Journal of Mathematical Research with Applications
Jul., 2012, Vol. 32, No. 4, pp. 476–484
DOI:10.3770/j.issn:2095-2651.2012.04.012
Http://jmre.dlut.edu.cn

Asymptotic Behavior of a Non-Local Hyperbolic Equation Modelling Ohmic Heating

Xianchao WANG[1,*], **Chengshun JIANG**[2]

1. *National Digital Switching System Engineering & Technological R & D Centre, Henan* 450002*, P. R. China;*
2. *Wuhan College, Zhongnan University of Economics and Law, Hubei* 430079*, P. R. China*

Abstract In this paper, the asymptotic behavior of a non-local hyperbolic problem modelling Ohmic heating is studied. It is found that the behavior of the solution of the hyperbolic problem only has three cases: the solution is globally bounded and the unique steady state is globally asymptotically stable; the solution is infinite when $t \to \infty$; the solution blows up. If the solution blows up, the blow-up is uniform on any compact subsets of $(0,1]$ and the blow-up rate is $\lim_{t\to T^*-} u(x,t)(T^*-t)^{\frac{1}{\alpha+\beta p-1}} = (\frac{\alpha+\beta p-1}{1-\alpha})^{\frac{1}{1-\alpha-\beta p}}$, where T^* is the blow-up time.

Keywords non-local hyperbolic equation; asymptotical behavior; blow-up; blow-up rate.

MR(2010) Subject Classification 35L04; 35L60

1. Introduction

In this paper, the asymptotic behavior of the following non-local hyperbolic problem is studied

$$\begin{aligned} &u_t + u_x = \lambda u^\alpha \Big/ \Big(\int_0^1 (u+1)^{-\beta} \mathrm{d}x\Big)^p, \quad 0 < x < 1,\ t > 0; \\ &u(0,t) = 0, \quad t > 0; \\ &u(x,0) = u_0(x) \ge 0, \quad 0 < x < 1, \end{aligned} \tag{1}$$

where $0 \le \alpha < 1,\ \beta > 0,\ p > 0,\ \lambda > 0,\ u_0 = O(x^{\frac{1}{1-\alpha}})$ as $x \to 0$.

The problem comes from one method for sterilizing food. This method is to heat the food rapidly electrically. The food is passed through a conduit, part of which lies between two electrodes. A high electric current flowing between the electrodes results in Ohmic heating of the food which quickly gets hot. The problem was considered by Please, Schwendeman and Hagen [1] who observed the stability of models allowing for different types of flow. Both homogeneous and inhomogeneous cases were discussed. In [1] it was found that heat convection dominates heat conduction, which brought about the form of the left hand side of the equation (1). Concerning the source term of this equation, see [2], and the references therein. In the paper we consider

Received December 28, 2010; Accepted December 19, 2011
Supported by the National High Technology Research and Development Program of China (Grant No. 2012AA011603).
* Corresponding author
E-mail address: wxc1980@yahoo.com.cn (Xianchao WANG); csjiang2005@126.com (Chengshun JIANG)

the basic one-dimensional, non-local, hyperbolic model for the simplest case of a plug flow of a homogeneous material.

Lacey has originally studied the problem with source term of the form $\frac{\lambda f(u)}{(\int_0^1 f(u)\mathrm{d}x)^2}$, $f, -f' > 0$ (see [3–5]). Tzanetis et al. also obtained many good results towards this direction [6–11]. This paper applies the comparison principle to studying the asymptotical behavior of the solution of the problem for any $p > 0$.

We describe our main results as follows (assume u is the solution of problem (1)):

1) $0 < p < 1$. u is globally bounded and the unique steady state is globally asymptotically stable.

2) $p = 1$. (i) $\alpha + \beta > 1$. If $0 < \lambda < C_1 = \frac{1}{1-\alpha}\int_0^\infty (s^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}s$, u is globally bounded and the unique steady state is globally asymptotically stable; if $\lambda \geq C_1 = \frac{1}{1-\alpha}\int_0^\infty (s^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}s$, u is infinite as $t \to \infty$, (ii) If $\alpha + \beta \leq 1$, u is globally bounded and the unique steady state is globally asymptotically stable.

