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Finite p-Groups with a Cyclic Subgroup of Index p^3

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Abstract We classify up to isomorphism those finite p-groups, for odd primes p, which contain a cyclic subgroup of index p^3.

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1. Introduction

Classifying certain classes of finite p-groups defined by their subgroup structure is important in the study of finite p-groups. For example, finite p-groups with "large cyclic subgroups" have been investigated by many authors. A well-known important result is the classification of finite p-groups with a cyclic subgroup of index p, which was obtained by Burnside [5] in 1897. Hua and Tuan [7] classified finite p-groups with a cyclic subgroup of index p^2 in terms of generators and defining relations for $p > 2$ in 1940, and Bai [1] did this for $p = 2$ in 1985. Ninomiya [14] in 1994 also classified these p-groups. Berkovich and Janko [2, pp. 274–276] in 2008 classified again these p-groups in a structural form, Berkovich for $p > 2$, and Janko for $p = 2$. It is natural to classify finite p-groups with a cyclic subgroup of index p^3. In fact, early in the last century, Neikirk [13] classified these p-groups for $p > 2$, and McKelden [12] for $p = 2$. However, their results are incorrect and some groups are missing from their papers. Titov [16] in 1980 classified these p-groups in some special cases for $p > 3$. The objective in this paper is to classify these p-groups completely in terms of generators and define relations for $p > 2$ up to isomorphism. This also solves Problem 12.11.13 proposed by Xu and Qu in [18].

For convenience, we introduce some new notation. Assume G is a group of order p^n. We say G is a $\mathcal{C}_t$-group if G has a cyclic subgroup of index p^t and all subgroups of index p^{t-1} of G are not cyclic. In other words, G is a $\mathcal{C}_t$-group if $\exp G = p^{n-t}$.

We sketch the classification: If G is a regular $\mathcal{C}_3$-group of order p^n, then the type of G is one of the following: $(e, 3)$, $(e, 2, 1)$ or $(e, 1, 1, 1)$, where $e = n - 3$. If the type of G is $(e, 3)$, then G

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is a metacyclic p-group. Metacyclic p-groups have been classified by Xu in [19]. So it is enough to determine which ones have type $(e,3)$. If the type of G is $(e,2,1)$, then G was classified by Ji et al. in [10]. However, the list of groups given there is incorrect and we correct their results. If the type of G is $(e,1,1,1)$, then G was classified by Zhang et al. in [21], so it suffices for us to classify irregular $\mathcal{C}_3$-groups of order p^n with p odd.

If G is an irregular $\mathcal{C}_3$-group of order p^n, then we classify $\mathcal{C}_3$-groups using different methods. First we prove that $p=3$. We then proceed by examining two cases, depending on whether $|G|<3^7$, or $|G|\geq 3^7$. If $|G|<3^7$, then the desired groups are completely listed in the "SmallGroups" library of Magma [3,4], and we only need to select those that satisfy our conditions. If $|G|\geq 3^7$, we classify the desired groups by considering whether $Z(G)$ is cyclic or not. The methods we use are cyclic extensions and central extensions, respectively.

2. Preliminaries

Let G be a finite p-group. Then G is inner abelian if G is non-abelian, but every proper subgroup of G is abelian; G is metabelian if $G''=1$; G is regular if $(ab)^p=a^pb^p{c_3}^p\cdots {c_m}^p$ for arbitrary $a,b\in G$, where $c_i\in\langle a,b\rangle'$; and G is p^s-abelian if for arbitrary $a,b\in G$, $(ab)^{p^s}=a^{p^s}b^{p^s}$, where s is a positive integer.

Assume H and N are finite groups. Then G is an extension of N by H if there exists a normal subgroup $M\lhd G$ such that $N\cong M$ and $G/M\cong H$. if H is cyclic, we say that G is a cyclic extension; if $M\subseteq Z(G)$, we say G is a central extension. And we say G is a central extension of degree p if G is a central extension of N by H and $|N|=p$.

If G is a finite group, then $\exp G$ denotes the smallest positive integer n such that $g^n=1$ for all $g\in G$, $c(G)$ denotes the nilpotency class of G, and $o(b)$ denotes the order of an element b of G. We use G_n to denote the nth term of the lower central series of G.

Assume A and B are subgroups of a group G. We say that G is a central product of A and B if $G=AB$ and $[A,B]=1$, we denote this by $A*B$.

Assume G is a finite p-group, $\exp G=p^e$. For $0\leq s\leq e$, let

$$\Omega_s(G)=\langle g\in G|g^{p^s}=1\rangle,\quad \mho_s(G)=\langle g^{p^s}|g\in G\rangle.$$

Let $p^{\omega_s(G)}=|\Omega_s(G)/\Omega_{s-1}(G)|$. Then $(\omega_1,\omega_2,\dots,\omega_e)$ is an invariant of G. For arbitrary integer i, $1\leq i\leq\omega$, let e_i be the number satisfying $\omega_t\geq i$ for $\omega_t\in\{\omega_1,\omega_2,\dots,\omega_e\}$. Then $e_1\geq e_2\geq\cdots\geq e_\omega$. The *type* of G is $(e_1,e_2,\dots,e_\omega)$.

Let G be a p-group and let $b_1,\dots,b_\omega$ be elements of G. We call $(b_1,\dots,b_\omega)$ a uniqueness basis (a U.B.) of G if every $g\in G$ can be uniquely expressed in the following form:

$$g=b_1^{\alpha_1}b_2^{\alpha_2}\cdots b_\omega^{\alpha_\omega},$$

where $0\leq\alpha_j<o(b_j)$, $j=1,\dots,\omega$.

For convenience, we summarize known results which are used in this paper.

Lemma 1 ([15]) *Assume G is an inner abelian p-group. Then G is one of the following pairwise* *non-isomorphic groups:*

(1) Q_8;

(2) $M_p(n,m) = \langle a, b | a^{p^n} = b^{p^m} = 1,\ a^b = a^{1+p^{n-1}} \rangle$, $n \geq 2$ *(metacyclic); or*

(3) $M_p(n,m,1) = \langle a,b,c | a^{p^n} = b^{p^m} = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1\rangle$, $n \geq m$ *and* $p = 2$, $m + n \geq 3$ *(non-metacyclic).*

Lemma 2 ([17, Lemma 3]) *Assume G is a metabelian p-group, $a, b \in G$. For arbitrary integers i, j, let*

$$[ia, jb] = [a, b, \underbrace{a, \ldots, a}_{i-1}, \underbrace{b, \ldots, b}_{j-1}].$$

Then for arbitrary integers m, n,

$$[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^{\binom{m}{i}\binom{n}{j}},$$

$$(ab^{-1})^m = a^m \Pi_{i+j \leq m} [ia, jb]^{\binom{m}{i+j}} b^{-m}, \quad m \geq 2.$$

Theorem 3 ([8]) *Assume G is a group of order p^n, $\exp G = p^{n-\alpha}$, $p \geq 3$, $n \geq 2\alpha + 1$. Then*

(1) There exist $\alpha + 1$ elements $b, b_1, b_2, \ldots, b_\alpha$ in G such that for all $g \in G$, g can be uniquely expressed as $g = b_\alpha^{\lambda_\alpha} \ldots b_1^{\lambda_1} b^\lambda$, $1 \leq \lambda_\alpha \leq p, \ldots, 1 \leq \lambda_1 \leq p$, $1 \leq \lambda \leq p^{n-\alpha}$, where $o(b) = p^{n-\alpha}$, $o(b_i) \leq p^i$.

(2) For all $b_1, b_2 \in G$, $(b_1 b_2)^{p^\alpha} = b_1^{p^\alpha} b_2^{p^\alpha}$.

(3) $|G'| \leq p^\alpha$, $|G_3| \leq p^{\alpha-1}$.

(4) $b^{p^\alpha} \in Z(G)$.

