative of $\gamma_n(s;x)$ at $s_1 := s(1;x)$ can be derived in similar fashion:

$$\ddot{\gamma}_n(s_1;x) = (n-1)/(n\theta_{n-1}(x)\sin\theta_{n-1}(x))r_n(x) - x_n.$$

Note that $\ddot{\gamma}_n(0;x)$ and $\ddot{\gamma}_n(s_1;x)$ are curvature vectors of $\beta_n(t;x)$ at endpoints, which complete the proof. $\square$

For each $p \in E^{m+1}$, define, for all $q \in E^{m+1}$,

$$I_p(q) := 2\langle q, p\rangle p - q. \tag{7}$$

Then $I_p(q)$ is a reflection of q about the one-dimensional subspace spanned by p. For $n \geq 3$, take $x^i, x^{i+1} \in \widetilde{\mathcal{C}}^n$. Set

$$\tilde{x}^i_{n-1} := I_{x^i_n}(x^{i+1}_1)$$

and

$$\tilde{x}^i := (x^i_1, \ldots, x^i_{n-2}, \tilde{x}^i_{n-1}, x^i_n).$$

We have the following results.

Lemma 3 *If (2) is satisfied then*

(i) $\dot{\beta}_1(1, x^{i+1}_0, x^{i+1}_1) = \dot{\beta}_1(0, \tilde{x}^i_{n-1}, x^i_n) - 2\theta_0(x^{i+1})\sin\theta_0(x^{i+1})x^i_n$;

(ii) $\dot{\beta}_1(0, x^{i+1}_1, x^{i+1}_2) = 2\lambda(x^{i+1})x^i_n - \dot{\beta}_1(0, \tilde{x}^i_{n-1}, I_{x^i_n}(x^{i+1}_2))$;

(iii) $\lambda(x^{i+1}) = \langle \dot{\beta}_1(0, \tilde{x}^i_{n-1}, I(x^{i+1}_2)), x^i_n\rangle$.

Proof Part (i) follows from (1) and (2) since

$$x^{i+1}_1 = 2\cos\theta_0(x^{i+1})x^i_n - \tilde{x}^i_{n-1}.$$

The following calculation proves (ii):

$$I_{x^i_n}(\dot{\beta}_1(0, x^{i+1}_1, x^{i+1}_2)) = \dot{\beta}_1(0, \tilde{x}^i_{n-1}, I_{x^i_n}(x^{i+1}_2)) = 2\lambda(x^{i+1})x^i_n - \dot{\beta}_1(0, x^{i+1}_1, x^{i+1}_2).$$

Part (iii) is proved by (ii) and (2). $\square$

Once x_0^{i+1} and x_1^{i+1} have been constrained subject to G^1 continuity, the control point x_2^{i+1} can be defined by

$$\dot{\beta}_1(0, \tilde{x}_{n-1}^i, I_{x_n^i}(x_2^{i+1})) = -\frac{\theta_0(x^{i+1})\sin\theta_0(x^{i+1})}{\theta_{n-1}(x^i)\sin\theta_{n-1}(x^i)} r_n(x^i) + \tau_2^i \dot{\beta}_1(0, \tilde{x}_{n-1}^i, x_n^i) \tag{8}$$

with constrain condition

$$\left\| -\frac{\theta_0(x^{i+1})\sin\theta_0(x^{i+1})}{\theta_{n-1}(x^i)\sin\theta_{n-1}(x^i)} r_n(x^i) + \tau_2^i \dot{\beta}_1(0, \tilde{x}_{n-1}^i, x_n^i) \right\| < \pi, \tag{9}$$

to guarantee G^2 continuity for a given shape parameter $\tau_2^i \in \mathbb{R}$. The above inequality can be satisfied by choosing τ_1^i and τ_2^i properly. Then by (1) and (8),

$$x_2^{i+1} = \begin{cases} I_{x_n^i}(\tilde{x}_{n-1}^i), & \text{if } \sigma^i = 0, \\ I_{x_n^i}(\sin\|\sigma^i\|/\|\sigma^i\|\sigma^i + \cos\|\sigma^i\|\tilde{x}_{n-1}^i), & \text{if } \sigma^i \neq 0, \end{cases} \tag{10}$$

where σ^i denotes the right-hand side of (8).