3) $p > 1$. If $\alpha+\beta > 1$, there is a λ^* such that when $0 < \lambda \leq \lambda^*$, u blows up when u_0 is large enough, and when $\lambda > \lambda^*$, u blows up for any initial condition; if $\alpha + \beta \leq 1$, there are three cases: (i) $\alpha + \beta p < 1$. There is a unique steady state which is globally asymptotically stable, (ii) $\alpha + \beta p = 1$. When $0 < \lambda < C_2 = \frac{1}{1-\alpha}\left(\frac{p}{p-1}\right)^p$, ($M$ is the maximum of solution of stationary problem (3) which is introduced in Section 3) there is a unique steady state which is globally asymptotically stable; when $\lambda \geq C_2 = \frac{1}{1-\alpha}(\frac{p}{p-1})^p$, then $u \to \infty$ as $t \to \infty$, (iii) $\alpha + \beta p > 1$. There is a λ^* such that when $0 < \lambda \leq \lambda^*$, u blows up for u_0 large enough and when $\lambda > \lambda^*$, u blows up for any initial condition.

4) If u blows up, the blow-up is uniformly on any compact subsets of $(0, 1]$. We have the result

$$\lim_{t\to T^*-}\frac{u^{1-\alpha}(t,x)}{G(t)} = \lim_{t\to T^*-}\frac{|u^{1-\alpha}(t,x)|_\infty}{G(t)} = 1,$$

where T^* is the blow-up time, $G(t) = \int_0^t g(t)\mathrm{d}t$ and $g(t) = \lambda/(\int_0^1 (u+1)^{-\beta}\mathrm{d}x)^p$.

5) Applying the character of uniform blow-up, we obtain the uniform rate of the solution of problem (1)

$$\lim_{t\to T^*-} u(x,t)(T^*-t)^{\frac{1}{\alpha+\beta p-1}} = \Big(\frac{\alpha+\beta p-1}{1-\alpha}\Big)^{\frac{1}{1-\alpha-\beta p}}.$$

This paper is organized as follows: the next three sections are preliminaries. Section 5 concerns the asymptotical behavior and blow-up of the solutions. In the last section the blow-up rate of solution of the problem (1) is given.

2. Local existence and a comparison principle

At first we set $v = u^{1-\alpha}$, then (1) changes into

$$\begin{aligned} &v_t + v_x = \lambda(1-\alpha)\Big/\Big(\int_0^1 (v^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x\Big)^p, \quad 0 < x < 1,\ t > 0; \\ &v(0,t) = 0, \quad t > 0; \\ &v(x,0) = u_0^{1-\alpha}(x), \quad 0 < x < 1. \end{aligned} \tag{2}$$

It is obvious that the character of problem (1) can be gotten from problem (2). Next we study the character of problem (2).

The local existence of a unique solution of (2) follows from employing standard methods, such as Picard iteration [3, 4]. Non-existence can only come from blow-up with v becoming infinite after finite time T^*.

Next a comparison lemma is given, which will be used in the following sections.

Lemma 2.1 *If $\underline{v}$ is the lower solution of (2) and $\overline{v}$ is the upper solution of (2), then $\underline{v} \le v \le \overline{v}$.*

For the proof, see [3]. This comparison lemma is the main tool of this paper.

3. Steady states of (2)

The steady state of the problem (2) plays an important role in the description of the asymptotic behavior of the solution of (2) and the construction of the lower and upper solutions, so we first consider the stationary problem of (2).

The stationary problem of (2) is

$$\begin{aligned} &w'(x) = \lambda(1-\alpha)\Big/\Big(\int_0^1 (w^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x\Big)^p, \quad 0<x<1, \\ &w(0)=0. \end{aligned} \tag{3}$$

Set

$$\mu = \lambda(1-\alpha)\Big/\Big(\int_0^1 (w^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x\Big)^p,$$

then problem (3) turns into

$$w'(x)=\mu, \quad 0<x<1, \ w(0)=0. \tag{4}$$

Denote its solution by $w(x;\mu)$. Integrating on $(0,x)$, we have

$$w(x;\mu)=\mu x.$$

For $w_x>0$, set $M=\max_{0<x\le 1} w(x)=\mu$. The following relation between λ and M can be obtained

$$\lambda = \frac{1}{1-\alpha} M\Big(\int_0^1 ((Mx)^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x\Big)^p = \Lambda(M). \tag{5}$$

Set $s=Mx$, then

$$(1-\alpha)^{\frac{1}{p}}\lambda^{\frac{1}{p}} = M^{\frac{1-p}{p}}\int_0^M (s^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}s. \tag{6}$$

It is clear $\lambda=0$ as $M=0$. But when $M\to\infty$, the situations are more complicated. Next we discuss them.