Theorem 4 ([7]) *Assume G is a group of order p^{n+2}, $\exp G = p^n$, where $p \geq 3$, $n \geq 4$. Then G is one of the following pairwise non-isomorphic groups:*

$H_1 = \langle a, b, c \mid a^{p^n} = 1, b^p = 1, c^p = 1, [a,b] = [a,c] = [b,c] = 1\rangle$;

$H_2 = \langle a, b \mid a^{p^n} = 1, b^{p^2} = 1, [a,b] = 1\rangle$;

$H_3 = \langle a, b, c \mid a^{p^n} = 1, b^p = 1, c^p = 1, [a,b] = a^{p^{n-1}}, [a,c] = [b,c] = 1\rangle$;

$H_4 = \langle a, b, c \mid a^{p^n} = 1, b^p = 1, c^p = 1, [b,c] = a^{p^{n-1}}, [a,b] = [a,c] = 1\rangle$;

$H_5 = \langle a, b, c \mid a^{p^n} = 1, b^p = 1, [a,b] = c, c^p = 1, [a,c] = [b,c] = 1\rangle \cong M_3(n,1,1)$;

$H_6 = \langle a, b, c \mid a^{p^n} = 1, b^p = 1, [a,b] = c, c^p = 1, [a,c] = a^{p^{n-1}}, [b,c] = 1\rangle$;

$H_7 = \langle a, b, c \mid a^{p^n} = 1, b^p = 1, [a,b] = c, c^p = 1, [b,c] = a^{p^{n-1}}, [a,c] = 1\rangle$;

$H_8 = \langle a, b, c \mid a^{p^n} = 1, b^p = 1, [a,b] = c, c^p = 1, [b,c] = a^{\nu p^{n-1}}, [a,c] = 1\rangle$ *where ν is a fixed quadratic non-residue modulo p;*

$H_9 = \langle a, b \mid a^{p^n} = 1, b^{p^2} = 1, [a,b] = a^{p^{n-1}}\rangle \cong M_3(n,2)$;

$H_{10} = \langle a, b \mid a^{p^n} = 1, b^{p^2} = 1, [a,b] = a^{p^{n-2}}\rangle$;

$H_{11} = \langle a, b \mid a^{p^n} = 1, b^{p^2} = 1, [a,b] = b^p\rangle \cong M_3(2,n)$;

$H_{12} = \langle a, b \mid a^{p^n} = b^{p^2} = 1, [a,b] = a^{p^{n-2}} b^p, [a, b^p] = a^{p^{n-1}}\rangle$.

Lemma 5 ([9, p. 322, Satz 10.2]) *Assume G is a finite p-group. If G satisfies one of the following conditions, then G is regular:*

(1) $c(G) < p$,

(2) $p > 2$ *and* G' *is cyclic,*

(3) $\exp G = p$,

(4) $|G/\mho_1(G)| < p^p$.

Lemma 6 ([18, p. 132, Theorem 5.2.2, p. 134, Theorem 5.2.11])

(1) Assume G *is finite 3-group generated by two elements. Then* G *is regular if and only if* G' *is cyclic.*

(2) A finite 3-group is regular if and only if every subgroup generated by two elements has a cyclic derived subgroup.

Lemma 7 ([18, p. 71, Theorem 2.2.15]) *Assume* G *is a finite* p*-group. If* $Z(G')$ *is cyclic, then* G' *is cyclic.*

Lemma 8 ([18, p. 78, Corollary 2.4.5]) *Assume* G *is a finite* p*-group,* $p > 2$*. If* G *can be expressed as a product of two cyclic subgroups, then* G *is metacyclic.*

Lemma 9 ([6]) *Assume* G *is a regular* p*-group with the type* $(e_1, e_2, \ldots, e_\omega)$*. Then* G *has a uniqueness basis* $(b_1, b_2, \ldots, b_r)$*, where* $r = \omega$ *and* $o(b_i) = p^{e_i}$.

Lemma 10 ([19, 20]) *Every metacyclic* p*-group* G *(*p *an odd prime) has the following presentation:*

$$\langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, [a, b] = a^{p^r} \rangle,$$

where r, s, t, u *are non-negative integers with* $r \geq 1$ *and* $u \leq r$*. Different values of the parameters* r, s, t, u *with the above conditions give non-isomorphic metacyclic* p*-groups. Furthermore,* G *is split if and only if either* $s = 0$*, or* $t = 0$*, or* $u = 0$*. Also* $|G| = p^{2r+2s+t+u}$ *and* $\exp G = p^{r+s+t+u}$.

3. A classification of finite regular $\mathcal{C}_3$-groups

Assume G is a regular $\mathcal{C}_3$-group of order p^n, $p > 2$, $e = n - 3$. Obviously, G is a $\mathcal{C}_3$-group if and only if the type of G is one of the following: $(e, 3)$, $(e, 2, 1)$ or $(e, 1, 1, 1)$, where $e = n - 3$. So classifying regular $\mathcal{C}_3$-groups of order p^n is equivalent to classifying regular p-groups whose types are $(e, 3)$, $(e, 2, 1)$ or $(e, 1, 1, 1)$, respectively. The following three theorems give the classification of regular $\mathcal{C}_3$-groups.

Theorem 11 *Assume* G *is a* p*-group of order* p^n*,* $p > 2$*,* $e = n - 3$*. Then* G *is a regular* p*-group whose type is* $(e, 3)$ *if and only if* G *is one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b \mid a^{p^3} = 1, b^{p^e} = 1, [a, b] = a^p \rangle$, $e \geq 3$;

(2) $\langle a, b \mid a^{p^4} = 1, b^{p^{e-1}} = a^{p^3}, [a, b] = a^p \rangle$, $e \geq 4$;

(3) $\langle a, b \mid a^{p^3} = 1, b^{p^e} = 1, [a, b] = a^{p^2} \rangle$, $e \geq 3$;

(4) $\langle a, b \mid a^{p^4} = 1, b^{p^{e-1}} = a^{p^3}, [a, b] = a^{p^2} \rangle$, $e \geq 4$;

(5) $\langle a, b \mid a^{p^5} = 1, b^{p^{e-2}} = a^{p^3}, [a, b] = a^{p^2} \rangle$, $e \geq 5$;

(6) $\langle a, b \mid a^{p^3} = 1, b^{p^e} = 1, [a, b] = 1 \rangle$, $e \geq 3$;

(7) $\langle a,b \mid a^{p^4}=1, b^{p^{e-1}}=a^{p^3}, [a,b]=a^{p^3}\rangle$, $e\geq 4$;

(8) $\langle a,b \mid a^{p^5}=1, b^{p^{e-2}}=a^{p^3}, [a,b]=a^{p^3}\rangle$, $e\geq 5$;

(9) $\langle a,b \mid a^{p^6}=1, b^{p^{e-3}}=a^{p^3}, [a,b]=a^{p^3}\rangle$, $e\geq 6$.

Proof Since G is regular and the type of G is $(e,3)$, G has a uniqueness basis (b_1,b_2) such that $G=\langle b_1\rangle\langle b_2\rangle$. Since $p>2$, G is metacyclic by Lemma [8]. By Lemma [10],

$$G\cong\langle a,b \ \mid \ a^{p^{r+s+u}}=1,\ b^{p^{r+s+t}}=a^{p^{r+s}},\ [a,b]=a^{p^r}\rangle,$$

where r,s,t,u are non-negative integers with $r\geq 1$ and $u\leq r$. Different values of the parameters r, s, t and u give non-isomorphic metacyclic p-groups. Furthermore, $|G|=p^{2r+2s+t+u}$ and $\exp G=p^{r+s+t+u}$.

Since the type invariant of G is $(e,3)$, we have $e=r+s+t+u$, $r+s=3$.

Obviously, $e\geq r+s+u$. Then $t=e-r-s-u\geq 0$ is uniquely determined by r, s, u. Since $r+s=3$, $r\geq 1$, $u\leq r$, we obtain the groups listed in the theorem by considering all possible values for r, s, u.