Theorem 5 *Spherical Bézier curves $\beta_n(t; x^i)$ and $\beta_n(t; x^{i+1})$ meet with G^2 continuity at $\beta_n(1; x^i)$ if and only if we choose the control points satisfying (2), (3) and (8).*

Proof Set $\Delta := \ddot{\gamma}_n(s_1; x^i) - \ddot{\gamma}_n(0; x^{i+1})$. Then by Lemma 3,

$$\Delta = \frac{n-1}{n\theta_{n-1}(x^i)\sin\theta_{n-1}(x^i)} r_n(x^i) - \frac{n-1}{n\theta_0(x^{i+1})\sin\theta_0(x^{i+1})} v, \tag{11}$$

where

$$v = -\dot{\beta}_1(0, \tilde{x}_{n-1}^i, I(x_2^{i+1})) + \frac{\langle \dot{\beta}_1(0, \tilde{x}_{n-1}^i, I(x_2^{i+1})), x_n^i \rangle}{\theta_0(x^{i+1})\sin\theta_0(x^{i+1})} \dot{\beta}_1(0, \tilde{x}_{n-1}^i, x_n^i).$$

Suppose (2), (3) and (8) hold. Take the inner product of (8) with x_n^i, and then by (5),

$$\langle \dot{\beta}_1(0, \tilde{x}_{n-1}^i, I(x_2^{i+1})), x_n^i \rangle = \tau_2^i \theta_0(x^{i+1}) \sin\theta_0(x^{i+1}). \tag{12}$$

So by (8) and (12),

$$v = \frac{\theta_0(x^{i+1})\sin\theta_0(x^{i+1})}{\theta_{n-1}(x^i)\sin\theta_{n-1}(x^i)} r_n(x^i).$$

Therefore $\Delta = 0$.

Conversely, if $\beta_n(t; x^i)$ and $\beta_n(t; x^{i+1})$ meet with G^2 continuity at $\beta_n(1; x^i)$ (namely $\Delta = 0$), then, in particular, (2) and (3) hold. So by (11)

$$\dot{\beta}_1(0, \tilde{x}_{n-1}^i, I(x_2^{i+1})) = -\frac{\theta_0(x^{i+1})\sin\theta_0(x^{i+1})}{\theta_{n-1}(x^i)\sin\theta_{n-1}(x^i)} r_n(x^i) + \frac{\langle \dot{\beta}_1(0, \tilde{x}_{n-1}^i, I(x_2^{i+1})), x_n^i \rangle}{\theta_0(x^{i+1})\sin\theta_0(x^{i+1})} \dot{\beta}_1(0, \tilde{x}_{n-1}^i, x_n^i).$$

Therefore (8) is satisfied. □

Similarly, the given points z_{i-1}, z_i and z_{i+1} are interpolated if (4) is satisfied.

4. Shape parameters

Let $\beta_n(t; x^i)$ and $\beta_n(t; x^{i+1})$ be two spherical Bézier cubic (namely with $n = 3$) curves meeting with G^2 continuity at $\beta_n(1; x^i)$. Suppose that the initial velocity and covariant acceleration of $\beta_n(t; x^i)$ are given, then the remaining control points x_1^{i+1} and x_2^{i+1} are uniquely determined by the shape parameters τ_1^i and τ_2^i. We now give a method for choosing τ_1^i and τ_2^i in an optimal way, such that the resulting piecewise curve performs well in terms of uniformity of speed and magnitude of covariant acceleration.

Define, for all $1 \le i \le r-1$ and all $\tau_1^i \in (0,\pi)$,

$$E_1(\tau_1^i) := (\|\dot{\beta}_n(0;x^{i+1})\| - \|\dot{\beta}_n(0;x^i)\|)^2 + (\|\dot{\beta}_n(0;x^{i+1})\| - \|\dot{\beta}_n(1;x^i)\|)^2.$$

Then $E_1(\tau_1^i)$ is the sum of squares of differences between the initial speed of the $(i+1)$-th curve segment and the initial (final) speed of the i-th curve segment. By Theorem 1(iii), (iv) and Lemma 1(ii),

$$E_1(\tau_1^i) = 2n^2(\tau_1^i)^2 - 2n^2(\theta_0(x^i)+\theta_{n-1}(x^i))\tau_1^i + n^2(\theta_0(x^i)^2 + \theta_{n-1}(x^i)^2).$$

In the absence of any other conditions additional to τ_1^i, it is easy to verify that E_1 achieves its minimum at

$$\tau_1^i = (\theta_0(x^i)+\theta_{n-1}(x^i))/2.$$

However, by Lemma 1(ii), (3), (5) and (7),

$$\|\dot{\beta}_1(0,\tilde{x}^i_{n-1}, I_{x^i_n}(x_2^{i+1}))\|^2 = \frac{\theta_0(x^{i+1})^2 \sin^2\theta_0(x^{i+1})}{\theta_{n-1}(x^i)^2\sin^2\theta_{n-1}(x^i)}\|r_n(x^i)\|^2 + (\tau_2^i)^2\theta_0(x^{i+1})^2. \tag{13}$$

