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Entire Function Sharing Small Function with Its Difference Operators or Shifts

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Abstract In this paper, we give some interesting results concerning the entire function $f(z)$ sharing a small function a CM with its difference operators or shifts.

Keywords Brück conjecture; shift; difference; shared value.

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1. Introduction and main results

In this paper, we adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [1–4]. We denote the exponent of convergence of zeros of $f(z)$ by $\lambda(f)$ which is defined as follows

$$\lambda(f)=\overline{\lim_{r\to\infty}}\frac{\log N(r,\frac{1}{f})}{\log r}.$$

In addition, for any given nonconstant meromorphic function $f(z)$, we denote by $S(r,f)$ any quantity satisfying

$$\lim_{r\to\infty}\frac{S(r,f)}{T(r,f)}=0,\quad r\notin E,$$

where $E\subset(0,\infty)$ is of finite logarithmic measure. A meromorphic function $a(z)$ is said to be a small function of $f(z)$ if $T(r,a)=S(r,f)$. In addition, we say that two meromorphic functions $f(z)$ and $g(z)$ share a small function a CM, provided that $f(z)-a$ and $g(z)-a$ have the same zeros counting multiplicities. For a fixed, nonzero complex constant η, we define difference operators as

$$\Delta_\eta f(z)=f(z+\eta)-f(z)\text{ and }\Delta_\eta^n f(z)=\Delta_\eta^{n-1}(\Delta_\eta f(z)),\quad n\in\mathbb{N},\ n\geq 2.$$

In particular, we use a general difference notation $\Delta_\eta^n f(z)=\Delta^n f(z)$ for $\eta=1$.

In 1977, Rubel and Yang started to consider the uniqueness of meromorphic functions sharing values with their derivatives in [5]. Here we recall a well-known conjecture by Brück [6].

Conjecture ([6]) *Let $f(z)$ be a nonconstant entire function such that $\rho_2(f)<\infty$ and $\rho_2(f)\notin\mathbb{N}$.*

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If $f(z)$ and $f'(z)$ share a finite value a CM, then

$$\frac{f'(z)-a}{f(z)-a}=c,$$

where c is a nonzero constant and $\rho_2(f)$ is the hyper-order of $f(z)$ which is defined by

$$\rho_2(f)=\varlimsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

This conjecture has been well studied, although it remains open in full generality. For the case that $a=0$ and that $N(r,\frac{1}{f'})=S(r,f)$, the conjecture had been proved by Brück himself [6]. In [7], Gundersen and Yang proved the above conjecture is true, provided that $f(z)$ is of finite order. Moreover, Chen and Shon [8] proved that the conjecture still holds for the case that $\rho_2(f)<\frac{1}{2}$. Recently, Heittokangas et al. [9] proved a shifted analogue of Brück conjecture as the following Theorem A.

Theorem A ([9]) *Let $f(z)$ be a meromorphic function of order of growth $\rho(f)<2$, and let $\eta\in\mathbb{C}$. If $f(z)$ and $f(z+\eta)$ share the values $a\in\mathbb{C}$ and ∞ CM, then*

$$\frac{f(z+\eta)-a}{f(z)-a}=\tau$$

for some constant τ.

Then Li and Gao [10] proved the following result.

Theorem B ([10]) *Let $f(z)$ be a non-periodic transcendental entire function of finite order $\rho(f)<\infty$. If $f(z)$ and $\Delta_\eta^n f(z)$ share a nonzero finite value a CM, then $1\le\rho(f)\le\lambda(f-a)+1$, that is, $f(z)$ is of the form*

$$f(z)=P(z)e^{Q(z)}+a,$$

where $P(z)$ is an entire function such that $\rho(P)=\lambda(f-a)$ and $Q(z)$ is a polynomial such that $\deg(Q)\le\rho(P)+1$.

A new question is: What happens if the entire function $f(z)$ shares a small function $a(z)$ with its difference operator $\Delta_\eta^n f(z)$ or shift $f(z+\eta)$? Considering this question, we improve Theorems A and B by the following results.