Conversely, by checking we know the conclusion is true. □

Theorem 12 *Assume G is a p-group of order p^n, $p>2$. Then G is a regular p-group whose type invariant is $(e,2,1)$ if and only if G is isomorphic to one of the following pairwise non-isomorphic groups, where ν denotes a fixed quadratic non-residue modulo p.*

(1) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=[c,b]=1\rangle$, *where* $p\geq 3, e\geq 2$;

(2) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=1, [c,b]=a^{p^{e-1}}\rangle$, *where* $p\geq 5, e\geq 2$;

(3) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=1, [c,b]=b^p\rangle$, *where* $p\geq 5, e\geq 2$;

(4) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=1, [c,b]=a^{\nu p^{e-1}}\rangle$, *where* $p\geq 5, e\geq 2$;

(5) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=a^{p^{e-1}}, [c,b]=1\rangle$, *where* $p\geq 5, e\geq 3$;

(6) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=b^p, [c,b]=1\rangle$, *where* $p\geq 5, e\geq 3$;

(7) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=b^{\nu p}, [c,b]=1\rangle$, *where* $p\geq 5, e\geq 3$;

(8) $\langle a,b,c \mid a^{p^2}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=b^{-p}, [c,b]=a^pb^{hp}\rangle$, *where* $p\geq 5$, $h=0,\ldots,\frac{p-1}{2}$;

(9) $\langle a,b,c \mid a^{p^2}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=b^{-\nu p}, [c,b]=a^{\nu p}b^{2\nu p}\rangle$, *where* $p\geq 5$;

(10) $\langle a,b,c \mid a^{p^2}=1, b^{p^2}=1, c^p=1, [b,a]=c, [c,a]=b^{-p}, [c,b]=a^{\nu p}b^{hp}\rangle$, *where* $p\geq 5$, $h=0,\ldots,\frac{p-1}{2}$;

(11) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [b^p,a]=1, [c,a]=b^p, [c,b]=a^{p^{e-1}}\rangle$, *where* $p\geq 5, e\geq 3$;

(12) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [b^p,a]=1, [c,a]=b^{\nu p}, [c,b]=a^{p^{e-1}}\rangle$, *where* $p\geq 5$, $e\geq 3$;

(13) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [b^p,a]=1, [c,a]=b^p, [c,b]=a^{\nu p^{e-1}}\rangle$, *where* $p\geq 5$, $e\geq 3$;

(14) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [b^p,a]=1, [c,a]=b^{\nu p}, [c,b]=a^{\nu p^{e-1}}\rangle$, *where* $p\geq 5$, $e\geq 3$;

(15) $\langle a,b,c \mid a^{p^e}=1, b^{p^2}=1, c^p=1, [b,a]=c, [b^p,a]=1, [c,a]=a^{ip^{e-1}}, [c,b]=b^p\rangle$, *where* $p\geq 5$, $e\geq 3$, $i=1,\dots,p-1$;

(16) $\langle a,b,c \mid a^{p^2}=1, b^{p^2}=1, c^p=1, [b,a]=c, [b^p,a]=1, [c,a]=a^p, [c,b]=b^p\rangle$, *where* $p\geq 5$;

(17) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^{p^2}=1, [b,a]=c, c^p=a^{p^{e-1}}, [c,a]=1, [c,b]=a^{kp^{e-1}}\rangle$, *where* $p\geq 3$, $e\geq 3$, $k=0,\dots,p-1$;

(18) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^{p^2}=1, [b,a]=c, c^p=a^{p^{e-1}}, [c,a]=a^{p^{e-1}}, [c,b]=1\rangle$, *where* $p\geq 3$, $e\geq 3$;

(19) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^{p^2}=1, [b,a]=c, c^p=a^{p^{e-1}}, [c,a]=b^p, [c,b]=a^{kp^{e-1}}\rangle$, *where* $p\geq 5$, $e\geq 3$, $k=0,\dots,p-1$;

(20) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^{p^2}=1, [b,a]=c, c^p=a^{p^{e-1}}, [c,a]=b^{\nu p}, [c,b]=a^{kp^{e-1}}\rangle$, *where* $p\geq 5, e\geq 3$, $k=0,\dots,p-1$;

(21) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-1}}, [c,a]=[c,b]=1\rangle$, *where* $p\geq 3$, $e\geq 2$;

(22) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,c]=a^{p^{e-1}}, [b,a]=[c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 2$;

(23) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,c]=b^p, [b,a]=[c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 2$;

(24) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=b^p, [c,a]=[c,b]=1\rangle$, *where* $p\geq 3$, $e\geq 3$;

(25) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [c,a]=a^{p^{e-1}}, [b,a]=[c,b]=1\rangle$, *where* $p\geq 3$, $e\geq 3$;

(26) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [c,a]=b^p, [b,a]=[c,b]=1\rangle$, *where* $p\geq 3$, $e\geq 3$;

(27) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=1, [b,c]=a^{p^{e-1}}b^{hp}, [c,a]=b^p\rangle$, *where* $p\geq 3$, $e\geq 2$, $h=0,\dots,\frac{p-1}{2}$;

(28) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=1, [b,c]=a^{p^{e-1}}b^{hp}, [c,a]=b^{\nu p}\rangle$, *where* $p\geq 3$, $e\geq 2$, $h=0,\dots,\frac{p-1}{2}$;

(29) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=b^p, [b,c]=1, [c,a]=a^{p^{e-1}}\rangle$, *where* $p\geq 3$, $e\geq 2$;

(30) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-1}}, [b,c]=1, [c,a]=b^p\rangle$, *where* $p\geq 3$, $e\geq 2$;

(31) $\langle a,b,c \mid a^{p^2}=b^{p^2}=c^p=1, [b,a]=1, [b,c]=b^{-p}, [c,a]=a^p\rangle$, *where* $p\geq 3$;

(32) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-1}}, [b,c]=b^p, [c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 3$;

(33) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=b^p, [b,c]=a^{p^{e-1}}, [c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 3$;

(34) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-2}}, [b,c]=a^{p^{e-1}}, [c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 3$;

(35) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-2}}, [b,c]=[c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 3$;

(36) $\langle a,b,c | a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-2}}b^p, [b^p,a]=a^{p^{e-1}}, [b,c]=a^{p^{e-1}}, [c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 4$;

(37) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-2}}b^p, [b^p,a]=a^{p^{e-1}}, [b,c]=[c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 4$;

(38) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-2}}, [b,c]=1, [c,a]=b^p\rangle$, *where* $p\geq 5$, $e\geq 3$;

(39) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-2}}, [b,c]=a^{p^{e-1}}, [c,a]=b^p\rangle$, *where* $p\geq 5$, $e\geq 3$;

(40) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=a^{p^{e-2}}, [b,c]=a^{\nu p^{e-1}}, [c,a]=b^p\rangle$, *where* $p\geq 5$, $e\geq 3$;

(41) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1, [b,a]=[b,c]=[c,a]=1\rangle$, *where* $p\geq 3$, $e\geq 2$.

Proof The groups satisfying the hypothesis were classified incorrectly in [10]. We correct this work. The errors are as follows.

(i) The defining relation $[b^p, a] = 1$ in the following 5 groups listed in Table 1 of [10] is missing. We add it and get groups (11)–(15).

(11) $\langle a, b \mid a^{p^e} = 1, b^{p^2} = 1, c^p = 1, [b, a] = c, [c, a] = b^p, [c, b] = a^{p^{e-1}}\rangle$, where $p \geq 5$, $e \geq 3$;

(12) $\langle a, b \mid a^{p^e} = 1, b^{p^2} = 1, c^p = 1, [b, a] = c, [c, a] = b^{\nu p}, [c, b] = a^{p^{e-1}}\rangle$, where $p \geq 5$, $e \geq 3$;

(13) $\langle a, b \mid a^{p^e} = 1, b^{p^2} = 1, c^p = 1, [b, a] = c, [c, a] = b^p, [c, b] = a^{\nu p^{e-1}}\rangle$, where $p \geq 5$, $e \geq 3$;

(14) $\langle a, b \mid a^{p^e} = 1, b^{p^2} = 1, c^p = 1, [b, a] = c, [c, a] = b^{\nu p}, [c, b] = a^{\nu p^{e-1}}\rangle$, where $p \geq 5$, $e \geq 3$;

(15) $\langle a, b \mid a^{p^e} = 1, b^{p^2} = 1, c^p = 1, [b, a] = c, [c, a] = a^{ip^{e-1}}, [c, b] = b^p\rangle$, where $p \geq 5$, $e \geq 3$, $i = 1, \ldots, p-1$.

