Journal of Mathematical Research with Applications
Jul., 2012, Vol. 32, No. 4, pp. 493–504
DOI:10.3770/j.issn:2095-2651.2012.04.014
Http://jmre.dlut.edu.cn

Low-Dimensional Semisimple Hopf Algebras of Frobenius Type

Jingcheng DONG
College of Engineering, Nanjing Agricultural University, Jiangsu 210031*, P. R. China*

Abstract Let k be an algebraically closed field of characteristic zero. This paper proves that semisimple Hopf algebras over k of dimension $66, 70$ and 78 are of Frobenius type.

Keywords semisimple Hopf algebras; characters; Frobenius type.

MR(2010) Subject Classification 16W30

1. Introduction

Throughout this paper, we will work over an algebraically closed field k of characteristic zero.

A semisimple Hopf algebra H is called of Frobenius type if the dimensions of the simple H-modules divide the dimension of H. Kaplansky conjectured that every finite dimensional semisimple Hopf algebra is of Frobenius type [1, Appendix 2]. In general, the conjecture is still open. It is well-known that if H is a group algebra, then H is of Frobenius type. This is a classical result by Frobenius.

Many examples show that a positive answer to Kaplansky's conjecture would be very helpful in the classification of semisimple Hopf algebras. For example, Natale [2] completed the classification of semisimple Hopf algebras of dimension $pq^2 < 100$ by showing that these Hopf algebras are of Frobenius type.

Quite recently, Natale proved [3] that semisimple Hopf algebras of dimension less than 60 are of Frobenius type. Based on this fact, Natale then proved that all these Hopf algebras are semisolvable in the sense introduced by Montgomery and Witherspoon [4].

The present paper is devoted to extending Natale's result to semisimple Hopf algebras of dimension less than 80. By [5, Theorem 3.5], semisimple Hopf algebras of dimension 60 are of Frobenius type. By [6], semisimple Hopf algebras of dimension 72 are of Frobenius type. By [7, 8], semisimple Hopf algebras of dimension $61, 62, 65, 67, 69, 71, 73, 74, 77$ and 79 are group algebras or their duals. Hence, these Hopf algebras are of Frobenius type. By [2, Section 5.5], semisimple Hopf algebras of dimension $63, 68, 75$ and 76 are of Frobenius type. By [4], semisimple Hopf

Received November 18, 2010; Accepted October 31, 2011
Supported by the Jiangsu Planned Projects for Postdoctoral Research Funds (Grant No. 1102041C).
E-mail address: dongjc@njau.edu.cn

algebras of dimension 64 are of Frobenius type. Therefore, it suffices to consider the semisimple Hopf algebras of dimension $66, 70$ and 78 in this paper.

The paper is organized as follows. In Section 2, we recall some basic results on characters of a semisimple Hopf algebra. Some useful lemmas are also contained in this section.

In Section 3, we discuss the algebra types of semisimple Hopf algebras of dimension $66, 70$ and 78, respectively. The notion of algebra types of a semisimple Hopf algebra is recalled in Section 2. Based on the study of the character algebra of a semisimple Hopf algebra, we obtain all possible algebra types of these Hopf algebras. As a consequence, all these Hopf algebras are of Frobenius type.

Throughout this paper, all modules are left modules. Moreover they are finite dimensional over k. $\otimes$, dim mean $\otimes_k$, $\dim_k$, respectively. Our references for the theory of Hopf algebras are [9] or [10]. The notation for Hopf algebras is standard. For example, the group of group-like elements in H is denoted by $G(H)$.

2. Characters of a semisimple Hopf algebra

Throughout this section, H will be a semisimple Hopf algebra over k.

Let V be an H-module. The character of V is the element $\chi = \chi_V \in H^*$ defined by $\langle \chi, h\rangle = \mathrm{Tr}_V(h)$ for all $h \in H$. The degree of χ is defined to be the integer $\deg\chi = \chi(1) = \dim V$. We shall use X_t to denote the set of all irreducible characters of H of degree t. If U is another H-module, we have

$$\chi_{U\otimes V} = \chi_U \chi_V, \quad \chi_{V^*} = S(\chi_V),$$

where S is the antipode of H^*.

Hence, the irreducible characters, namely, the characters of the simple H-modules, span a subalgebra $R(H)$ of H^*, which is called the character algebra of H. The antipode S induces an anti-algebra involution $* : R(H) \to R(H)$, given by $\chi \to \chi^* := S(\chi)$. The character of the trivial H-module is the counit ε.

The properties of $R(H)$ have been intensively studied in [11]. We recall some of them here, and will use them freely in this paper [3, Section 1.2].

Let $\chi_U, \chi_V \in R(H)$ be the characters of the H-modules U and V, respectively. The integer $m(\chi_U, \chi_V) = \dim\mathrm{Hom}_H(U, V)$ is defined to be the multiplicity of U in V. This can be extended to a bilinear form $m : R(H) \times R(H) \to k$.

Let $\widehat{H}$ denote the set of irreducible characters of H. If $\chi \in R(H)$, we may write $\chi = \sum_{\alpha\in\widehat{H}} m(\alpha, \chi)\alpha$. Let $\chi, \psi, \omega \in R(H)$. Then $m(\chi, \psi\omega) = m(\psi^*, \omega\chi^*) = m(\psi, \chi\omega^*)$ and $m(\chi, \psi) = m(\chi^*, \psi^*)$ (see [11, Theorem 9]).

For each group-like element g in $G(H^*)$, we have $m(g, \chi\psi) = 1$, if $\psi = \chi^* g$ and 0 otherwise for all $\chi, \psi \in \widehat{H}$. In particular, $m(g, \chi\psi) = 0$ if $\deg(\chi) \neq \deg(\psi)$. Let $\chi \in \widehat{H}$. Then for any group-like element g in $G(H^*)$, $m(g, \chi\chi^*) > 0$ iff $m(g, \chi\chi^*) = 1$ iff $g\chi = \chi$. The set of such group-like elements forms a subgroup of $G(H^*)$, of order at most $(\deg(\chi))^2$ (see [11, Theorem 10]). Denote this subgroup by $G[\chi]$. In particular, we have $\chi\chi^* = \sum_{g\in G[\chi]} g + \sum_{\alpha\in\widehat{H}, \deg\alpha>1} m(\alpha, \chi\chi^*)\alpha$.