1) $0<p<1$. $\lambda\to\infty$ as $M\to\infty$ (see Figure 1(a)).

2) $p=1$. If $\alpha+\beta>1$, $\lambda\to C_1=\frac{1}{1-\alpha}\int_0^\infty (s^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}s$ as $M\to\infty$ (see Figure 1(b)); if $\alpha+\beta\le 1$, $\lambda\to\infty$ as $M\to\infty$ (see Figure 1(a)).

3) $p>1$. If $\alpha+\beta>1$, $\lambda\to 0$ as $M\to\infty$ (see Figure 1(c)); if $\alpha+\beta\le 1$, there are three cases: (i) $\alpha+\beta p<1$, $\lambda\to\infty$ as $M\to\infty$ (see Figure 1(a)), (ii) $\alpha+\beta p=1$, $\lambda\to C_2=\frac{1}{1-\alpha}(\frac{p}{p-1})^p$ as

$M \to \infty$ (see Figure 1(b)), (iii) $\alpha + \beta p > 1$, $\lambda \to 0$ as $M \to \infty$ (see Figure 1(c)). Here $C_1 > 0$, $C_2 > 0$ and they are all constants.

In 1), the result is obvious. In 2), if $\alpha + \beta > 1$, $\int_0^M (s^{\frac{1}{1-\alpha}} + 1)^{-\beta} \mathrm{d}s < \infty$, so $\lambda \to C_1 = \frac{1}{1-\alpha} \int_0^\infty (s^{\frac{1}{1-\alpha}} + 1)^{-\beta} \mathrm{d}s$ as $M \to \infty$; if $\alpha + \beta \le 1$, $\int_0^M (s^{\frac{1}{1-\alpha}} + 1)^{-\beta} \mathrm{d}s \to \infty$, so $\lambda \to \infty$ as $M \to \infty$. In 3), if $\alpha + \beta > 1$, it is obvious that $\lambda \to 0$ as $M \to \infty$; if $\alpha + \beta \le 1$, $M^{\frac{1-p}{p}} \to \infty$ and $\int_0^M (s^{\frac{1}{1-\alpha}} + 1)^{-\beta} \mathrm{d}s \to \infty$ as $M \to \infty$, next we discuss this case.
From (6), we have

$$(1-\alpha)^{\frac{1}{p}} \lambda^{\frac{1}{p}} = \frac{\int_0^M (s^{\frac{1}{1-\alpha}} + 1)^{-\beta} \mathrm{d}s}{M^{\frac{p-1}{p}}}.$$

Applying L'Hospital's rule,

$$(1-\alpha)^{\frac{1}{p}} \lambda^{\frac{1}{p}} = \frac{(M^{\frac{1}{1-\alpha}} + 1)^{-\beta}}{\frac{p-1}{p} M^{\frac{-1}{p}}},$$

$$(1-\alpha)^{\frac{1}{p}} \lambda^{\frac{1}{p}} = \lim_{M \to \infty} \frac{p}{p-1} (M^{\frac{1}{1-\alpha}} + 1)^{-\beta} M^{\frac{1}{p}}.$$

Finally for $M \gg 1$, we have

$$(1-\alpha)^{\frac{1}{p}} \lambda^{\frac{1}{p}} \sim \frac{p}{p-1} M^{\frac{1-\alpha-\beta p}{(1-\alpha)p}}.$$

Therefore, if $\alpha + \beta \ge 1$, there are three cases for $\alpha + \beta p < 1$, $\alpha + \beta p = 1$ and $\alpha + \beta p > 1$.