(ii) By [21, Theorem 5.1], the following group in Table 1 of [10] is missing, which is group (16).

$\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, c^p = 1, [b, a] = c, [b^p, a] = 1, [c, a] = a^p, [c, b] = b^p\rangle$, where $p \geq 5$;

(iii) The authors of [10] omit the case $k = 0$ of the groups (1), (2), (4) listed in Table 2 of [10], so the following 3 groups are missing. They are the case $k = 0$ of groups (17), (19), (21).

$\langle a, b \mid a^{p^e} = b^{p^2} = c^{p^2} = 1, [b, a] = c, c^p = a^{p^{e-1}}, [c, a] = 1, [c, b] = 1\rangle$, where $p \geq 3$, $e \geq 3$;

$\langle a, b \mid a^{p^e} = b^{p^2} = c^{p^2} = 1, [b, a] = c, c^p = a^{p^{e-1}}, [c, a] = b^p, [c, b] = 1\rangle$, where $p \geq 5$, $e \geq 3$;

$\langle a, b \mid a^{p^e} = b^{p^2} = c^{p^2} = 1, [b, a] = c, c^p = a^{p^{e-1}}, [c, a] = b^{\nu p}, [c, b] = 1\rangle$, where $p \geq 5$, $e \geq 3$.

(iv) One of the groups (11) in Table 3 of [10] is isomorphic to one of groups (7), (8). The following is the proof.

The groups (11) are $\langle a, b, c \mid a^{p^e} = b^{p^2} = c^p = 1, [b, a] = 1, [b, c] = b^{hp}, [c, a] = a^{p^{e-1}}\rangle$, where $p \geq 3$, $e \geq 3$, $h = 1, \ldots, p-1$.

Replacing a by ab, we have $\langle a, b, c \mid a^{p^e} = b^{p^2} = c^p = 1, [b, a] = 1, [b, c] = b^{hp}, [c, a] = a^{p^{e-1}} b^{-hp}\rangle$. Replacing b by $b^{-h} a^{p^{e-2}}$, we have $\langle a, b, c \mid a^{p^e} = b^{p^2} = c^p = 1, [b, a] = 1, [b, c] = b^{hp} a^{-hp^{e-1}}, [c, a] = b^p\rangle$.

Letting s be an integer satisfying $-sh \equiv 1 (\mathrm{mod}\, p)$ and replacing b by b^s, we have $\langle a, b, c \mid a^{p^e} = b^{p^2} = c^p = 1, [b, a] = 1, [b, c] = b^{hp} a^{p^{e-1}}, [c, a] = b^{s^{-1}p}\rangle$, where s^{-1} is the inverse of s in the field Z_p.

Let t be an integer satisfying $s^{-1}t^2 \equiv 1$ or $\nu (\mathrm{mod}\, p)$, where ν is a fixed quadratic non-residue modulo p. Replacing a by a^t, c by c^t, we have $\langle a, b, c \mid a^{p^e} = b^{p^2} = c^p = 1, [b, a] = 1, [b, c] = b^{thp} a^{p^{e-1}}, [c, a] = b^p\rangle$, or $\langle a, b, c \mid a^{p^e} = b^{p^2} = c^p = 1, [b, a] = 1, [b, c] = b^{thp} a^{p^{e-1}}, [c, a] = b^{\nu p}\rangle$.

If $th < p/2$, the groups are isomorphic to some groups in (7) and (8). If $th > p/2$, replacing a by a^{-1}, c by c^{-1}, then the groups are also isomorphic to some groups in (7) and (8).

(v) By [21, Theorem 5.1], the following group in Table 3 of [10] is missing, which is group (31).

$\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [b, a] = 1, [b, c] = b^{-p}, [c, a] = a^p\rangle$, where $p \geq 3$.

(vi) The defining relation $[b^p, a] = a^{p^{e-1}}$ in the following 2 groups listed in the Table 4 of [10] is missing. We add it and get groups (36), (37).

(3) $\langle a, b, c | a^{p^e} = b^{p^2} = c^p = 1, [b, a] = a^{p^{e-2}} b^p, [b, c] = a^{p^{e-1}}, [c, a] = 1\rangle$, where $p \geq 3$, $e \geq 4$;

(4) $\langle a, b, c \mid a^{p^e} = b^{p^2} = c^p = 1, [b, a] = a^{p^{e-2}} b^p, [b, c] = [c, a] = 1\rangle$, where $p \geq 3$, $e \geq 4$.

(vii) The order of groups (5) in Table 4 of [10] is not p^{e+3}, so we remove them.

The groups (5) are $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1,[b,a]=a^{p^{e-2}},[b,c]=b^p,[c,a]=1\rangle$, where $p\geq 3$, $e\geq 3$. It is easy to see that $|\langle a,b\rangle|=p^{e+2}$ and $G/\langle a,b\rangle\cong\langle c\rangle$. Thus, if the order of groups (5) is p^{e+3}, then $[b^c,a^c]=(a^{p^{e-2}})^c$. On the other hand, it follows from G is regular and Lemma 2 that $[b^c,a^c]=[b^{1+p},a]=([b,a])^p[b^p,a]=a^{p^{e-2}}a^{p^{e-1}}\neq(a^{p^{e-2}})^c$, a contradiction.

(viii) In the following 3 groups in the Table 4 of [10] $p\geq 3$ should replace $p\geq 5$. Thus we get the groups (38), (39), (40) of Theorem.

(6) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1,[b,a]=a^{p^{e-2}},[b,c]=1,[c,a]=b^p\rangle$, where $p\geq 3$, $e\geq 3$;

(7) $\langle a,b,c \mid a^{p^e}=b^{p^2}=c^p=1,[b,a]=a^{p^{e-2}},[b,c]=a^{p^{e-1}},[c,a]=b^p\rangle$, where $p\geq 3$, $e\geq 3$;

(8) $\langle a,b,c| a^{p^e}=b^{p^2}=c^p=1,[b,a]=a^{p^{e-2}},[b,c]=a^{\nu p^{e-1}},[c,a]=b^p\rangle$, where $p\geq 3$, $e\geq 3$. The reason is as follows: if $p=3$, then $\langle c,a\rangle'$ is not cyclic. By Lemma 6, G is irregular.

Finally, those groups listed in the statement of the theorem are pairwise non-isomorphic and satisfy all hypotheses. $\square$

Theorem 13 *Assume G is a p-group of order p^n. Then G is a regular p-group whose type is $(e,1,1,1)$ if and only if G is isomorphic to one of the following pairwise non-isomorphic groups, where ν denotes a fixed quadratic non-residue modulo p and $p\geq 5$, $e\geq 2$ unless otherwise stated.*

(1) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=c,[c,a]=1,[c,b]=d,[d,a]=[d,b]=1\rangle$;

(2) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=c,[c,a]=d,[c,b]=1,[d,a]=[d,b]=1\rangle$; *where* $e\geq 1$;

(3) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,\ [b,a]=c,\ [c,a]=d,\ [c,b]=a^{ip^{e-1}},\ [d,a]=[d,b]=1\rangle$, *where* $i=1$ *or* ν;

(4) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=c,[c,a]=a^{p^{e-1}},[c,b]=d,[d,a]=[d,b]=1\rangle$;

(5) *If* $p\equiv 3(\bmod 4)$, $\langle a,b,c,d|a^{p^e}=b^p=c^p=d^p=1,[b,a]=c,[c,a]=d,[c,b]=a^{ip^{e-1}},[d,a]=a^{p^{e-1}},[d,b]=[d,c]=1\rangle$, *where* $i=0,1$ *or* ν;