Theorem 1.1 *Let $f(z)$ be a transcendental entire function of finite order and $a(z)$ be an entire small function of $f(z)$ such that $\rho(a)<\rho(f)$. If $f(z)$ and $\Delta_\eta^n f(z)$ share the entire small function a CM, then $\rho(f)\ge 1$. What's more, if $\rho(\Delta_\eta^n a-a)<1$, then we have $1\le\rho(f)\le\lambda(f-a)+1$.*

Example 1.1 (1) Let

$$f(z)=e^{-z\ln 2}+\frac{3}{2}z+\frac{3}{2},\quad a(z)=\frac{z}{2}+\frac{3}{2}.$$

Then $\rho(f)=1>0=\rho(a)$, and the functions $f(z)$ and $\Delta f(z)$ share the small function a CM;

(2) Let

$$f(z)=2e^{z}+(e-2),\quad a(z)\equiv e-1.$$

Then $\rho(f)=1>0=\rho(a)$, and the functions $f(z)$ and $\Delta f(z)$ share the constant function a CM;

(3) Let

$$f(z) = 2e^{z\ln 3}h(z) + e^{\frac{\ln 2}{\eta}z}, \quad a(z) = e^{\frac{\ln 2}{\eta}z}.$$

If $h(z)$ is a periodic entire function with period η such that $1 < \rho(h) < \infty$, then $\rho(f) = \rho(h) > 1 = \rho(a)$, and the functions $f(z)$ and $\Delta_\eta f(z)$ share the small function a CM. Here we note that for any $1 \le \sigma < \infty$, there exists a prime periodic entire function $w(z)$ such that $\rho(w) = \sigma$ by Theorem 1 in [11]. This implies the existence of $h(z)$.

Theorem 1.2 *Let $f(z)$ be a transcendental entire function of finite order such that $\Delta_\eta f(z) \not\equiv 0$, and let $a(z)$ be an entire small function of $f(z)$ such that $\rho(a) < \rho(f)$. If $f(z)$ and $f(z+\eta)$ share the entire small function a CM, then $\rho(f) \ge 1$. What's more, if $a(z)$ is a periodic function with period η, especially, a constant function, then $f(z_0 + k\eta) = a(z_0)$ holds for all $k \in \mathbb{Z}$ provided that $f(z_0) = a(z_0)$.*

Example 1.2 (1) Let

$$f(z) = e^{z^2}\sin z + \cos z, \quad a(z) = \cos z, \quad \eta = 2\pi.$$

Then $\rho(f) = 2 > 1 = \rho(a)$, $\Delta_\eta f(z) \not\equiv 0$ and the functions $f(z)$ and $f(z+\eta)$ share the entire small function a CM;

(2) Let

$$f(z) = 3e^z + 2, \quad a(z) \equiv 2.$$

Then $\rho(f) = 1 > 0 = \rho(a)$, $\Delta f(z) \not\equiv 0$ and the functions $f(z)$ and $f(z+1)$ share the constant function a CM.

Remark Theorems 1.1 and 1.2 improve Theorems A and B. Our method in the proof of them is quite different from that in [10] and seems more simple.

2. Proof of Theorem 1.1

Lemma 2.1 ([12]) *Let $f(z)$ be a meromorphic function with $\rho(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E \subset [0, +\infty)$ with finite linear measure $mE < \infty$, such that for all z satisfying $|z| = r \notin [0,1] \cup E$, and r sufficiently large,*

$$\exp\{-r^{\alpha+\varepsilon}\} \le |f(z)| \le \exp\{r^{\alpha+\varepsilon}\}.$$

Lemma 2.2 ([13]) *Let $f(z)$ be a meromorphic function with finite order $\rho(f) = \rho < 1$. Then for any given $\varepsilon > 0$, and integers $0 \le j < k$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, so that for all z satisfying $|z| = r \notin E \cup [0,1]$, we have*

$$\Big|\frac{\Delta^k f(z)}{\Delta^j f(z)}\Big| \le |z|^{(k-j)(\rho-1)+\varepsilon}.$$

We need the following observation.