Lemma 1 *Let $\chi \in \widehat{H}$ be an irreducible character of H. Then*

(1) The order of $G[\chi]$ divides $(\deg\chi)^2$.

(2) The order of $G(H^*)$ divides $n(\deg\chi)^2$, where n is the number of non-isomorphic simple H-modules of dimension $\deg\chi$.

Proof It follows from Nichols-Zoeller Theorem [12, Lemma 2.2.2].

Let $1 = d_1, d_2, \ldots, d_s$, $n_1, n_2, \ldots, n_s$ be positive integers, with $d_1 < d_2 < \cdots < d_s$. H is said to be of type $(d_1, n_1; \ldots; d_s, n_s)$ as an algebra if $d_1, d_2, \ldots, d_s$ are the dimensions of the simple H-modules and n_i is the number of the non-isomorphic simple H-modules of dimension d_i. That is, as an algebra, H is isomorphic to a direct product of full matrix algebras

$$H \cong k^{(n_1)} \times \prod_{i=2}^{s} M_{d_i}(k)^{(n_i)}.$$

If H^* is of type $(d_1, n_1; \ldots; d_s, n_s)$ as an algebra, then H is said to be of type $(d_1, n_1; \ldots; d_s, n_s)$ as a coalgebra. □

The following result is due to [13, Lemma 11].

Lemma 2 *If H is of type $(1, 1; d_2, n_2; \ldots, d_s, n_s)$ as an algebra, then $\{d_i | d_i > 1\}$ has at least three elements.*

A subalgebra A of $R(H)$ is called a standard subalgebra if A is spanned by irreducible characters of H. Let X be a subset of $\widehat{H}$. Then X spans a standard subalgebra of $R(H)$ if and only if the product of characters in X decomposes as a sum of characters in X. There is a bijection between standard subalgebras of $R(H)$ and quotient Hopf algebras of H (see [11, Theorem 6]). The following theorem is a finite-dimensional version of [11, Theorem 11].

Theorem 1 *If H has an irreducible character χ of degree 2, then at least one of the following conditions holds:*

(1) $G[\chi] \neq \{\varepsilon\}$;

(2) H has a quotient Hopf algebra of dimension 24, which has a character g of degree 1 of order 2 such that $g\chi \neq \chi$;

(3) H has a quotient Hopf algebra of dimension 12 or 60.

The next three lemmas are due to [6, Lemmas 2.3–2.5]. We give the proof here for completeness.

Lemma 3 *Let H have an irreducible character χ of degree 2.*

(1) If 12 does not divide $\dim H$ or H does not have irreducible characters of degree 3, then $G[\chi] \neq \{\varepsilon\}$.

(2) If $G(H^*) = \{\varepsilon\}$, then H has a quotient Hopf algebra of dimension 60.

Proof Part (1) is obvious from Theorem 1. Part (2) is a special case of [11, Theorem 11]. In fact, from the proof of [11, Theorem 11], we know if H has a quotient Hopf algebra of dimension 12 or 24, then $G(H^*)$ contains at least 2 elements. □

Lemma 4 *If H is of type $(1,m;t,n)$ as an algebra, then $t|m$. In particular, H is of Frobenius type.*

Proof Let χ_i $(1\leq i\leq n)$ be all distinct irreducible characters of degree t, s the order of group $G[\chi_1]$, and u the number of irreducible characters of degree t in the decomposition of $\chi_1\chi_1^*$. Then, we have $t^2=s+ut$ from $\chi_1\chi_1^*=\sum_{g\in G[\chi_1]}g+\sum_i m(\chi_i,\chi_1\chi_1^*)\chi_i$. It follows that $t|s$, which implies $t|m$. $\square$

Lemma 5 *Suppose that $|G(H^*)|=2$ and $s=|X_2|$. Then at least one of the following conditions holds:*

(a) H has a quotient Hopf algebra of dimension $12, 24$ or 60;

(b) $G[\chi_2]=G(H^*)$ for every $\chi_2\in X_2$, and $2+4s$ divides $\dim H$.

Proof The result follows from Lemma 1(1) and Lemma 3(1). $\square$

Lemma 6 *Suppose that H has an irreducible character of degree 2 and does not have irreducible characters of degree 4. If 12 does not divide $\dim H$ or H does not have irreducible characters of degree 3, then $n+4s$ divides $\dim H$, where $n=|G(H^*)|$, $s=|X_2|$.*

Proof By assumption and Lemma 3(1), we have $G[\chi]\neq\{\varepsilon\}$ for all $\chi\in X_2$. We shall show that $\chi'\chi^*$ is a sum of irreducible characters of degrees 1 and 2, for all $\chi',\chi\in X_2$. In fact, if there is an irreducible characters ψ of degree 3 such that $m(\psi,\chi'\chi^*)=1$, then there must exist a group-like element $g\in G(H^*)$ such that $m(g,\chi'\chi^*)=1$. Then we have $\chi'=g\chi$. It follows that

$$\chi'\chi^*=g\chi\chi^*=g(\varepsilon+g'+\phi)=g+gg'+g\phi,$$

where $g'\in G(H^*)$, ϕ is either a sum of two irreducible characters of degree 1 or an irreducible character of degree 2. Hence, $\chi'\chi^*$ is a sum of irreducible characters of degrees 1 and 2, which contradicts the assumption that ψ is an element in $\chi'\chi^*$.

As a result, all irreducible characters of degrees 1 and 2 span a standard subalgebra of $R(H)$. It follows that H has a quotient Hopf algebra of dimension $n+4s$. $\square$

Lemma 7 *Let G be a non-trivial subgroup of $G(H^*)$. If $G[\chi_t]=G$ for every $\chi_t\in X_t$, then $\chi_t\chi_t'$ is not irreducible for all $\chi_t,\chi_t'\in X_t$.*

Proof Let $g\in G$ and $\chi_t\in X_t$. Then, by assumption, $g\chi_t=\chi_t$ and $g^{-1}\chi_t^*=\chi_t^*$. This means that $g\chi_t=\chi_t g=\chi_t$.