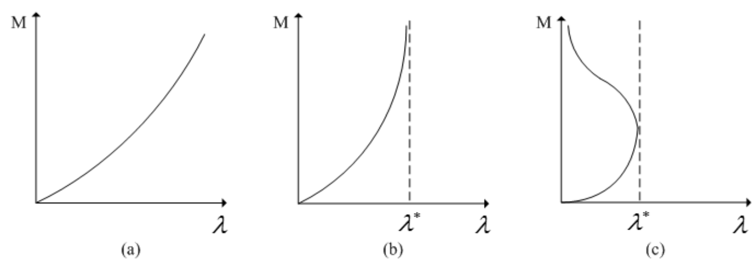

Figure 1 Bars joined with hinges

Now we examine the number of turning points that exist.

In 1) and 2), it is obvious that λ is increasing with respect to M. So there are no turning points in these two cases. In 3), from (5) we know

$$\lambda = \frac{1}{1-\alpha} M^{\frac{1-\alpha-\beta p}{1-\alpha}} \Big(\int_0^1 (x^{\frac{1}{1-\alpha}} + \frac{1}{M})^{-\beta} \mathrm{d}x \Big)^p.$$

Then λ is increasing with respect to M. So we deduce that there are no turning points in 3 (i) and (ii).

Next we study the case where there is only one turning point in 3 (iii) ($\lambda \to 0$ as $M \to \infty$). Using (6), we have

$$(1-\alpha)^{\frac{1}{p}} (\lambda^{\frac{1}{p}})' = M^{\frac{1-p}{p}} (M^{\frac{1}{1-\alpha}} + 1)^{-\beta} + \frac{1-p}{p} M^{\frac{1-2p}{p}} \int_0^M (s^{\frac{1}{1-\alpha}} + 1)^{-\beta} \mathrm{d}s.$$

We set

$$L(M) = M (M^{\frac{1}{1-\alpha}} + 1)^{-\beta} + \frac{1-p}{p} \int_0^M (s^{\frac{1}{1-\alpha}} + 1)^{-\beta} \mathrm{d}s.$$

Differentiating with respect to M gives

$$L'(M)=(M^{\frac{1}{1-\alpha}}+1)^{-\beta-1}((\frac{1}{p}-\frac{1}{(1-\alpha)\beta})M^{\frac{1}{1-\alpha}}+\frac{1}{p}).$$

$L'(M)=0$ has one positive solution at most, which implies that $L(M)=0$ has two positive solutions at most. So we can see there is only one turning point in 3) (iii).

According to the above analysis, we have relations between λ and M (see Fig. 1). From relations between λ and M, the existence theorem of the steady state of (3) is given.

Theorem 3.1 *The existence of the steady state of (3) for fixed $\lambda>0$:*

1) $0<p<1$ (see Fig. 1(a)). There exists a unique steady state for any λ.

2) $p=1$. (i) If $\alpha+\beta>1$ (see Fig. 1(b)), there exists a steady state when $\lambda<C_1$; no steady state exists when $\lambda\geq C_1$; (ii) if $\alpha+\beta\leq 1$ (see Fig. 1(a)), there exists a unique steady state for any λ.

3) $p>1$. If $\alpha+\beta>1$ (see Fig. 1(c)), there exist two steady states when $\lambda\leq\lambda^*$; there exists a unique steady state when $\lambda=\lambda^*$; no steady state exists when $\lambda>\lambda^*$; if $\alpha+\beta\leq 1$, there are three cases: (i) $\alpha+\beta p<1$ (see Fig. 1(a)), there exists a unique steady state, (ii) $\alpha+\beta p=1$ (see Fig. 1(b)), there exists a steady state when $\lambda<C_2$; no steady state exists when $\lambda\geq C_2$, (iii) $\alpha+\beta p>1$ (see Fig. 1(c)), there exist two steady states when $\lambda<\lambda^*$; there exists a unique steady state when $\lambda=\lambda^*$; no steady state exists when $\lambda>\lambda^*$.

4. Construction of the lower and upper solutions

In (4), let μ be a function of t. Following [3, 4, 12], we find the conditions on $\mu(t)$ for $w(x;\mu(t))$ to be a lower or an upper solution of (2).