Lemma 2.3 *Let*

$$P(z) = p_n z^n + p_{n-1}z^{n-1} + \cdots + p_0, \quad Q(z) = q_n z^n + q_{n-1}z^{n-1} + \cdots + q_0,$$

where n is a positive integer, $p_n = \alpha e^{i\theta}$, $q_n = \beta e^{i\varphi}$, $\alpha \geq \beta > 0$, $\theta, \varphi \in [-\pi, \pi)$. If $p_n \neq q_n$, then for any given $\varepsilon > 0$, there exists some $r_0 > 1$, such that for all $z = re^{-i\frac{\theta}{n}}$ satisfying $r \geq r_0$, we have

$$\mathrm{Re}\{P(re^{-i\frac{\theta}{n}})\} > \alpha(1-\varepsilon)r^n$$

and

$$\mathrm{Re}\{P(re^{-i\frac{\theta}{n}}) - Q(re^{-i\frac{\theta}{n}})\} > [\alpha - \beta\cos(\theta-\varphi)](1-\varepsilon)r^n.$$

Proof The first assertion holds because

$$\mathrm{Re}\{p_n(re^{-i\frac{\theta}{n}})^n\} = \alpha r^n$$

and for sufficiently large r,

$$|p_{n-1}z^{n-1}| + \cdots + |p_0| = o(r^n).$$

Next we prove the second assertion. Since $\alpha \geq \beta > 0$ and $\cos(\theta - \varphi) = 1$ if and only if $\theta = \varphi$, we see that, if $p_n \neq q_n$,

$$\alpha - \beta\cos(\theta - \varphi) > 0.$$

What's more, we have

$$\mathrm{Re}\{p_n(re^{-i\frac{\theta}{n}})^n - q_n(re^{-i\frac{\theta}{n}})^n\} = [\alpha - \beta\cos(\theta-\varphi)]r^n.$$

On the other hand, for $z = re^{i\psi}$, $|z| = r$, $\psi \in [-\pi, \pi)$, we have, as r sufficiently large,

$$|p_{n-1}z^{n-1}| + \cdots + |p_0| + |q_{n-1}z^{n-1}| + \cdots + |q_0| = o(r^n).$$

Now we can easily find that our conclusion is true. $\square$

The following Lemma was proved by Nevanlinna in [14], and can be seen in [4], Theorem 1.50.

Lemma 2.4 ([4, 14]) *Let $f_j(z)$ $(j = 1, 2, \ldots, n, n \geq 2)$ be meromorphic functions such that*

(1) $\sum_{j=1}^{n} C_j f_j(z) \equiv 0$, where C_j $(j = 1, 2, \ldots, n)$ are all constants;

(2) $f_j(z) \not\equiv 0$ $(j = 1, 2, \ldots, n)$ and $\frac{f_j}{f_k}$ are not constant functions for $1 \leq j < k \leq n$;

(3) $\sum_{j=1}^{n}(N(r, f_j) + N(r, \frac{1}{f_j})) = o(\tau(r))$, $(r \to \infty, r \notin E)$, where E is an exceptional set of finite linear measure, $\tau(r) = \min_{1\leq j<k\leq n}\{T(r, \frac{f_j}{f_k})\}$.

Then $C_j = 0$ $(j = 1, 2, \ldots, n)$.

Proof of Theorem 1.1 Without loss of generality, it may be assumed that $\eta = 1$. Denote $g(z) = f(z) - a(z)$, then $\rho(g) = \rho(f) = \rho$ and

$$\Delta^n f(z) = \Delta^n g(z) + \Delta^n a(z) = \sum_{j=0}^{n}(-1)^{n-j}C_n^j g(z+j) + \Delta^n a(z),$$

where $C_n^0 = C_n^n = 1, C_n^1 = C_n^{n-1} = n, C_n^2 = C_n^{n-2} = n(n-1)/2, \ldots$, are non-zero integers.

It follows from the assumption that

$$\frac{\Delta^n f(z) - a(z)}{f(z) - a(z)} = \frac{\Delta^n g(z) + \Delta^n a(z) - a(z)}{g(z)}$$

$$= \frac{\sum_{j=0}^{n}(-1)^{n-j}C_n^j g(z+j)+b(z)}{g(z)} = e^{P(z)}, \tag{2.1}$$

where $b(z)=\Delta^n a(z)-a(z)$ is an entire small function of $f(z)$ such that $\sigma=\rho(b)\le\rho(a)<\rho$, and $P(z)$ is an entire function satisfying

$$\rho(e^P)\le\rho(f)=\rho<\infty.$$

Therefore, we see that $P(z)$ is a polynomial with $0\le d=\deg(P)\le\rho$. Now set

$$P(z)=p_d z^d+p_{d-1}z^{d-1}+\cdots+p_0,$$

where $p_d\ne 0, p_{d-1},\dots,p_0$ are constants, $p_d=\alpha_d e^{i\theta_d}$, $\alpha_d>0$, $\theta_d\in[-\pi,\pi)$. Next, we divide the proof into two steps.