If $\chi_t\chi_t'=\psi$ is irreducible, we have

$$\begin{aligned}\psi\psi^*&=\chi_t(\chi_t'\chi_t'^*)\chi_t^*=\chi_t(\sum_{g\in G}g+\phi)\chi_t^*\\&=\sum_{g\in G}\chi_t g\chi_t^*+\chi_t\phi\chi_t^*=\sum_{g\in G}\chi_t\chi_t^*+\chi_t\phi\chi_t^*\\&=|G|\chi_t\chi_t^*+\chi_t\phi\chi_t^*,\end{aligned}$$

where we write $\chi_t'\chi_t'^*=\sum_{g\in G}g+\phi$. This means that $m(\varepsilon,\psi\psi^*)\geq|G|>1$, which is impossible. $\square$

Lemma 8 *Let H be a finite-dimensional semisimple Hopf algebra over k.*

(1) If 60 does not divide $\dim H$, then H cannot be of type $(1,1;2,m;\cdots)$ as an algebra.

(2) If 12 does not divide $\dim H$ and $2+4s$ does not divide $\dim H$, then H cannot be of type $(1,2;2,s;\ldots)$ as an algebra.

(3) If 12 does not divide $\dim H$, $m+4s$ does not divide $\dim H$ and H does not have irreducible characters of degree 4, then H cannot be of type $(1,m;2,s;\ldots)$ as an algebra.

(4) If $m+4s$ does not divide $\dim H$ and H does not have irreducible characters of degree 3 and 4, then H cannot be of type $(1,m;2,s;\cdots)$ as an algebra.

(5) If n_1 does not divide $\dim H$ or $n_i d_i^2$ $(2 \le i \le s)$, then H cannot be of type $(1,n_1;d_2,n_2;\ldots;d_s,n_s)$ as an algebra.

Proof Parts (1),(2) follow from Lemma 3(2), Lemma 5, respectively. Parts (3), (4) follow from Lemma 6. Part (5) follows from Lemma 1(2). $\square$

3. Semisimple Hopf algebras of dimension $66, 70$ and 78

3.1. Semisimple Hopf algebras of dimension 66

Theorem 2 *Let H be a semisimple Hopf algebra of dimension 66. Then, as an algebra, H is of one of the following types:*

$$(1,66),(1,3;3,7),(1,2;2,16),(1,6;2,15),(1,22;2,11).$$

In particular, H is of Frobenius type.

Proof According to Lemma 8 and a precise calculation, H has possibly one of the following types:

$$(1,2;8,1),(1,2;4,4),(1,1;4,1;7,1),(1,3;3,3;6,1),(1,6;2,3;4,3),$$

$$(1,66),(1,3;3,7),(1,2;2,16),(1,6;2,15),(1,22;2,11).$$

We shall eliminate first five types. The first two types can be eliminated by Lemma 4. The third type can be eliminated by Lemma 2.

Suppose that H is of type $(1,3;3,3;6,1)$ as an algebra. Let ψ be the unique irreducible character of degree 6. If there is an irreducible character χ of degree 3 such that $m(\chi,\psi^2)=1$, then $m(\chi,\psi^2)=m(\psi,\chi\psi)=1$. This means that $\chi\psi=\psi+\phi$, where $m(\psi,\phi)=0$ and $\deg\phi=12$. Since $|X_3|=3$, there must exist an irreducible character χ' of degree 3 such that $m(\chi',\phi)\ge 2$. Hence, $m(\chi',\chi\psi)=m(\chi'^*,\psi\chi^*)=m(\psi,\chi'^*\chi)\ge 2$. It is a contradiction.

It follows that we have four possible decompositions of ψ^2:

(1) $\psi^2=\varepsilon+g+g^2+3\chi_1+4\psi$;

(2) $\psi^2=\varepsilon+g+g^2+2\chi_1+3\chi_2+3\psi$;

(3) $\psi^2=\varepsilon+g+g^2+2\chi_1+2\chi_2+3\chi_3+2\psi$;

(4) $\psi^2=\varepsilon+g+g^2+3\chi_1+3\chi_2+3\chi_3+\psi$,

where χ_1,χ_2,χ_3 are distinct irreducible characters of degree 3, g is the generator of $G(H^*)$.

In case (1)–(3), the irreducible character (denoted by χ) of degree 3 such that $m(\chi, \psi^2) = 3$ is self-dual, since ψ^2 is self-dual. In case (4), we also have a self-dual irreducible character (also denoted by χ) of degree 3 such that $m(\chi, \psi^2) = 3$, since $|X_3|$ is odd and ψ^2 contains all irreducible characters of degree 3.

Then $m(\chi, \psi^2) = m(\psi, \chi\psi) = 3$ implies that $\chi\psi = 3\psi$. Multiplying on the left by χ, we have $(\varepsilon + g + g^2 + \phi)\psi = 9\psi$, where we write $\chi^2 = \varepsilon + g + g^2 + \phi$ with $\deg \phi = 6$. Hence, $\phi\psi = 6\psi$. Obviously, $\phi \neq \psi$ and $\phi = \chi' + \chi''$, where $\chi', \chi'' \in X_3$. It follows that $\chi'\psi = \chi''\psi = 3\psi$. If $\chi' = \chi'' = \chi$, then $\{\chi\} \cup \{\varepsilon, g, g^2\}$ spans a standard subalgebra of $R(H)$. It follows that H has a quotient Hopf algebra of dimension 12. It is impossible.

We then may assume that $\chi' \neq \chi$. From $m(\psi, \chi'\psi) = m(\chi', \psi^2) = 3$, we know that χ' also lies in the decomposition of ψ^2 with multiplicity 3. Hence, (4) is the only decomposition of ψ^2.

In this case, $m(\chi_i, \psi^2) = m(\psi, \chi_i\psi) = 3$ for every $\chi_i \in X_3$. We then reach a conclusion that $\chi_i\chi_j$ is a sum of irreducible characters of degrees 1 and 3. In fact, if $m(\psi, \chi_i\chi_j) = 1$, then $m(\chi_i, \psi\chi_j^*) = m(\chi_i^*, \chi_j\psi) = 1$, which contradicts the result above. Hence, all irreducible characters of degrees 1 and 3 span a standard subalgebra of $R(H)$. It follows that H has a quotient Hopf algebra of dimension 30. It is also impossible.