To have (9) satisfied, we need

$$\frac{\tau_1^i \sin\tau_1^i}{\theta_{n-1}(x^i)\sin\theta_{n-1}(x^i)}\|r_n(x^i)\| \le \pi - \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. Therefore, τ_1^i can be determined by the following constrained optimization problem

$$\begin{array}{ll} \min_{\tau_1^i} & E_1(\tau_1^i) \\ \text{s.t.} & \tau_1^i \sin\tau_1^i \|r_n(x^i)\|/(\theta_{n-1}(x^i)\sin\theta_{n-1}(x^i)) \le \pi-\varepsilon, \\ & \varepsilon \le \tau_1^i \le \pi-\varepsilon. \end{array} \tag{14}$$

Furthermore, define $E_2 : \mathbb{R} \to [0,+\infty)$ by

$$E_2(\tau_2^i) = \|D/dt|_{t=0}\dot{\beta}_n(t;x^{i+1})\|^2.$$

Then $E_2(\tau_2^i)$ is the square of magnitude of initial covariant acceleration of the $(i+1)$-th curve segment. By Corollary 1(i) of [20] and Lemma 1,

$$E_2(\tau_2^i) = \frac{n^2(n-1)^2}{\sin^2\theta_0(x^{i+1})}((\|\dot{\beta}_1(0,x_1^{i+1},x_2^{i+1})\|^2 + \sin^2\theta_0(x^{i+1}))\theta_0(x^{i+1})^2 + 2\theta_0(x^{i+1})\sin\theta_0(x^{i+1})\lambda(x^{i+1}) + \Big(1 - \frac{\theta_0(x^{i+1})^2}{\sin^2\theta_0(x^{i+1})}\Big)\lambda^2(x^{i+1})).$$

Lemma 4 *If (2), (3) and (8) are satisfied, then*

(i) $\lambda(x^{i+1}) = \tau_2^i\theta_0(x^{i+1})\sin\theta_0(x^{i+1})$;

(ii) $\|\dot{\beta}_1(0,x_1^{i+1},x_2^{i+1})\|^2 = \frac{\theta_0(x^{i+1})^2\sin^2\theta_0(x^{i+1})}{\theta_{n-1}(x^i)^2\sin^2\theta_{n-1}(x^i)}\|r_n(x^i)\|^2 + (\tau_2^i)^2\theta_0(x^{i+1})^2$.

Proof Part (i) follows from Lemma 3(iii) and (5) since

$$\theta_{n-1}(\tilde{x}^i) = \theta_0(x^{i+1}).$$

Lemma 3(ii) and (iii) give

$$\|\dot{\beta}_1(0, x_1^{i+1}, x_2^{i+1})\|^2 = \|\dot{\beta}_1(0, \tilde{x}_{n-1}^i, I_{x_n^i}(x_2^{i+1}))\|^2.$$

Then Part (ii) is proved by (13). □

Computing $\|E_2(\tau_2^i)\|^2$ by Lemma 4, we have

$$\|E_2(\tau_2^i)\|^2 = n^2(n-1)^2\theta_0(x^{i+1})^2((\tau_2^i)^2 + 2\tau_2^i + 1 + \frac{\theta_0(x^{i+1})^2}{\theta_{n-1}(x^i)^2 \sin^2\theta_{n-1}(x^i)}\|r_n(x^i)\|^2).$$

As noted in Section 3, x_2^{i+1} is well defined by (8) if (9) holds. Then by (13), τ_2^i can be determined by the following constrained optimization problem

$$\begin{array}{ll} \min_{\tau_2^i} & E_2(\tau_2^i) \\ \text{s.t.} & \frac{\theta_0(x^{i+1})^2 \sin^2\theta_0(x^{i+1})}{\theta_{n-1}(x^i)^2 \sin^2\theta_{n-1}(x^i)}\|r_n(x^i)\|^2 + (\tau_2^i)^2\theta_0(x^{i+1})^2 \le \pi^2 - \varepsilon. \end{array} \tag{15}$$

Constrained optimization problem (14) and (15) can be solved by the method of Lagrange multiplier.