If $p\equiv 1(\bmod 4)$,

$\langle a,b,c,d|a^{p^e}=b^p=c^p=d^p=1,[b,a]=c,[c,a]=d,[c,b]=a^{ip^{e-1}},[d,a]=a^{p^{e-1}},[d,b]=1\rangle$, *where* $i=0,1,\ \nu,\mu$ *or* ρ *and* $1,\nu,\mu,\rho$ *are the coset representations of the subgroup generated by biquadratic residues of* $\mathbb{Z}_p^*$;

(6) *If* $p\equiv 2(\bmod 4)$,

$\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=c,[c,a]=a^{kp^{e-1}},[c,b]=d,[d,a]=1,[d,b]=a^{p^{e-1}}\rangle$, *where* $k=0$ *or* 1;

If $p\equiv 1(\bmod 3)$,

$\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=c,[c,a]=a^{kp^{e-1}},[c,b]=d,[d,a]=1,[d,b]=a^{sp^{e-1}}\rangle$, *where* $k=0$ *or* 1, $s=1,\mu$ *or* ν *and* $1,\nu,\mu$ *are the coset representations of the subgroup generated by cubic residues of* $\mathbb{Z}_p^*$;

(7) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=d,[c,a]=[c,b]=1,[d,a]=[d,b]=[d,c]=1\rangle$, *where* $p\geq 3$, $e\geq 1$;

(8) $\langle a,b,c,d|a^{p^e}=b^p=c^p=d^p=1,[b,a]=1,[c,a]=1,[c,b]=d,[d,a]=[d,b]=[d,c]=1\rangle$, *where* $p\geq 3$;

(9) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=a^{p^{e-1}},[c,a]=d,[c,b]=1,[d,a]=[d,b]=[d,c]=1\rangle$, *where* $p\geq 3$;

(10) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=1,[c,a]=a^{p^{e-1}},[c,b]=d,[d,a]=[d,b]=[d,c]=1\rangle$, *where* $p\geq 3$;

(11) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=1,[c,a]=d,[c,b]=a^{p^{e-1}},[d,a]=[d,b]=[d,c]=1\rangle$, *where* $p\geq 3$;

(12) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=1,[c,a]=1,[c,b]=d,[d,a]=[d,b]=1,[d,c]=a^{p^{e-1}}\rangle$;

(13) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=1,[c,a]=a^{p^{e-1}},[c,b]=d,[d,a]=1,[d,b]=a^{p^{e-1}},[d,c]=1\rangle$;

(14) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=d,[c,a]=1,[c,b]=1,[d,a]=1,[d,b]=a^{ip^{e-1}},[d,c]=1\rangle$, *where* $i=1$ *or* ν;

(15) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=d,[c,a]=a^{p^{e-1}},[c,b]=1,[d,a]=1,[d,b]=a^{ip^{e-1}},[d,c]=1\rangle$, *where* $i=1$ *or* ν;

(16) $\langle a,b,c,d|a^{p^e}=b^p=c^p=d^p=1,[b,a]=d,[c,a]=1,[c,b]=1,[d,a]=a^{p^{e-1}},[d,b]=[d,c]=1\rangle$;

(17) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=d,[c,a]=1,[c,b]=a^{p^{e-1}},[d,a]=a^{p^{e-1}},[d,b]=[d,c]=1\rangle$;

(18) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=[c,a]=[c,b]=[d,a]=[d,b]=[d,c]=1\rangle$, *where* $p\geq 3$, $e\geq 1$;

(19) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=[c,a]=[c,b]=[d,a]=[d,c]=1,[d,b]=a^{p^{e-1}}\rangle$, *where* $p\geq 3$;

(20) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=[c,a]=[c,b]=[d,b]=[d,c]=1,[d,a]=a^{p^{e-1}}\rangle$, *where* $p\geq 3$;

(21) $\langle a,b,c,d \mid a^{p^e}=b^p=c^p=d^p=1,[b,a]=[c,a]=[d,b]=[d,c]=1,[c,b]=[d,a]=a^{p^{e-1}}\rangle$, *where* $p\geq 3$.

Proof By Theorem 3.4 in [21] we obtain the desired p-groups for $p\geq 5$. For $p=3$, we only need to select those regular 3-groups from that list.

If $e=1$, then G is one of the groups of order p^4 and $\exp G=p$. These groups occur in (2), (7) and (18). If $e\geq 2$, then by Theorem 3.4 in [21] we obtained the desired p-groups for the case of $p\geq 5$. For $p=3$, obviously, $d(G)\leq 4$. If $d(G)=2$, then, by using the same approach as [21, Theorem 3.1], we get G' is not cyclic. By Lemma 6(1) there do not exist such 3-groups satisfying the theorem's condition. If $d(G)=3$ or 4, we observe that the method in [21, Theorem 3.2, 3.3] is still effective for $p=3$. By checking the list of Theorems 3.2 and 3.3 in [21] using Lemma 6(2), we learn that these groups occur in (7)–(11) and (18)–(21).

The groups we obtained are pairwise non-isomorphic and satisfy the hypothesis. □

4. A classification of finite irregular $\mathcal{C}_3$-groups

Assume G is an irregular $\mathcal{C}_3$-group of order p^n. By Lemma 15 below we have $p=3$. Since our argument depends on Theorem 3, we proceed in two cases: $|G| \geq 3^7$ and $|G| < 3^7$.

Lemma 14 *Assume G is a $\mathcal{C}_3$-group of order p^n. If $p \geq 5$, then G is regular.*

Proof Since $\exp G = p^{n-3}$, there exists $a \in G$ such that $o(a) = p^{n-3}$. Since $\langle a^p \rangle \leq \mho_1(G)$ and $p \geq 5$, $|G/\mho_1(G)| \leq p^4 < p^p$. By Lemma 5(4), G is regular. □

Lemma 15 *Assume G is an irregular $\mathcal{C}_3$-group of order p^n and $p > 2$. Then*

(1) $p = 3$;

(2) G' *is not cyclic;*

(3) $c(G) \geq 3$;

(4) G *has a subgroup H generated by two elements with H' being not cyclic.*

Proof (1) follows by Lemma 14; (2) and (3) follows by Lemma 5; (4) follows by Lemma 6. □

Lemma 16 *If G is an irregular $\mathcal{C}_3$-group of order 3^n and $n \geq 7$, then*

(1) G' *is one of* $C_3 \times C_3$, $C_3 \times C_3 \times C_3$ *or* $C_9 \times C_3$,

(2) $\exp G_3 = 3$, *where G_3 is the third term of the lower central series of G,*

(3) G *is 9-abelian,*

(4) *if $a \in G$, then $a^9 \in Z(G)$.*

Proof (1) By Theorem 3(3) we have $|G'| \leq 3^3$. If G' is not abelian, then G' has order 3^3. So $Z(G')$ is cyclic. By Lemma 7 we have G' is cyclic, which contradicts Lemma 15(2). So G' is noncyclic abelian and the conclusion follows.

(2) We consider the quotient group $G/\Omega_1(G')$. Since $|(G/\Omega_1(G'))'| = |G'/\Omega_1(G')| \leq 3$, we have $|(G/\Omega_1(G'))_3| = 1$. Therefore, $G_3 \leq \Omega_1(G')$, that is, $\exp G_3 = 3$.

(3) and (4) follow from the formula of Lemma 2. □

Lemma 17 *Let H_i be the groups listed in Theorem 4. Then*

(1) H_i' *have the following possible cases:*

$H_1' \cong H_2' = 1$; $H_3' \cong H_4' \cong H_9' = \langle a^{p^{n-1}} \rangle \cong C_p$; $H_5' = \langle c \rangle \cong C_p$; $H_{11}' = \langle b^p \rangle \cong C_p$;

$H_6' \cong H_7' \cong H_8' = \langle c \rangle \times \langle a^{p^{n-1}} \rangle \cong C_p \times C_p$; $H_{10}' = \langle a^{p^{n-2}} \rangle \cong C_{p^2}$; $H_{12}' = \langle a^{p^{n-2}} b^p \rangle \cong C_{p^2}$.