Suppose that H is of type $(1, 6; 2, 3; 4, 3)$ as an algebra. Let χ_i $(1 \leq i \leq 3)$ be all distinct irreducible characters of degree 2. Since the order of $G[\chi_i]$ divides both $|G(H^*)|$ and 4, we have $|G[\chi_i]| = 1$ or 2. In addition, H does not have irreducible characters of degree 3. It follows that $|G[\chi_i]| = 2$ for all $1 \leq i \leq 3$. By [3, Proposition 1.2.6], we know that $G(H^*)$ is abelian. Let G be the unique subgroup of $G(H^*)$ of order 2. As a result, we have $G[\chi_i] = G$ for all $1 \leq i \leq 3$. By Lemma 7, $\chi_i\chi_j$ is not irreducible. Hence, the irreducible characters of degree 1 and 2 span a standard subalgebra of $R(H)$. It follows that H has a quotient Hopf algebra of dimension 18. It is impossible by Nichols-Zoeller Theorem. This completes the proof. □

Remark The computation in the proof of Theorem 10 is partly handled by a computer. For example, it is easy to write a computer program by which one finds out all possible positive integers $1 = d_1, d_2, \ldots, d_s$ and $n_1, n_2, \ldots, n_s$ such that $66 = \sum_{i=1}^{s} n_i d_i^2$, and then one can eliminate those which cannot be algebra types of H by using Lemma 8. The computations in the followings are similar.

Corollary 1 *Let H be a semisimple Hopf algebra of dimension 66. If H is not a dual group algebra, then $G(H^*)$ is abelian.*

Proof The corollary follows from Theorem 2 and [3, Proposition 1.2.6]. □

3.2. Semisimple Hopf algebras of dimension 70

Theorem 3 *Let H be a semisimple Hopf algebra of dimension 70. Then, as an algebra, H is of one of the following types:*

$$(1, 70), (1, 14; 2, 14), (1, 10; 2, 15), (1, 2; 2, 17).$$

In particular, H is of Frobenius type.

Proof According to Lemma 8 and a precise calculation, H has possibly one of the following types:

$$(1,7;3,7),(1,2;4,2;6,1),(1,2;3,2;5,2),(1,2;3,4;4,2),$$

$$(1,14;2,14),(1,10;2,15),(1,2;2,17),(1,70).$$

We shall eliminate first four types. The first type can be eliminated by Lemma 4.

Suppose that H is of type $(1,2;4,2;6,1)$ as an algebra. Let χ_4,χ_4' be distinct irreducible characters of degree 4, χ_6 the unique irreducible character of degree 6, and $\{\varepsilon,g\}=G(H^*)$. Then $\{\varepsilon,g,\chi_4,\chi_4',\chi_6\}$ is a basis of $R(H)$. Let $\{e_1,e_2,\ldots,e_n\}$ be the set of primitive orthogonal idempotents in $R(H)$ such that e_1 is an integral in H^*. Note that ε,g commute with χ_4,χ_4',χ_6.

Suppose that $\chi_4^*=\chi_4',\chi_4'^*=\chi_4$. There is only one decomposition of $\chi_4\chi_4^*$:

$$\chi_4\chi_4^*=\chi_4\chi_4'=\varepsilon+g+\chi_4+\chi_4'+\chi_6.$$

From $m(\chi_6,\chi_4\chi_4')=m(\chi_4,\chi_6\chi_4'^*)=1$, we have $\chi_6\chi_4=\chi_4+2\chi_4'+2\chi_6$. From $m(\chi_4',\chi_6\chi_4)=m(\chi_4'^*,\chi_4^*\chi_6^*)=m(\chi_4,\chi_4'\chi_6)=m(\chi_4',\chi_4\chi_6)=2$, we have $\chi_4\chi_6=\chi_4+2\chi_4'+2\chi_6$. Hence, χ_4 commutes with χ_6. In addition, $\chi_4'\chi_4'^*=\chi_4'\chi_4=\varepsilon+g+\chi_4+\chi_4'+\chi_6$. This means that χ_4 also commutes with χ_4'. Therefore, χ_4 appears in the center of $R(H)$. Similarly, we can show that χ_4' also appears in the center of $R(H)$. Hence, $R(H)$ is commutative.

Suppose that $\chi_4^*=\chi_4,\chi_4'^*=\chi_4'$. There are three possible decompositions of $\chi_4\chi_4^*$:

$$\chi_4\chi_4^*=\varepsilon+g+2\chi_4+\chi_6;\chi_4\chi_4^*=\varepsilon+g+2\chi_4'+\chi_6;\chi_4\chi_4^*=\varepsilon+g+\chi_4+\chi_4'+\chi_6.$$

In all cases, we have $m(\chi_6,\chi_4\chi_4^*)=m(\chi_4,\chi_6\chi_4)=1$, which implies that $\chi_6\chi_4=\chi_4+2\chi_4'+2\chi_6$. Applying "$*$" on both sides, we have $\chi_4\chi_6=\chi_4+2\chi_4'+2\chi_6$. Hence, χ_4 commutes with χ_6. From $m(\chi_4',\chi_6\chi_4)=m(\chi_6,\chi_4'\chi_4)=2$, we have $\chi_4'\chi_4=2\chi_6+\varphi$, where φ is χ_4 or χ_4'. Applying "$*$" on both sides, we have $\chi_4\chi_4'=2\chi_6+\varphi$. Hence, χ_4 also commutes with χ_4'. Therefore, χ_4 appears in the center of $R(H)$. Similarly, we can show that χ_4' also appears in the center of $R(H)$. Hence, $R(H)$ is commutative.

Now, we reach a conclusion that $R(H)$ is semisimple and commutative. Hence, $\{e_1,e_2,\ldots,e_n\}$ is also a basis of $R(H)$ and $n=5$. By the well-known "Class Equation" [8, Theorem 1], $\dim(e_iH^*)$ divides $\dim H=70$ and

$$\dim H=1+\sum_{i=2}^{5}\dim(e_iH^*).$$

A direct check shows that it cannot happen.