Step 1. We prove that $\rho\ge 1$. Otherwise, we have $\rho<1$ and hence $P(z)\equiv C\in\mathbb{C}$. By Lemma 2.1, for any given $\varepsilon_1(0<\varepsilon_1<\min\{\frac{\rho-\sigma}{3},\frac{1-\rho}{2}\})$, there exists a set $E_1\subset[0,+\infty)$ with finite linear measure, such that for all z satisfying $|z|=r\notin[0,1]\cup E_1$, and r sufficiently large,

$$\exp\{-r^{\sigma+\varepsilon_1}\}\le|b(z)|\le\exp\{r^{\sigma+\varepsilon_1}\}. \tag{2.2}$$

By Lemma 2.2, for the given ε_1, there exists a set $E_2\subset(1,\infty)$ of finite logarithmic measure, so that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, we have

$$\Big|\frac{\Delta^n g(z)}{g(z)}\Big|\le|z|^{n(\rho-1)+\varepsilon_1}. \tag{2.3}$$

Choose an infinite sequence of points $\{z_k=r_k e^{i\theta_k}\}$ such that

$$|g(z_k)|=M(r_k,g)\ge\exp\{r_k^{\rho-\varepsilon_1}\},\quad r_k\notin E_1\cup E_2. \tag{2.4}$$

(2.1)–(2.4) give a contradiction that

$$|e^C|\le\Big|\frac{\Delta^n g(z_k)}{g(z_k)}\Big|+\frac{|b(z_k)|}{M(r_k,g)}\le {r_k}^{n(\rho-1)+\varepsilon_1}+o(1)=o(1).$$

Step 2. We prove that $\rho\le\lambda(f-a)+1$, if $\rho(\Delta^n a-a)<1$. Otherwise, we have $\rho>\lambda(f-a)+1$ and hence $\rho(g)>\lambda(g)+1$. It follows from the Hadamard factorization theorem that,

$$g(z)=h(z)e^{Q(z)},$$

where $Q(z)$ is a polynomial such that (we will need this special form to simplify our proof in the following)

$$Q(z)=-(q_l z^l+q_{l-1}z^{l-1}+\cdots+q_0),$$

where $q_l\ne 0,q_{l-1},\dots,q_0$ are constants, $q_l=\beta_l e^{i\varphi_l}$, $\beta_l>0$, $\varphi_l\in[-\pi,\pi)$, and $h(z)$ is an entire function satisfying $\rho(h)=\lambda(g)<\rho-1=l-1$.

Rewrite (2.1) as

$$\frac{b(z)}{h(z)e^{Q(z)}}=e^{P(z)}-\frac{\sum_{j=0}^{n}(-1)^{n-j}C_n^j h(z+j)e^{Q(z+j)}}{h(z)e^{Q(z)}}.$$

Observe that for each $j \in \{0, \ldots, n\}$, $\deg(Q(z+j) - Q(z)) = l - 1$, and $\rho(\frac{b(z)}{h(z)e^{Q(z)}}) = l$, then by the equation above, we can easily get $l \le d$. Thus we have $d = l$.