(2) $Z(H_i)$ *have the following possible cases:*

$Z(H_1) = H_1, Z(H_2) = H_2$; $Z(H_3) \cong Z(H_5) = \langle a^p \rangle \times \langle c \rangle \cong C_{p^{n-1}} \times C_p$;

$Z(H_9) \cong Z(H_{11}) = \langle a^p \rangle \times \langle b^p \rangle \cong C_{p^{n-1}} \times C_p$; $Z(H_4) = \langle a \rangle \cong C_{p^n}$;

$Z(H_6) \cong Z(H_7) \cong Z(H_8) = \langle a^p \rangle \cong C_{p^{n-1}}$; $Z(H_{10}) \cong Z(H_{12}) = \langle a^{p^2} \rangle \cong C_{p^{n-2}}$.

(3) $c(H_i)$ *have the following possible cases:*

$c(H_1) = c(H_2) = 1$; $c(H_3) = c(H_4) = c(H_5) = c(H_9) = c(H_{11}) = 2$;

$c(H_6) = c(H_7) = c(H_8) = c(H_{10}) = c(H_{12}) = 3$.

(4) $\Omega(H_i)$ *have the following possible cases:*

$\Omega_i(H_1) \cong \Omega_i(H_3) \cong \Omega_i(H_4) \cong \Omega_i(H_5) \cong \Omega_i(H_6) \cong \Omega_i(H_7) \cong \Omega_i(H_8) = \langle a^{p^{n-i}}, b, c \rangle$;

$\Omega_i(H_2) \cong \Omega_i(H_9) \cong \Omega_i(H_{10}) \cong \Omega_i(H_{11}) \cong \Omega_i(H_{12}) = \langle a^{p^{n-i}}, b^{p^{2-i}} \rangle$, *where* $1 \leq i \leq 2$;

$\Omega_i(H_2) \cong \Omega_i(H_9) \cong \Omega_i(H_{10}) \cong \Omega_i(H_{11}) \cong \Omega_i(H_{12}) = \langle a^{p^{n-i}}, b \rangle$, *where* $i > 2$.

(5) $\mho(H_i)$ *have the following possible cases:*
$\mho_i(H_1) \cong \mho_i(H_3) \cong \mho_i(H_4) \cong \mho_i(H_5) \cong \mho_i(H_6) \cong \mho_i(H_7) \cong \mho_i(H_8) = \langle a^{p^i}, b^{p^i}, c^{p^i}\rangle$;
$\mho_i(H_2) \cong \mho_i(H_9) \cong \mho_i(H_{10}) \cong \mho_i(H_{11}) \cong \mho_i(H_{12}) = \langle a^{p^i}, b^{p^i}\rangle$.

Proof It is straightforward by checking the list of groups listed in Theorem 4. □

4.1. Irregular $\mathcal{C}_3$-groups of order $\geq 3^7$ whose center is not cyclic

Lemmas 18, 19 and 20 below are simple, but we use them several times.

Lemma 18 *Assume G is a finite p-group, $N \leq Z(G)$, $|N| = p$, $G/N = \langle \bar{x_1}, \bar{x_2}, \ldots, \bar{x_s}\rangle$, $M = \langle x_1, x_2, \ldots, x_s\rangle$. Then $G = M$ or $G = M \times N$. Furthermore, $G = M$ if and only if $d(G) = d(G/N)$; and $G = M \times N$ if and only if $d(G) = d(G/N) + 1$.*

Lemma 19 *Assume G is a finite p-group, $N \leq Z(G)$, $|N| = p$ and $G/N \cong H$. Then $H' \cong G'$ or $H' \cong G'/N$.*

Lemma 20 *Assume G is a $\mathcal{C}_3$-group of order p^n whose center is not cyclic, $p > 2$. Then there exists a central subgroup N of order p in G such that $G/N \cong H_i$, where H_i is one of the groups listed in Theorem 4.*

Proof By hypothesis there exists $b \in G$ such that $o(b) = p^{n-3}$. Since $Z(G)$ is not cyclic, there exists $N \leq \Omega_1(Z(G))$, $|N| = p$ and $N \cap \langle b\rangle = 1$. Thus $\langle b\rangle N/N \cong \langle b\rangle/N \cap \langle b\rangle \cong \langle b\rangle$ is a cyclic subgroup of order p^{n-3} of G/N. That is, $\exp(G/N) = p^{n-2}$. Since $p > 2$, G/N is isomorphic to some H_i, where H_i is one of the groups listed in Theorem 4. □

Theorem 21 *Assume G is an irregular group of order 3^{n+3} whose center is not cyclic, $n \geq 4$ and $G' \cong C_3 \times C_3$. Then G is a $\mathcal{C}_3$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b, c, x \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [a, c] = a^{3^{n-1}}, [b, c] = 1, x^3 = 1, [x, a] = [x, b] = 1\rangle$;

(2) $\langle a, b, c, x \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [b, c] = a^{3^{n-1}}, [a, c] = 1, x^3 = 1, [x, a] = [x, b] = 1\rangle$;

(3) $\langle a, b, c, x \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [b, c] = a^{2\times 3^{n-1}}, [a, c] = 1, x^3 = 1, [x, a] = [x, b] = 1\rangle$;

(4) $\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [c, b] = 1, [c, a] = b^3\rangle$;

(5) $\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [c, b] = 1, [c, a] = b^{2\times 3}\rangle$;

(6) $\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [c, b] = b^3, [c, a] = 1\rangle$;

(7) $\langle a, b, c, d \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [c, b] = d, [c, a] = 1, d^3 = 1, [d, a] = [d, b] = 1\rangle$;

(8) $\langle a, b, c, d \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [c, a] = d, [c, b] = 1, d^3 = 1, [d, a] = [d, b] = 1\rangle$;

(9) $\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [a, c] = a^{3^{n-1}}, [b, c] = 1\rangle$;

(10) $\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [a, c] = 1, [b, c] = a^{3^{n-1}}\rangle$;

(11) $\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [a, c] = 1, [b, c] = a^{2\times 3^{n-1}}\rangle$.

Proof By Lemma 20, G has a central subgroup N of order 3 such that $G/N \cong H_i$, where H_i is one of the groups listed in Theorem 4, $1 \le i \le 12$. For convenience, let $N = \langle x\rangle$. By Lemma 18, $G = M$ or $G = M \times N$, where M is the group listed in Lemma 18.

Case 1 $G = M \times N$.

Since G is irregular and $G/N \cong M \cong H_i$, H_i is irregular. Thus, H_i is one of H_6, H_7, or H_8. Therefore, G is isomorphic to $H_6 \times C_3$, $H_7 \times C_3$, or $H_8 \times C_3$, the groups (1), (2) and (3) listed in the theorem.

Case 2 $G = M$.

Subcase 1 $G/N \cong H_1$, H_2, H_{10} or H_{12}.

If $G/N \cong H_1$ or H_2, then by Lemma 17 we have ${H_1}' = 1$ and ${H_2}' = 1$. By Lemma 19, $|G'| = 1$ or $|G'| = 3$. This contradicts the hypothesis. If $G/N \cong H_{10}$ or H_{12}, then ${H_{10}}' \cong {H_{12}}' \cong C_9$ by Lemma 17. This contradicts the hypothesis again. Thus, this subcase is impossible.

Subcase 2 $G/N \cong H_3$ or H_4.

If $G/N \cong H_3$, by Lemma 4 we can assume $G/N = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^{3^n} = \bar{1}, \bar{b}^3 = \bar{1}, \bar{c}^3 = \bar{1}, [\bar{a}, \bar{b}] = \bar{a}^{3^{n-1}}, [\bar{a}, \bar{c}] = [\bar{b}, \bar{c}] = \bar{1}\rangle$. Then $G = M = \langle a, b, c\rangle$. By Lemma 17 we have $(G/N)' = \langle \bar{a}^{3^{n-1}}\rangle$. It follows that $G' \le \langle a^{3^{n-1}}, N\rangle$. By Theorem 16(4) we have $\langle a^{3^{n-1}}\rangle \le Z(G)$. So $G' \le Z(G)$, hence $c(G) = 2$. This contradicts Lemma 15(3). If $G/N \cong H_4$, then a contradiction arises by a similar argument. So this subcase is likewise impossible.