Suppose that H is of type $(1,2;3,2;5,2)$ as an algebra. Let χ be an irreducible character of degree 3. By counting degrees, we have $\chi\chi^*=\varepsilon+\chi'+\psi$, where $\chi'\in X_3,\psi\in X_5$. From $m(\psi,\chi\chi^*)=m(\chi,\psi\chi)=1$, we have $\psi\chi=\chi+\phi$, where $m(\chi,\phi)=0$ and $\deg\phi=12$. It follows that $\phi=4\chi''$, where $\chi\neq\chi''\in X_3$. It is impossible since $m(\chi'',\psi\chi)=m(\psi,\chi''\chi^*)\le 1$.

Suppose that H is of type $(1,2;3,4;4,2)$ as an algebra. Let χ_3 be an irreducible character of degree 3, and $\{\varepsilon,g\}=G(H^*)$. Then we have two possible decompositions of $\chi_3\chi_3^*$:

(1) $\chi_3\chi_3^* = \varepsilon + 2\chi_4$; (2) $\chi_3\chi_3^* = \varepsilon + \chi_4 + \chi_4'$,

where χ_4, χ_4' are distinct irreducible characters of degree 4.

In case (1), we have $\chi_4\chi_3 = 2\chi_3 + m\chi_3' + n\chi_3''$ from $m(\chi_3, \chi_4\chi_3) = m(\chi_4, \chi_3\chi_3^*) = 2$, where m, n are non-negative integers such that $m + n = 2$, and χ_3', χ_3'' are different from χ_3. If $\chi_4\chi_3 = 2\chi_3 + \chi_3' + \chi_3''$ with $\chi_3' \neq \chi_3''$, then $\chi_3'\chi_3^* = \chi_4 + \chi_4' + g$ from $m(\chi_3', \chi_4\chi_3) = m(\chi_4, \chi_3'\chi_3^*) = 1$ since $\chi_3' \neq \chi_3$. Hence, $\chi_3' = g\chi_3$. Similarly, we obtain that $\chi_3'' = g\chi_3$. It follows that $\chi_3' = \chi_3''$, a contradiction. If $\chi_4\chi_3 = 2\chi_3 + 2\chi_3'$, then $m(\chi_3', \chi_4\chi_3) = m(\chi_4, \chi_3'\chi_3^*) = 2$ implies that $\chi_3'\chi_3^* = 2\chi_4 + g$ since $\chi_3' \neq \chi_3$. Hence, $\chi_3' = g\chi_3$. Then $\chi_3'\chi_3^* = g\chi_3\chi_3^* = g(\varepsilon + 2\chi_4)$ implies that $g\chi_4 = \chi_4$. Multiplying $\chi_4\chi_3 = 2\chi_3 + 2g\chi_3$ on the right by χ_3^* and using $g\chi_4 = \chi_4$, we have $2\chi_4^2 = 2\varepsilon + 2g + 7\chi_4$, which is impossible.

In case (2), we have $\chi_4\chi_3 = \chi_3 + \chi_3' + \chi_3'' + \chi_3'''$ or $\chi_4\chi_3 = \chi_3 + 2\chi_3' + \chi_3''$ from $m(\chi_4, \chi_3\chi_3^*) = m(\chi_3, \chi_4\chi_3) = 1$, where $\chi_3, \chi_3', \chi_3'', \chi_3'''$ are distinct irreducible characters of degree 3. For the first case, $m(\chi_3', \chi_4\chi_3) = m(\chi_4, \chi_3'\chi_3^*) = 1$ implies that $\chi_3'\chi_3^* = \chi_4 + \chi_4' + g$. Hence, $\chi_3' = g\chi_3$. Similarly, we have $\chi_3'' = g\chi_3$. It follows that $\chi_3' = \chi_3''$, a contradiction. For the second case, we have $\chi_3'' = g\chi_3$ similarly. In addition, we also have $\chi_3' = g\chi_3$ from $m(\chi_3', \chi_4\chi_3) = m(\chi_4, \chi_3'\chi_3^*) = 2$. Hence, $\chi_3' = \chi_3''$. It is also a contradiction. This completes the proof. □

Corollary 2 *Let H be a semisimple Hopf algebra of dimension 70. If H is not a dual group algebra, then $G(H^*)$ is abelian.*

Proof The result follows from [3, Proposition 1.2.6] and Theorem 3.

3.3. Semisimple Hopf algebras of dimension 78

Theorem 4 *Let H be a semisimple Hopf algebra of dimension 78. Then, as an algebra, H is of one of the following types:*

$$(1,2;2,1;6,2),(1,6;6,2),(1,2;2,1;3,4;6,1),(1,6;3,4;6,1),(1,2;2,1;3,8),$$

$$(1,6;3,8),(1,2;2,19),(1,6;2,18),(1,26;2,13),(1,78).$$

In particular, H is of Frobenius type.

Proof According to Lemma 8 and a precise calculation, H has possibly one of the following types:

$$(1,3;5,3),(1,1;3,3;5,2),(1,1;3,5;4,2),(1,1;4,1;5,1;6,1),(1,1;3,1;4,2;6,1),$$

$$(1,3;3,3;4,3),(1,1;3,4;4,1;5,1),(1,6;2,6;4,3),(1,2;2,6;4,1;6,1),(1,2;2,6;3,4;4,1),$$

$$(1,2;2,1;6,2),(1,6;6,2),(1,2;2,1;3,4;6,1),(1,6;3,4;6,1),(1,2;2,1;3,8),$$

$$(1,6;3,8),(1,2;2,19),(1,6;2,18),(1,26;2,13),(1,78).$$

We shall rule out first ten types. The first type can be ruled out by Lemma 4. The second and third types can be ruled out by Lemma 2.