5. Examples

First of all, we describe how to solve the interpolation problem mentioned at the end of Section 2 by using a piecewise spherical Bézier cubic curve. The algorithm can be stated as follows:

1) Set $x_0^1 := z_0$ and $x_3^i := z_i$ for $i = 1, \dots, r$;
2) Define x_1^1 and x_2^1 by the given initial velocity and covariant acceleration [20];
3) For $i = 1$, determine τ_1^i by the constrained optimization problem (14);
4) Define x_1^{i+1} by (3);
5) Determine τ_2^i by the constrained optimization problem (15);
6) Define x_2^{i+1} by (10);
7) Replace i by $i + 1$ and return to Step 3).

We now interpolate the given points by using a G^2 piecewise spherical Bézier cubic curve and compare it with other constructions.

Example 1 Consider the interpolation points given in [20]

$$\begin{aligned} z_0 &= (1.0000000, 0.0000000, 0.0000000), \\ z_1 &= (-0.995244, 0.0497622, 0.0837469), \\ z_2 &= (0.9909890, 0.0990989, 0.0901104), \\ z_3 &= (-0.988840, 0.1483260, 0.0139545), \\ z_4 &= (0.9778890, 0.1955920, -0.0740123). \end{aligned}$$

The control points and shape parameters defined by the algorithm above are shown in Table 1. The resulting G^2 piecewise spherical Bézier cubic curve constructed by the presented method is shown in Figure 1. Figure 2 shows the C^2 piecewise spherical Bézier cubic curve which is obtained by the method in [20]. Figure 3 shows the G^2 piecewise spherical curve constructed by the method in [13]. The interpolation points z_i are labeled by i in the figures. The three curves are distinct in appearance. Obviously, our curve performs better than the other two in uniform speed and magnitude of covariant acceleration.

We now consider another example involving the data in which the construction in [20] is unavailable, but our method does work.

Example 2 Consider the interpolation points $z_0, \dots, z_4$ given in Example 1, but suppose that the other two control points of the first curve segment $\beta_3(t; x^1)$ are

$$x_1^1 = (0.9567641, 0.1485841, 0.2500584),$$
$$x_2^1 = (0.7726255, 0.3243098, 0.5457945).$$

Noting that the inequality (16) of [20] does not hold for the above control points, and the C^2 piecewise curve interpolating the desired data cannot be constructed by [20]. Using our presented algorithm, we can still construct a G^2 piecewise spherical Bézier cubic curve to interpolate the given points. Table 2 shows the control points and the shape parameters. Figure 4 shows the resulting curve constructed by the presented method. Figure 5 shows the G^2 piecewise spherical curve constructed by the method in [13]. The interpolation points z_i are labeled by i in the figures. Obviously, our curve still performs better than the one in [13].

i	1	2	3	4
x_0^i	(1,0,0)	(-0.9952,0.0498,0.0837)	(0.9910,0.0991,0.0901)	(-0.9888,0.1483,0.0140)
x_1^i	(0.4769,0.0686,0.8763)	(-0.4538,-0.0491,-0.8898)	(0.3816,0.1845,0.9057)	(-0.6115,-0.0695,-0.7882)
x_2^i	(-0.1823,0.0837,0.9797)	(0.6559,-0.0707,-0.7515)	(-0.6407,0.2417,0.7287)	(0.2873,-0.1564,-0.9450)
x_3^i	(-0.9952,0.0498,0.0837)	(0.9910,0.0991,0.0901)	(-0.9888,0.1483,0.0140)	(0.9779,0.1956,-0.0740)
τ_1^i	1.1867	1.0723	0.9480	
τ_2^i	-1.0000	-1.0000	-1.0000	

Table 1 Control points and shape parameters of spherical Bézier curve segments

i	1	2	3	4
x_0^i	(1.0000,0,0)	(-0.9952,0.0498,0.0837)	(0.9910,0.0991,0.0901)	(-0.9888,0.1483,0.0140)
x_1^i	(0.9568,0.1486,0.2501)	(-0.3360,-0.4811,-0.8097)	(0.4067,0.5162,0.7538)	(-0.6622,-0.3807,-0.6454)
x_2^i	(0.7726,0.3243,0.5458)	(0.8321,-0.2833,-0.4768)	(-0.5709,0.5358,0.6221)	(0.2106,-0.5812,-0.7861)
x_3^i	(-0.9952,0.0498,0.0837)	(0.9910,0.0991,0.0901)	(-0.9888,0.1483,0.0140)	(0.9779,0.1956,-0.0740)
τ_1^i	1.3257	1.0215	0.9405	
τ_2^i	-1.0000	-1.0000	-1.0000	