Subcase 3 $G/N \cong H_5, H_9$ or H_{11}.

By Theorem 4, $H_5 \cong M_3(n, 1, 1), H_9 \cong M_3(n, 2)$, and $H_{11} \cong M_3(2, n)$. By hypothesis, we have $N \le Z(G)$ and $|N| = p$. Hence G is a central extension of degree p of an inner abelian p-group. Such groups were classified by [11]. So we need only to pick those $\mathcal{C}_3$-groups G from [11, Theorems 10, 11] that satisfy $G' \cong C_3 \times C_3$. We get the following five groups:

$\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [c, b] = 1, [c, a] = b^3\rangle$;

$\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [c, b] = 1, [c, a] = b^{2\times 3}\rangle$;

$\langle a, b, c \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = c, c^3 = 1, [c, b] = b^3, [c, a] = 1\rangle$;

$\langle a, b, c, d \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [c, b] = d, [c, a] = 1, d^3 = 1, [d, a] = [d, b] = 1\rangle$;

$\langle a, b, c, d \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [c, a] = d, [c, b] = 1, d^3 = 1, [d, a] = [d, b] = 1\rangle$.

These are the groups (4)–(8).

Subcase 4 $G/N \cong H_6$, H_7 or H_8.

If $G/N \cong H_6$, then by Theorem 4 we have $G = M = \langle a, b, c, x \mid a^{3^n} = x^i, b^3 = x^j, [a, b] = cx^k, c^3 = x^l, [a, c] = a^{3^{n-1}}x^m, [b, c] = x^h, x^3 = 1, [x, a] = [x, b] = 1\rangle$, where $i, j, k, l, m, h =$ 0,1 or 2 and they are not all simultaneously zero.

Since G is a $\mathcal{C}_3$-group, $a^{3^n} = 1$. Since $G' \cong C_3 \times C_3$, $[b, c] = 1$ and $c^3 = 1$. Let $c_1 = cx^k$. Then $G = \langle a, b, c_1, x \mid a^{3^n} = 1, b^3 = x^j, [a, b] = c_1, c_1^3 = 1, [a, c_1] = a^{3^{n-1}}x^m, [b, c_1] = 1, x^3 = 1, [x, a] = [x, b] = 1\rangle$.

If $b^3 = 1$, then $m \neq 0$. This is group (8). If $b^3 \neq 1$, then $j \neq 0$. Thus $jk \equiv 1 (\bmod 3)$ has a solution, say j. Let $x_1 = x^j$. Then $G = \langle a, b \mid a^{3^n} = 1, b^3 = x_1, [a, b] = c_1, c_1^3 = 1, [a, c_1] = a^{3^{n-1}} x_1^{mj}, [b, c_1] = 1, x_1^3 = 1, [x_1, a] = [x_1, b] = 1\rangle$. Obviously, $mj = 0, 1$ or 2. If $mj = 0$, this is group (9). If $mj = 1$, replacing b by $a^{3^{n-2}} b$, then we get group (5). If $mj = 2$, replacing b by $a^{-3^{n-2}} b$, then we get group (4).

If $G/N \cong H_7$ or H_8, we get groups (6), (7), (10) and (11) by a similar argument.

We prove the groups (1)–(11) are pairwise non-isomorphic.

It is easy to see that $\Phi(G) = \langle a^3, c\rangle$ for groups (1)–(3), so $d(G) = 3$ for these groups (1)–(3). On the other hand, $d(G) = 2$ for groups (4)–(11). Thus it is enough that we prove that groups $(1)-(3)$ are pairwise non-isomorphic, and similarly for groups (4) through (11).

We know that H_6, H_7, H_8 are pairwise non-isomorphic. So $H_6 \times C_3, H_7 \times C_3, H_8 \times C_3$ are pairwise non-isomorphic. That is, groups (1),(2) and (3) are pairwise non-isomorphic.

By Lemma 16(3) we know G is 9-abelian. Hence the following are true:

$\Omega_2(G) = \langle a^{3^{n-2}}, b, c\rangle \cong C_{3^2} \times C_{3^2} \times C_3$ for groups (4), (5) and (9);

$\Omega_2(G) = \langle a^{3^{n-2}}, b, c, d\rangle \cong C_{3^2} \times C_3 \times C_3 \times C_3$ for group (8);

$\Omega_2(G) = \langle a^{3^{n-2}}, b, c\rangle \cong C_{3^2} \times M_3(2, 1)$ for group (6);

$\Omega_2(G) = \langle a^{3^{n-2}}, b, c, d\rangle \cong C_{3^2} \times M_3(1, 1, 1)$ for group (7);

$\Omega_2(G) = \langle a^{3^{n-2}}, b, c\rangle \cong C_{3^2} *_{C_3} M_3(2, 1, 1)$ for group (10), (11).

Observing that $\Omega_2(G)$ is either abelian or not, we know that none of (4), (5), (8) or (9) is isomorphic to any one of (6), (7), (10) or (11).

By checking $\Omega_2(G)$, we know that groups (4), (5) and (9) are not isomorphic to group (8).

We observe that groups (4) and (5) have a maximal subgroup $\langle a, c\rangle$ which is isomorphic to $M_3(n, 1, 1)$. On the other hand, no maximal subgroup of group (9) is isomorphic to $M_3(n, 1, 1)$. It follows that group (9) is neither isomorphic to group (4) nor (5). Moreover, by [11, Theorem 11], we know that (4) is not isomorphic to (5). Thus the groups (4), (5), (8) and (9) are pairwise non-isomorphic.

For group (7), $\Omega_1(\Omega_2(G)) \cong C_3^4$. For group (6), (10) and (11), we have $\Omega_1(\Omega_2(G)) \cong C_3^3$. It follows that group (7) is not isomorphic to any of (6), (10) and (11). We consider again $\Omega_2(G)$ for groups (6), (10) and (11). Observe that $C_{3^2} * M_3(2, 1, 1)$ has a maximal subgroup which is isomorphic to $M_3(2, 1, 1)$. But no maximal subgroup of $C_{3^2} \times M_3(2, 1)$ is isomorphic to $M_3(2, 1, 1)$. It follows that (6) is not isomorphic to either of (10) or (11).

Finally, assume there exists an isomorphism σ from the group (10) to the group (11). As $o(b) = 9$, by Lemma 16 we can assume $\sigma : a \to a^{i_1} b^{j_1} c^{k_1}$, $b \to a^{i_2 3^{n-2}} b^{j_2} c^{k_2}$. From $o(a) = 3^n$, then $3 \nmid i_1$. Since $c^\sigma = [a^\sigma, b^\sigma] = [a^{i_1} b^{j_1} c^{k_1}, a^{i_2 3^{n-2}} b^{j_2} c^{k_2}] \equiv [a, b]^{i_1 j_2} (\bmod G_3)$, we conclude $c^\sigma \equiv c^{i_1 j_2} (\bmod G_3)$. Since $[b^\sigma, c^\sigma] = [b^{j_2} c^{k_2}, c^{i_1 j_2}] = [b, c]^{i_1 j_2^2} = a^{2 i_1 j_2^2 3^{n-1}} = (a^\sigma)^{3^{n-1}} = a^{i_1 3^{n-1}}$, $2 j_2^2 \equiv 1 (\bmod G_3)$, a contradiction. Thus, (10) is not isomorphic to (11) either.