Suppose that H is of type $(1,1;4,1;5,1;6,1)$ as an algebra. Let χ_4,χ_5,χ_6 be the irreducible characters of degree $4,5,6$, respectively. Then $\chi_4\chi_4^*=\varepsilon+\chi_4+\chi_5+\chi_6$ or $\chi_4\chi_4^*=\varepsilon+3\chi_5$. If the first case holds, then $m(\chi_6,\chi_4\chi_4^*)=m(\chi_4,\chi_6\chi_4)=1$. This implies that $\chi_6\chi_4=\chi_4+4\chi_5$. It follows that $m(\chi_5,\chi_6\chi_4)=m(\chi_6,\chi_5\chi_4)=4$. It is impossible. If the second case holds, then $m(\chi_5,\chi_4\chi_4^*)=m(\chi_4,\chi_5\chi_4)=3$. This implies that $\chi_5\chi_4=3\chi_4+\psi$, where $m(\chi_4,\psi)=0$ and $\deg\psi=8$. It is also impossible.

Suppose that H is of type $(1,1;3,1;4,2;6,1)$ as an algebra. Let χ_3 be the unique irreducible character of degree 3. Then $\chi_3\chi_3^*=\varepsilon+\chi_4+\chi_4'$ or $\chi_3\chi_3^*=\varepsilon+2\chi_4$, where χ_4,χ_4' are distinct irreducible characters of degree 4. If the first case holds, then $m(\chi_4,\chi_3\chi_3^*)=m(\chi_3,\chi_4\chi_3)=1$. This implies that $\chi_4\chi_3=\chi_3+\psi$, where $m(\chi_3,\psi)=0$ and $\deg\psi=9$. It is impossible. If the second case holds, then $m(\chi_4,\chi_3\chi_3^*)=m(\chi_3,\chi_4\chi_3)=2$. This implies that $\chi_4\chi_3=2\chi_3+\chi_6$, where χ_6 is the unique irreducible character of degree 6. Then $m(\chi_6,\chi_4\chi_3)=m(\chi_3,\chi_6\chi_4)=1$. This implies that $\chi_6\chi_4=\chi_3+\psi$, where $m(\chi_3,\psi)=0$ and $\deg\psi=21$. It is also impossible.

Suppose that H is of type $(1,3;3,3;4,3)$ as an algebra. Let χ_3 be an irreducible character of degree 3 and $G(H^*)=\{\varepsilon,g_1,g_2\}$. Then $\chi_3\chi_3^*=\varepsilon+\chi_4+\psi$ or $\chi_3\chi_3^*=\varepsilon+2\chi_4$ or $\chi_3\chi_3^*=\varepsilon+g_1+g_2+\varphi_1+\varphi_2$, where χ_4,ψ are distinct irreducible characters of degree 4, φ_1,φ_2 are irreducible characters of degree 3.

If the first case holds, then $m(\chi_4,\chi_3\chi_3^*)=m(\chi_3,\chi_4\chi_3)=1$. This implies that $\chi_4\chi_3=\chi_3+2\chi_3'+\chi_3''$, where χ_3,χ_3',χ_3'' are distinct irreducible characters of degree 3. From $m(\chi_3'',\chi_4\chi_3)=m(\chi_4,\chi_3''\chi_3^*)=1$, we have

(i) $\chi_3''\chi_3^*=\chi_4+\varphi+g_1+g_2$, where $\varphi\in X_3$;

(ii) $\chi_3''\chi_3^*=\chi_4+\chi_4'+g_3$, where $\chi_4\neq\chi_4'\in X_4$, $\varepsilon\neq g_3\in G(H^*)$.

In case (i), $\chi_3''=g_1\chi_3=g_2\chi_3$. This implies that $g_1^{-1}g_2$ lies in the decomposition of $\chi_3\chi_3^*$. It is a contradiction.

In case (ii), $\chi_3''=g_3\chi_3$. On the other hand, $m(\chi_4,\chi_3'\chi_3^*)=m(\chi_3',\chi_4\chi_3)=2$. This implies that $\chi_3'\chi_3^*=2\chi_4+g_4$, where $\varepsilon\neq g_4\in G(H^*)$. Hence, $\chi_3'=g_4\chi_3$. Multiplying $\chi_4\chi_3=\chi_3+2g_4\chi_3+g_3\chi_3$ on the right by χ_3^*, we have $\chi_4^2+\chi_4\chi_4'=\varepsilon+2g_4+g_3+2g_4\chi_4+2g_4\chi_4'+g_3\chi_4+g_3\chi_4'$. If ε lies in the decomposition of χ_4^2, then $\chi_4=\chi_4^*$, and hence g_3,g_4 do not lie in the decomposition of χ_4^2 since $G[\chi_4]$ is trivial. Counting degrees on both sides, we know that it is impossible. Similarly, the case that ε lies in the decomposition of $\chi_4\chi_4'$ is also impossible.

If the second case holds, then $m(\chi_3,\chi_4\chi_3)=m(\chi_4,\chi_3\chi_3^*)=2$. This implies that

(i) $\chi_4\chi_3=2\chi_3+2\chi_3'$, where $\chi_3\neq\chi_3'\in X_3$.

(ii) $\chi_4\chi_3=2\chi_3+\chi_3'+\chi_3''$, where χ_3,χ_3',χ_3'' are distinct elements in X_3.

In case (i), $m(\chi_3',\chi_4\chi_3)=m(\chi_4,\chi_3'\chi_3^*)=2$. This implies that $\chi_3'\chi_3^*=2\chi_4+g$ for some $\varepsilon\neq g\in G(H^*)$. Hence, $\chi_3'=g\chi_3$. Multiplying $\chi_4\chi_3=2\chi_3+2g\chi_3$ on the right by χ_3^*, we have $2\chi_4^2=2\varepsilon+2g+3\chi_4+4g\chi_4$. This means that both ε and g lie in the decomposition of χ_4^2. It is impossible.

In case (ii), we have $\chi_3'\chi_3^*=\chi_4+\chi_4'+g$ and $\chi_3''\chi_3^*=\chi_4+\chi_4''+h$, where $\chi_4',\chi_4''\in X_4$, and $\{\varepsilon,g,h\}=G(H^*)$. Hence, $\chi_3'=g\chi_3$, $\chi_3''=h\chi_3$. Multiplying $\chi_4\chi_3=2\chi_3+g\chi_3+h\chi_3$ on the right by χ_3^*, we have $2\chi_4^2=2\varepsilon+g+h+3\chi_4+2g\chi_4+2h\chi_4$. It is also impossible.