Denote $k(x,t)\equiv w(x;\mu(t))$. From (4), we have $k_x=\mu(t)$. Integrating (4) from 0 to x, we get

$$k(x,t)=\mu(t)x.$$

Differentiating with respect to t gives

$$k_t=\mu'(t)x.$$

So $k(x,t)$ satisfies

$$k_t+k_x-\frac{\lambda(1-\alpha)}{(\int_0^1(k^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x)^p}=\dot{\mu}(t)x+\mu(t)-\frac{\lambda(1-\alpha)}{(\int_0^1((\mu(t)x)^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x)^p}.$$

Let $\mu(t)$ be the solution of

$$\dot{\mu}=\frac{\lambda(1-\alpha)}{(\int_0^1((\mu x)^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x)^p}-\mu,\quad \mu(0)=\mu_0. \tag{7}$$

If there exists μ_0 such that

$$\lambda\leq\mu\frac{1}{1-\alpha}\Big(\int_0^1((\mu x)^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x\Big)^p\quad\text{and}\quad w(x;\mu_0)\geq v_0(x),$$

then $\mu(t)$ is decreasing, therefore $k(x,t)$ is decreasing and satisfies

$$k_t + k_x - \frac{\lambda(1-\alpha)}{(\int_0^1 (k^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x)^p} \geq 0.$$

So $k(x,t)$ is a decreasing upper solution.

If there exists μ_0 such that

$$\lambda \geq \mu \frac{1}{1-\alpha}\Big(\int_0^1 ((\mu x)^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x\Big)^p \quad \text{and} \quad w(x;\mu_0) \leq v_0(x),$$

then $\mu(t)$ is increasing, therefore $k(x,t)$ is increasing and satisfies

$$k_t + k_x - \frac{\lambda(1-\alpha)}{(\int_0^1 (k^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x)^p} \leq 0.$$

So $k(x,t)$ is an increasing lower solution.

5. Asymptotic behavior of problem (1)

With preparations of the above two sections in hand, we can discuss the asymptotical behavior and blow-up properties of the solution of (1). At first we discuss the asymptotical behavior of (2), then the asymptotical behavior of (1) is obvious because $v(x,t) = u^{1-\alpha}(x,t)$. Next we discuss the asymptotical behavior of (2) for three cases: $0 < p < 1$, $p = 1$, $p > 1$.

Theorem 5.1 *If $0 < p < 1$, then the solution of (2) is bounded and the unique steady state is globally asymptotically stable.*

Proof From Theorem 3.1, for fixed λ, (3) has a unique steady state $w(x;\mu_1)$ where μ_1 is the solution of the equation (5).

$$\lambda = \frac{1}{1-\alpha}\mu\Big(\int_0^1 ((\mu x)^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x\Big)^p = \Lambda(\mu). \tag{8}$$

Take $\overline{\mu}(t)$ satisfying (7) with $\mu(0) = \overline{\mu}_0$. For any initial data $v_0(x) > 0$, since

$$\Lambda(\mu) \to \infty, \quad \text{as} \quad \mu \to \infty,$$

we can select $\overline{\mu}_0$ sufficiently large such that

$$\lambda \leq \Lambda(\overline{\mu}_0) \quad \text{and} \quad w(x;\overline{\mu}_0) \geq v_0(x).$$

For $u_0 = O(x^{\frac{1}{1-\alpha}})$ as $x \to 0$, $\overline{\mu}_0$ must exist. Thus $\overline{\mu}(t)$ is decreasing and $\overline{\mu}(t) \to \mu_1+$ as $t \to \infty$. So $\overline{k}(x,t) = w(x;\overline{\mu}(t))$ is a decreasing upper solution of the problem and

$$\overline{k}(x,t) \to w(x,\mu_1)+, \quad \text{as} \quad t \to +\infty.$$

On the other hand, take $\underline{\mu}(t)$ satisfying (7) with $\mu(0) = \underline{\mu}_0$. Since

$$\Lambda(\mu) \to 0, \quad \text{as} \quad \mu \to 0,$$

we can select $\underline{\mu}_0$ sufficiently small such that

$$\lambda \geq \Lambda(\underline{\mu}_0) \quad \text{and} \quad w(x;\underline{\mu}_0) \leq v_0(x).$$