We claim that $p_d = q_d$. Otherwise, $p_d \ne q_d$, and we may assume that $\alpha_d \ge \beta_d > 0$. In what follows, set $Q_j^*(z) = Q(z+j) + q_d z^d$, $P^*(z) = P(z) - p_d z^d$. Then we obtain from (2.1) that

$$e^{P(z)} - \frac{b(z)e^{-Q(z)}}{h(z)} = \frac{\sum\limits_{j=0}^{n} (-1)^{n-j} C_n^j h(z+j) e^{Q_j^*(z)}}{h(z)e^{Q_0^*(z)}}. \tag{2.5}$$

Set $b_1(z) = b(z)$, $b_2(z) = h(z)$, $\sigma_j = \rho(b_j)$ $(j = 1, 2)$, then by Lemma 2.1 again, for any given $\varepsilon_2 (0 < \varepsilon_2 < \min\{\frac{\rho-\sigma_1}{2}, \frac{\rho-\sigma_2}{2}, \frac{1}{2}\})$, there exists a set $E_3 \subset [0, +\infty)$ with finite linear measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$, and r sufficiently large,

$$\exp\{-r^{\sigma_j+\varepsilon_2}\} \le |b_j(z)| \le \exp\{r^{\sigma_j+\varepsilon_2}\}. \tag{2.6}$$

By Lemma 2.3, we see that, for sufficiently large r,

$$\begin{aligned} \Big|\frac{e^{-Q(re^{-i\frac{\theta_d}{d}})}}{e^{P(re^{-i\frac{\theta_d}{d}})}}\Big| &= \exp\{-\{\mathrm{Re}\{P(re^{-i\frac{\theta_d}{d}}) - [-Q(re^{-i\frac{\theta_d}{d}})]\}\}\} \\ &< \exp\{-[\alpha_d - \beta_d\cos(\theta_d - \varphi_d)](1-\varepsilon_2)r^d\}. \end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we can easily get that

$$\begin{aligned} \Big|\frac{b(re^{-i\frac{\theta_d}{d}})e^{-Q(re^{-i\frac{\theta_d}{d}})}}{h(re^{-i\frac{\theta_d}{d}})e^{P(re^{-i\frac{\theta_d}{d}})}}\Big| &< \exp\{-[\alpha_d - \beta_d\cos(\theta_d - \varphi_d)](1-\varepsilon_2)r^d + r^{\sigma_1+\varepsilon_2} + r^{\sigma_2+\varepsilon_2}\} \\ &< \exp\{-[\alpha_d - \beta_d\cos(\theta_d - \varphi_d)](1-\varepsilon_2)r^d + 2r^{d-\varepsilon_2}\}, \end{aligned} \tag{2.8}$$

as $r \notin [0, 1] \cup E_3$, and r sufficiently large.

Since $p_d \ne q_d$ and $\alpha_d \ge \beta_d > 0$, we have $\alpha_d - \beta_d\cos(\theta_d - \varphi_d) > 0$. It now follows by (2.8) that

$$\Big|\frac{b(re^{-i\frac{\theta_d}{d}})e^{-Q(re^{-i\frac{\theta_d}{d}})}}{h(re^{-i\frac{\theta_d}{d}})}\Big| = o(|e^{P(re^{-i\frac{\theta_d}{d}})}|), \tag{2.9}$$

as $r \notin [0, 1] \cup E_3$, and $r \to \infty$.

Applying Lemma 2.3 again, with (2.5) and (2.9), we have

$$\begin{aligned} \frac{1}{2}\exp\{(1-\varepsilon_2)\alpha_d r^d\} \le \frac{1}{2}|e^{P(re^{-i\frac{\theta_d}{d}})}| &< |e^{P(re^{-i\frac{\theta_d}{d}})}| - \Big|\frac{b(re^{-i\frac{\theta_d}{d}})e^{-Q(re^{-i\frac{\theta_d}{d}})}}{h(re^{-i\frac{\theta_d}{d}})}\Big| \\ &\le \Big|\frac{\sum\limits_{j=0}^{n} (-1)^{n-j} C_n^j h(re^{-i\frac{\theta_d}{d}} + j) e^{Q_j^*(re^{-i\frac{\theta_d}{d}})}}{h(re^{-i\frac{\theta_d}{d}})e^{Q_0^*(re^{-i\frac{\theta_d}{d}})}}\Big| \\ &< \exp\{r^{d-\frac{1}{2}}\}, \end{aligned}$$

as $r \notin [0, 1] \cup E_3$, and $r \to \infty$, which is impossible.