If the third case holds, we consider the decomposition of $\varphi\chi_3^*$, where $\chi_3 \neq \varphi \in X_3$. If there exists an irreducible character χ_4 of degree 4 such that $m(\chi_4, \varphi\chi_3^*) > 0$, we have $\varphi\chi_3^* = \chi_4+\chi_4'+g$ or $\varphi\chi_3^* = 2\chi_4 + h$ or $\varphi\chi_3^* = \chi_4 + \chi_3' + g_1 + g_2$, where $\chi_4 \neq \chi_4' \in X_4$, $g, h \in G(H^*)$ and $\chi_3' \in X_3$. Then $g\chi_3 = \varphi$ or $h\chi_3 = \varphi$ or $g_1\chi_3 = \varphi$. It follows that $\chi_3\chi_3^* = g^{-1}\chi_4 + g^{-1}\chi_4' + \varepsilon$ or $\chi_3\chi_3^* = 2h^{-1}\chi_4 + \varepsilon$ or $\chi_3\chi_3^* = g_1^{-1}\chi_4 + g_1^{-1}\chi_3' + \varepsilon + g_1^{-1}g_2$, which contradicts the assumption. Hence, $G(H^*) \cup X_3$ spans a standard subalgebra of $R(H)$. It follows that H has a quotient Hopf algebra of dimension 30. It is impossible by Nichols–Zoeller Theorem.

Suppose that H is of type $(1,6;2,6;4,3)$ as an algebra. By Proposition 1.2.6 in [3], $G(H^*)$ is abelian. Let $G = \{1, g\}$ be the unique subgroup of $G(H^*)$ of order 2. Then $G[\chi_2] = G$ for all $\chi_2 \in X_2$. By Lemma 8, $\chi_2\chi_2'$ is not irreducible for all $\chi_2, \chi_2' \in X_2$. Hence, the irreducible character of degree $1,2$ spans a standard subalgebra of $R(H)$. It follows that H has a quotient Hopf algebra of dimension 30. It is also impossible by Nichols-Zoeller Theorem.

Suppose that H is of type $(1,2;2,6;4,1,6,1)$ as an algebra. In this case, $G[\chi_2] = G(H^*)$ for all $\chi_2 \in X_2$ and $\chi_2\chi_2'$ is not irreducible for all $\chi_2, \chi_2' \in X_2$.

We now consider the decomposition of $\chi_4\chi_4^*$, where χ_4 is the unique element in X_4. If there exists $\chi_2 \in X_2$ such that $m(\chi_2, \chi_4\chi_4^*) = 1$, then $m(\chi_4, \chi_2\chi_4) = 1$. This implies that $\chi_2\chi_4 = \chi_4 + \chi_2' + \chi_2''$, where $\chi_2', \chi_2'' \in X_2$. If $\chi_2' = \chi_2''$, then $m(\chi_2', \chi_2\chi_4) = m(\chi_4, \chi_2'^*\chi_2) = 2$, which is impossible. Hence, $\chi_2' \neq \chi_2''$, and $m(\chi_4, \chi_2'^*\chi_2) = 1$. This implies that $\chi_2'^*\chi_2 = \chi_4$ is irreducible, which contradicts the result above. Hence, $\chi_4\chi_4^* = \varepsilon + g + 2\chi_2 + 2\chi_2' + \chi_6$ or $\chi_4\chi_4^* = \varepsilon + g + 2\chi_4 + \chi_6$ or $\chi_4\chi_4^* = \varepsilon + g + 2\chi_2 + \chi_4 + \chi_6$, where $\chi_2, \chi_2' \in X_2$, $\chi_6 \in X_6$ and $\{\varepsilon, g\} = G(H^*)$.

From $m(\chi_6, \chi_4\chi_4^*) = m(\chi_4, \chi_6\chi_4) = 1$, we have $\chi_6\chi_4 = \chi_4 + \varphi$, where $m(\chi_4, \varphi) = 0$ and $\deg\varphi = 20$. If there exists $\chi_2 \in X_2$ such that $m(\chi_2, \chi_6\chi_4) = m \geq 0$, then $m(\chi_2, \chi_6\chi_4) = m(\chi_6, \chi_2\chi_4)$ implies that $0 \leq m \leq 1$. If $m = 1$, then $\chi_2\chi_4 = \chi_6 + \chi_2''$ for some $\chi_2'' \in X_2$. This will deduce that $\chi_4 = \chi_2^*\chi_2''$, which is a contradiction. Therefore, φ only contains several copies of χ_6, which is absurd since $\deg\varphi = 20$.

Suppose that H is of type $(1,1;3,4;4,1;5,1)$ as an algebra. Let χ_3 be an irreducible character of degree 3, and χ_4, χ_5 be the unique irreducible characters of degree $4,5$, respectively. If $\chi_3\chi_3^* = \varepsilon + 2\chi_4$, then $m(\chi_4, \chi_3\chi_3^*) = m(\chi_3, \chi_4\chi_3) = 2$ implies that $\chi_4\chi_3 \overset{(1)}{=} 2\chi_3 + \varphi_1 + \varphi_2$,where $\chi_3, \varphi_1, \varphi_2$ are distinct elements in X_3. $m(\varphi_1, \chi_4\chi_3) = m(\chi_4, \varphi_1\chi_3^*) = 1$ implies that $\varphi_1\chi_3^* = \chi_4 + \chi_5$. $m(\chi_5, \varphi_1\chi_3^*) = m(\varphi_1, \chi_5\chi_3) = 1$ implies that $\chi_5\chi_3 = \varphi_1 + \varphi_3 + \chi_4 + \chi_5$, where $\varphi_1 \neq \varphi_3 \in X_3$. $m(\chi_4, \chi_5\chi_3) = m(\chi_5, \chi_4\chi_3^*) = 1$ implies that $\chi_4\chi_3^* \overset{(2)}{=} \chi_5 + \chi_4 + \varphi_4$, where $\varphi_4 \in X_3$. $m(\chi_4, \chi_4\chi_3^*) = m(\chi_4, \chi_3\chi_4) = m(\chi_3, \chi_4^2) = 1$ implies that $\chi_4^2 = \varepsilon + \chi_3 + \omega$, where $m(\chi_3, \omega) = 0$ and $\deg\omega = 12$. Hence, we have

(i) $\chi_4^2 = \varepsilon + \chi_3 + 3\chi_4$, or

(ii) There exists $\chi_3 \neq \chi_3' \in X_3$ such that $m(\chi_3', \chi_4^2) > 0$.