For $u_0 = O(x^{\frac{1}{1-\alpha}})$ as $x \to 0$, $\underline{\mu}_0$ must exist. Thus $\underline{\mu}(t)$ is increasing and $\underline{\mu}(t) \to \mu_1-$ as $t \to \infty$. So $\underline{k}(x,t) = w(x;\underline{\mu}(t))$ is an increasing lower solution of the problem and

$$\underline{k}(x,t) \to w(x,\mu_1)-, \quad \text{as} \quad t \to +\infty.$$

Applying Lemma 2.1, we deduce that

$$v(x,t) \to w(x,\mu_1), \quad \text{as} \quad t \to +\infty.$$

The proof is completed. □

Theorem 5.2 *When $p = 1$, there are two cases:*

1) $\alpha + \beta > 1$. If $0 < \lambda < C_1$, the solution of (2) is globally bounded and the unique steady state is globally asymptotically stable; if $\lambda \geq C_1$, the solution of (2) is infinite as $t \to \infty$.

2) $\alpha + \beta \leq 1$. The solution of (2) is globally bounded and the unique steady state is globally asymptotically stable.

Proof 1) By Theorem 3.1, if $0 < \lambda < C_1$, the proof is the same as that of Theorem 5.1; if $\lambda \geq C_1$, we can construct an increasing lower solution $\underline{k}(x,t)$ like the construction in Section 4.

From (7) we know

$$\dot{\mu} = \frac{\lambda(1-\alpha)}{\int_0^1((\mu x)^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x} - \mu,$$

then

$$\dot{\mu}\int_0^1((\mu x)^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x = (1-\alpha)(\lambda - \frac{1}{1-\alpha}\mu\int_0^1((\mu x)^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x).$$

So

$$\frac{\dot{\mu}}{\mu}\int_0^{\mu}(s^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}s = (1-\alpha)(\lambda - \frac{1}{1-\alpha}\mu\int_0^1((\mu x)^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x).$$

For $\alpha + \beta > 1$, $\underline{k}(x,t) \to \infty$ as $t \to \infty$. Applying Lemma 2.1, we get the result that the solution of problem (2) is infinite as $t \to \infty$.

2) The proof is the same as that of Theorem 5.1. The proof is completed. □

Theorem 5.3 *For $p > 1$, there are two cases:*

1) $\alpha + \beta > 1$. There are two cases:

(i) If $0 < \lambda \leq \lambda^*$, the solution of (2) is globally bounded and the unique steady state is globally asymptotically stable when v_0 is small enough; the solution of (2) blows up when v_0 is large enough.

(ii) If $\lambda > \lambda^*$, the solution of (2) blows up for any initial condition.

2) $\alpha + \beta \leq 1$. There are three cases:

(i) $\alpha + \beta p < 1$. The solution of (2) is globally bounded and the unique steady state is globally asymptotically stable.

(ii) $\alpha + \beta p = 1$. When $0 < \lambda < C_2$, the solution of (2) is globally bounded and the unique steady state is globally asymptotically stable; when $\lambda \geq C_2$, the solution of (2) is infinite as $t \to \infty$.

(iii) $\alpha + \beta p > 1$. *If* $0 < \lambda \le \lambda^*$, *the solution of (2) is globally bounded and the unique steady state is globally asymptotically stable when* v_0 *is small enough; the solution of (2) blows up when* v_0 *is large enough; if* $\lambda > \lambda^*$, *the solution of (2) blows up for any initial condition.*

Proof 1) From Theorem 3.1 (see Fig. 1(c)) we know that if $0 < \lambda \le \lambda^*$, there are two solutions for equation (8). Suppose that the solutions are μ_1 and μ_2 with $\mu_1 < \mu_2$.

When $v_0 = u_0^{1-\alpha} \le w(x;\mu_2)$, the solution of (2) has a unique steady state which is globally asymptotically stable. The method is similar to that of Theorem 5.1. When $v_0 = u_0^{1-\alpha} > w(x;\mu_2)$, the problem only has an increasing lower solution of the types considered in Section 4.