Table 2 Control points and shape parameters of spherical Bézier curve segments

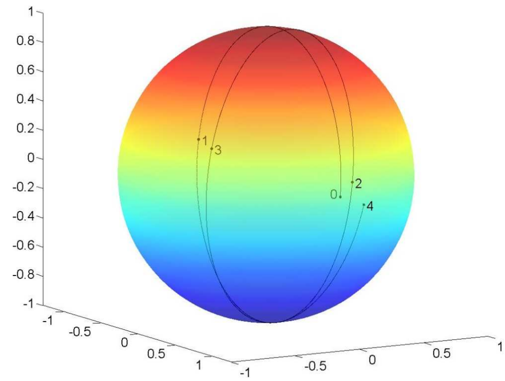

Figure 1 Our result: G^2 piecewise spherical Bézier curve

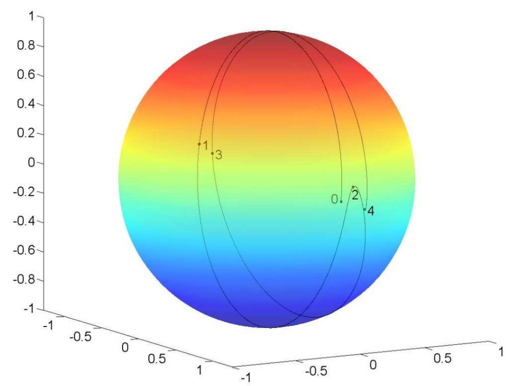

Figure 2 C^2 piecewise spherical Bézier curve constructed by the method in [20]

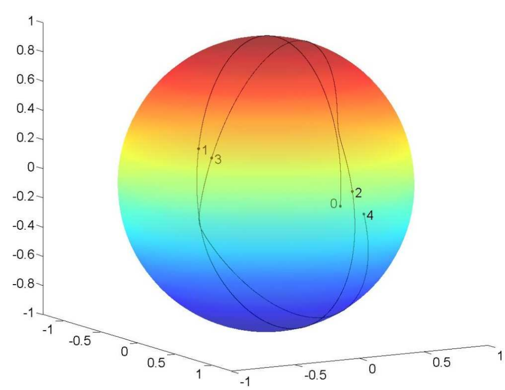

Figure 3 G^2 piecewise spherical curve constructed by the method in [13]

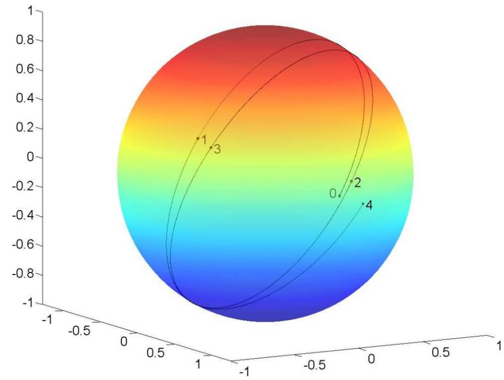

Figure 4 Our result: G^2 piecewise spherical Bézier curve

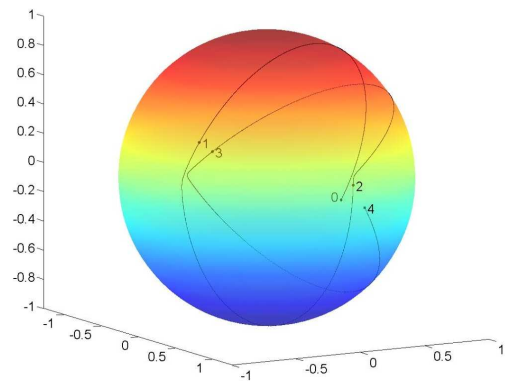

Figure 5 G^2 piecewise spherical curve constructed by the method in [13]

6. Conclusion

As we know, the use of parametric continuity disallows many parameterizations which may generate geometrically smooth curves. The nth-geometric continuity or G^n, a relaxed form of C^n continuity, is an intrinsic property of the curve. It is independent of parameterizations and thus has widespread application in CAGD. Therefore, it is necessary and exigent to construct spherical interpolation curves which are geometrically continuous. In this paper, geometrically continuous interpolation in spheres is developed. The basic idea is to construct spherical Bézier curves to interpolate every two adjacent points and then piece the curves together with G^1 and G^2 continuity. The construction introduces two degrees of freedom, called shape parameters, for every curve segment. We also give two criteria for choosing the shape parameters. The criteria are natural performance measurements and the involving constrained optimization problems can be solved by the method of Lagrange multiplier. Further research should be focused on higher order geometrically continuous interpolation in spheres.

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