In case (i), $\chi_3 = \chi_3^*$ since χ_4^2 is self-dual. Then (1) and (2) give a contradiction.

In case (ii). If $\chi_3 = \chi_3^*$, then (1) and (2) also give a contradiction. So, there must exist $\chi_3 \neq \chi_3'' \in X_3$ such that $\chi_3 = \chi_3''^*$ and $m(\chi_3'', \chi_4^2) = m > 0$. Then $m(\chi_3'', \chi_4\chi_4^*) = m(\chi_4, \chi_3''\chi_4) = m(\chi_4, \chi_4\chi_3''^*) = m$. This implies that $\chi_4\chi_3''^* = \chi_4\chi_3 = m\chi_4 + \psi$, where $m(\chi_4, \psi) = 0$ and

$\deg \psi = 12 - 4m$, which contradicts (1).

As a conclusion, for all $\chi_3 \in X_3$, we have $\chi_3\chi_3^* = \varepsilon + \chi_3' + \chi_5$, where $\chi_3' \in X_3$. Then $m(\chi_5, \chi_3\chi_3^*) = m(\chi_3, \chi_5\chi_3) = 1$ implies that $\chi_5\chi_3 = \chi_3 + \chi_4 + \chi_5 + \varphi_1$, where $\chi_3 \neq \varphi_1 \in X_3$. $m(\chi_4, \chi_5\chi_3) = m(\chi_5, \chi_4\chi_3^*) = 1$ implies that $\chi_4\chi_3^* = \chi_5 + \chi_4 + \varphi_2$, where $\varphi_2 \in X_3$. $m(\chi_4, \chi_4\chi_3^*) = m(\chi_4, \chi_3\chi_4) = m(\chi_3, \chi_4^2) = 1$ implies that χ_3 lies in the decomposition of χ_4^2 with multiplicity 1 for every $\chi_3 \in X_3$. Hence, $\chi_4^2 = \varepsilon + \chi_3^1 + \chi_3^2 + \chi_3^3 + \chi_4^4 + \psi$, where $\{\chi_3^1, \chi_3^2, \chi_3^3, \chi_4^4\} = X_3$ and $\psi \notin X_3$. It is impossible.

Suppose that H is of type $(1,2;2,6;3,4;4,1)$ as an algebra. By Theorem 1 and Lemma 7, $G[\chi] = G(H^*)$ for all $\chi \in X_2$ and $\chi_2\chi_2'$ is not irreducible for all $\chi_2, \chi_2' \in X_2$.

Let χ_4 be the unique irreducible character of degree 4. If there exists $\chi_3 \in X_3$ such that $m(\chi_3, \chi_4\chi_4^*) = 1$, then $m(\chi_4, \chi_3\chi_4) = 1$. This implies that $\chi_3\chi_4 = \chi_4 + \varphi$, where $m(\chi_4, \varphi) = 0$ and $\deg\varphi = 8$. If there exists $\chi_2 \in X_2$ such that $m(\chi_2, \varphi) = 1$, then $m(\chi_2, \chi_3\chi_4) = m(\chi_3, \chi_2\chi_4) = 1$. This implies that $\chi_2\chi_4 = \chi_3 + \chi_2' + \chi_3'$ for some $\chi_2' \in X_2, \chi_3' \in X_3$. This will deduce a contradiction $\chi_2^*\chi_2' = \chi_4$. If there exists $\chi_2 \in X_2$ such that $m(\chi_2, \varphi) = 2$, then $m(\chi_2, \chi_3\chi_4) = m(\chi_3, \chi_2\chi_4) = 2$. This implies that $\chi_2\chi_4 = 2\chi_3 + \chi_2'$ for some $\chi_2' \in X_2$. There is also a contradiction $\chi_2^*\chi_2' = \chi_4$. Hence, φ is a sum of irreducible characters of degree 3. It is impossible since $\deg \varphi = 8$.

If there exists $\chi_3 \in X_3$ such that $m(\chi_3, \chi_4\chi_4^*) = 2$, then $\chi_3\chi_4 = 2\chi_4 + \chi_2' + \chi_2''$ for some $\chi_2', \chi_2'' \in X_2$. It is impossible by the discussion above.

If there exists $\chi_2 \in X_2$ such that $m(\chi_2, \chi_4\chi_4^*) = 1$, then $\chi_1\chi_4 = \chi_4 + \chi_2' + \chi_2''$ for some $\chi_2', \chi_2'' \in X_2$. It is impossible, too.

Hence, we reach a conclusion that every irreducible character of degree 2 or 3 which appears in the decomposition of $\chi_4\chi_4^*$ must have multiplicity 2 or 3. Counting degrees, we find this cannot happen. This completes the proof. $\square$

Corollary 3 *Let H be a semisimple Hopf algebra of dimension* 78. *If H is not a dual group algebra, then $G(H^*)$ is abelian.*

Proof The result follows from [3, Proposition 1.2.6] and Theorem 4.

Remark Let q, r, p be distinct prime numbers such that q and r divide $p-1$. Andruskiewitsch and Natale [15] constructed two classes of non-trivial non-isomorphic semisimple Hopf algebras of dimension pqr. As an algebra, one is of type $(1, rq; q, \frac{r(p-1)}{q})$ and the other is of type $(1, rq; r, \frac{q(p-1)}{r})$. Therefore, non-trivial semisimple Hopf algebras of dimension 78 do exist. However, the existence of non-trivial semisimple Hopf algebras of dimension 66 and 70 is still unknown.

Acknowledgments The author would like to thank the referees for their valuable comments and suggestions.

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