Applying equation (7), we have

$$\dot{\mu}\Big(\int_0^1 ((\mu x)^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x\Big)^p = (1-\alpha)\Big(\lambda - \frac{1}{1-\alpha}\mu\Big(\int_0^1 ((\mu x)^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x\Big)^p\Big),$$

then

$$\dot{\mu}\Big(\int_0^\mu (s^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}s\Big)^p = (1-\alpha)\Big(\lambda - \frac{1}{1-\alpha}\mu\Big(\int_0^1 ((\mu x)^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x\Big)^p\Big).$$

For $\alpha + \beta > 1$ and $p > 1$, it is obvious that the solution of (2) blows up. When $\lambda > \lambda^*$, the problem only has an increasing lower solution. Proceeding with the same arguments as above, we can deduce the solution of (2) blows up.

2) The proof is similar to that of Theorem 5.1. □

6. The blow-up rate of the solution of (1)

If the solution of (2) blows up, the blow-up is uniform on any compact subsets of $(0,1]$, and then we have the following lemma [12,13].

Lemma 6.1 *Let* v *be the solution of (2) and assume that* $T^* < \infty$. *Then we have*

$$\lim_{t\to T^*-} \frac{v(t,x)}{G(t)} = \lim_{t\to T^*-} \frac{|v(t,x)|_\infty}{G(t)} = 1,$$

where T^* *is the blow-up time,* $G(t) = \int_0^t g(t)\mathrm{d}t$ *and* $g(t) = \lambda/(\int_0^1 (v^{\frac{1}{1-\alpha}} + 1)^{-\beta}\mathrm{d}x)^p$.

For $v(x,t) = u^{1-\alpha}(x,t)$ we obtain Corollary 6.2.

Corollary 6.2 *Let* u *be the solution of (1) and assume that* $T^* < \infty$. *Then we have*

$$\lim_{t\to T^*-} \frac{u^{1-\alpha}(t,x)}{G(t)} = \lim_{t\to T^*-} \frac{|u^{1-\alpha}(t,x)|_\infty}{G(t)} = 1,$$

where T^* *is the blow-up time,* $G(t) = \int_0^t g(t)\mathrm{d}t$ *and* $g(t) = \lambda/(\int_0^1 (u+1)^{-\beta}\mathrm{d}x)^p$.

At last, applying the result of Corollary 6.2 gives the uniform blow-up rate of the solution of (1).

Theorem 6.3 *Let* u *be the solution of (1) and assume that* $T^* < \infty$. *We have*

$$\lim_{t\to T^*-} u(x,t)(T^* - t)^{\frac{1}{\alpha+\beta p-1}} = \Big(\frac{\alpha+\beta p - 1}{1-\alpha}\Big)^{\frac{1}{1-\alpha-\beta p}}.$$

Proof From (2), we know

$$v_t + v_x = \lambda(1-\alpha)\Big/\Big(\int_0^1 (v^{\frac{1}{1-\alpha}}+1)^{-\beta}\mathrm{d}x\Big)^p.$$

Applying Corollary 6.2, when $t \to T^*$ we have

$$G'(t) \sim \lambda(1-\alpha)(G(t)^{\frac{1}{1-\alpha}}+1)^{\beta p},$$

then

$$\frac{G'(t)}{(G(t)^{\frac{1}{1-\alpha}}+1)^{\beta p}} \sim \lambda(1-\alpha), \quad 0 < T^* - t \ll 1.$$

Integrating from t to T^* yields

$$\int_{G(t)}^{\infty} \frac{\mathrm{d}s}{(s^{\frac{1}{1-\alpha}}+1)^{\beta p}} \sim \int_{G(t)}^{\infty} \frac{\mathrm{d}s}{s^{\frac{\beta p}{1-\alpha}}} \sim \lambda(1-\alpha)(T^*-t).$$

Thus

$$\frac{1-\alpha}{\alpha+\beta p-1} v^{\frac{1-\alpha-\beta p}{1-\alpha}} \sim \lambda(1-\alpha)(T^*-t).$$

Since $v = u^{1-\alpha}$, we deduce that

$$u(x,t)(T^*-t)^{\frac{1}{\alpha+\beta p-1}} \sim (\frac{\alpha+\beta p-1}{1-\alpha})^{\frac{1}{1-\alpha-\beta p}}.$$

The proof is completed. $\Box$

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