Now we have $p_d = q_d$. Then we obtain from (2.1) that

$$\frac{\sum_{j=0}^{n}(-1)^{n-j}C_n^j h(z+j)e^{Q_j^*(z)}}{h(z)e^{Q_0^*(z)}} = e^{p_d z^d}\left(e^{P^*(z)} - \frac{b(z)}{h(z)e^{Q_0^*(z)}}\right),$$

which infers that

$$e^{P^*(z)} - \frac{b(z)}{h(z)e^{Q_0^*(z)}} \equiv 0, \tag{2.10}$$

and hence we have

$$\sum_{j=0}^{n}(-1)^{n-j}C_n^j h(z+j)e^{Q_j^*(z)} = 0 \Longleftrightarrow \sum_{j=0}^{n}(-1)^{n-j}C_n^j g(z+j) = 0. \tag{2.11}$$

Then from (2.10), and $\rho(b) = \rho(\Delta^n a - a) < 1$, we see that $\rho(h) = \lambda(h) \le \lambda(b) \le \rho(b) < 1$. Since $d = l = \rho(g) > \lambda(g) + 1$, we have $d \ge 2$. Therefore, for $0 \le j < k \le n$, $\deg(Q_j^*(z) - Q_k^*(z)) \ge 1$. Applying Lemma 2.4 to (2.11), we get a contradiction that $(-1)^{n-j}C_n^j = 0,\ j = 0, 1, \ldots, n$. Hence, we prove that $1 \le \rho(f) \le \lambda(f-a) + 1$. $\square$

3. Proof of Theorem 1.2

Lemma 3.1 ([15]) *Let g be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \quad \frac{g(z+\eta)}{g(z)} \to 1 \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus E,$$

uniformly in η for $|\eta| \le h$. Further, E may be chosen so that for large z not in E the function g has no zeros or poles in $|\zeta - z| \le h$.

Proof of Theorem 1.2 Denote $g(z) = f(z) - a(z)$, then $\rho(g) = \rho(f) = \rho$. By assumption, we have

$$\frac{f(z+\eta) - a(z)}{f(z) - a(z)} = \frac{g(z+\eta) + b(z)}{g(z)} = e^{P(z)}, \tag{3.1}$$

where $P(z)$ is a polynomial with $d = \deg(P) \le \rho$ and $b(z) = a(z+\eta) - a(z)$ is an entire small function of $f(z)$ such that $\sigma = \rho(b) \le \rho(a) < \rho$.

We prove that $\rho \ge 1$. Otherwise, we have $\rho < 1$ and hence $P(z) \equiv C \in \mathbb{C}$. By Lemma 2.1, for any given ε $(0 < \varepsilon < \frac{\rho-\sigma}{3})$, there exists a set $E_1 \subset [0, +\infty)$ with finite linear measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, and r sufficiently large,

$$\exp\{-r^{\sigma+\varepsilon}\} \le |b(z)| \le \exp\{r^{\sigma+\varepsilon}\}. \tag{3.2}$$

By Lemma 3.1, there exists an ε-set F such that

$$\frac{g(z+\eta)}{g(z)} \to 1 \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus F. \tag{3.3}$$

Denote $E_2 = \{|z| : z \in F, |z| > 1\}$, then E_2 has finite logarithmic measure.

Choose an infinite sequence of points $\{z_k = r_k e^{i\theta_k}\}$ such that

$$|g(z_k)| = M(r_k, g) \ge \exp\{r_k^{\rho-\varepsilon}\}, \quad r_k \notin E_1 \cup E_2. \tag{3.4}$$

From (3.2)–(3.4), we can obtain

$$\frac{g(z_k+\eta)}{g(z_k)}+\frac{b(z_k)}{g(z_k)}\to 1,\quad r_k\notin E_1\cup E_2,\ r_k\to\infty.$$

From this and (3.1), we immediately get that $e^C\equiv 1$, which yields $f(z+\eta)=f(z)$ for all $z\in\mathbb{C}$, a contradiction to our assumption that $\Delta_\eta f(z)\not\equiv 0$. Hence, we prove that $\rho(f)\geq 1$.

Finally, suppose that $a(z)$ is a periodic function with period η, especially, a constant function. Now if there exists a point $z_0\in\mathbb{C}$ such that $f(z_0)=a(z_0)$, then also $f(z_0+\eta)=a(z_0)$, which implies that $f(z_0+k\eta)=a(z_0)$ holds for all $k\in\mathbb{Z}$. □

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