Conversely, it is easy to verify that these groups listed in the theorem satisfy all hypotheses. □

Theorem 22 *Assume G is an irregular group of order 3^{n+3} whose center is not cyclic, $n \geq 4$ and $|G'| = 3^3$. Then G is a $\mathcal{C}_3$-group if and only if G is isomorphic to one of the following pairwise* *non-isomorphic groups:*

(1) $\langle a,b,c,x \mid a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [a,c]=a^{3^{n-1}}, [b,c]=x, x^3=1, [x,a]=[x,b]=1\rangle$;

(2) $\langle a,b,c \mid a^{3^n}=1, b^{3^2}=1, [a,b]=c, c^3=1, [a,c]=a^{3^{n-1}}, [b,c]=b^3\rangle$;

(3) $\langle a,b,c \mid a^{3^n}=1, b^{3^2}=1, [a,b]=c, c^3=1, [a,c]=a^{3^{n-1}}, [b,c]=b^6\rangle$;

(4) $\langle a,b,c,x \mid a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [a,c]=x, [b,c]=a^{3^{n-1}}, x^3=1, [x,a]=[x,b]=1\rangle$;

(5) $\langle a,b,c \mid a^{3^n}=1, b^{3^2}=1, [a,b]=c, c^3=1, [a,c]=b^3, [b,c]=a^{3^{n-1}}\rangle$;

(6) $\langle a,b,c \mid a^{3^n}=1, b^{3^2}=1, [a,b]=c, c^3=1, [a,c]=b^6, [b,c]=a^{3^{n-1}}\rangle$;

(7) $\langle a,b,c,x \mid a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [a,c]=x, [b,c]=a^{2\times 3^{n-1}}, x^3=1, [x,a]=[x,b]=1\rangle$;

(8) $\langle a,b,c \mid a^{3^n}=1, b^{3^2}=1, [a,b]=c, c^3=1, [a,c]=b^3, [b,c]=a^{2\times 3^{n-1}}\rangle$;

(9) $\langle a,b,c \mid a^{3^n}=1, b^{3^2}=1, [a,b]=c, c^3=1, [a,c]=b^6, [b,c]=a^{2\times 3^{n-1}}\rangle$.

Proof By Lemma 20, G has a central subgroup N of order 3 such that $G/N\cong H_i$, where H_i is one of the groups listed in Theorem 4. For convenience, assume $N=\langle x\rangle$. Then $G=M$ or $G=M\times N$, where M is the group listed in Lemma 18.

Case I $G=M\times N$.

Since G is irregular and $G/N\cong M\cong H_i$, H_i is irregular. By inspection, H_i is one of H_6, H_7 or H_8. So G is isomorphic to one of $H_6\times C_3$, $H_7\times C_3$ or $H_8\times C_3$, but their derived subgroups are, in each of these cases, isomorphic to $C_3\times C_3$. This contradicts the hypothesis.

Case II $G=M$.

Subcase 1 $G/N\cong H_1, H_2, H_3, H_4, H_5, H_9, H_{10}, H_{11}$ or H_{12}.

If G/N is isomorphic to one of $H_1, H_2, H_3, H_4, H_5, H_9, H_{11}$, then, since Lemma 17, $|{H_i}'|=1$ or 3 for these H_i, we have $|G'|=1,3$ or 3^2 by Lemma 19. This contradicts $|G'|=3^3$.

If $G/N\cong H_{10}$, then by Theorem 4 we have that $G/N=\langle \bar a,\bar b \mid \bar a^{3^n}=\bar 1, \bar b^{3^2}=\bar 1, [\bar a,\bar b]=\bar a^{3^{n-2}}\rangle$. Then $G=M=\langle a,b\rangle$. By Lemma 17, $(G/N)'=\langle \bar a^{3^{n-2}}\rangle$. It follows that $G'\le\langle a^{3^{n-2}},N\rangle$. By Theorem 16(4), $\langle a^{3^{n-2}}\rangle\le Z(G)$. So $G'\le Z(G)$ and $c(G)=2$. This contradicts Lemma 15(3).

If $G/N\cong H_{12}$, then by Theorem 4 we have that $G=\langle a,b,x \mid a^{3^n}=x^i, b^{3^2}=x^j, [a,b]=a^{3^{n-2}}b^3x^k, [a,b^3]=a^{3^{n-1}}x^l, x^3=1, [x,a]=[x,b]=1\rangle$, where $i,j,k,l\in\{0,1,2\}$ and they are not all simultaneously zero. Since G is a $\mathcal{C}_3$-group, $a^{3^n}=1$. By Lemma 16(4), $a^{3^{n-2}}\in Z(G)$. Using the formula in Lemma 2, we have $[a,b,b]=1, [a,b,a]=[b^3,a]=[b,a]^3=[a,b]^{-3}=a^{-3^{n-1}}\in Z(G)$. It follows that $G'=\langle a^{3^{n-2}}b^3x^k\rangle\cong C_9$. This contradicts $|G'|=3^3$.

Subcase 2 $G/N\cong H_6$.

By Theorem 4, assume $G=M=\langle a,b \mid a^{3^n}=x^i, b^3=x^j, [a,b]=cx^k, c^3=x^l, [a,c]=a^{3^{n-1}}x^m, [b,c]=x^h, x^3=1, [x,a]=[x,b]=1\rangle$, where $i,j,k,l,m,h\in\{0,1,2\}$ and they are not all simultaneously zero.

Since G is a $\mathcal{C}_3$-group, $a^{3^n}=1$. We claim： $c^3=1$. If not, then by the formula in Lemma

2, we get: $[a,b^3]=[a,b]^3[a,b,b]^{3(3-1)/2}=c^3\neq 1$. On the other hand, $[a,b^3]=[a,x^j]=1$, a contradiction. Let $c_1=cx^k$. Then $[a,b]=c_1, c_1^3=1, [a,c_1]=a^{3^{n-1}}x^m, [b,c_1]=x^h$.

Since $|G'|=3^3$, $h\neq 0$. It follows that $(h,3)=1$, so $m+hy\equiv 0(\mathrm{mod}\ 3)$ has a solution, say t. Then $[ab^t,c_1]=a^{3^{n-1}}$. Let $a_1=ab^t, c_2=c_1x^{-ht}$. Then $G=\langle a_1,b,x,c_2\mid a_1^{3^n}=1, b^3=x^j, [a_1,b]=c_2, c_2^3=1, [a_1,c_2]=a_1^{3^{n-1}}, [b,c_2]=x^h, x^3=1, [x,a_1]=[x,b]=1\rangle$. If $b^3=1$, then, replacing x by x^h, we get group (1). If $b^3\neq 1$, then $j\neq 0$. Replacing x by x^j, and letting $h_1=hj$ we obtain $h_1\neq 0$. By calculation, $G=\langle a_1,b\mid a_1^{3^n}=1, b^{3^2}=1, [a_1,b]=c_2, c_2^3=1, [a_1,c_2]=a_1^{3^{n-1}}, [b,c_2]=b^{3h_1}\rangle$. If $h_1=1$, then we get group (2). If $h_1=2$, then we get group (3). If $h_1=4$, then it reduces to the case of $h_1=1$.

Subcase 3 $G/N\cong H_7$.

By Theorem 4, assume $G=M=\langle a,b,c,x\mid a^{3^n}=x^i, b^3=x^j, [a,b]=cx^k, c^3=x^l, [a,c]=x^m, [b,c]=a^{3^{n-1}}x^h, x^3=1, [x,a]=[x,b]=1\rangle$, where $i,j,k,l,m,h\in\{0,1,2\}$ and they are not all simultaneously zero.

By the same argument as in Subcase 2, $G=\langle a,b,c,x\mid a^{3^n}=1, b^3=x^j, [a,b]=c_1, c_1^3=1, [a,c_1]=x^m, [b,c_1]=a^{3^{n-1}}x^h, x^3=1, [x,a]=[x,b]=1\rangle$. Since $|G'|=3^3$, $m\neq 0$.

Subcase 3.1 $x^h=1$.

If $b^3=1$, then, letting $x_1=x^m$, we get group (4). If $b^3\neq 1$, then $j\neq 0$. Let $x_1=x^j$ and $m_1=mj$. Then $G=\langle a,b,c_1\mid a^{3^n}=1, b^{3^2}=1, [a,b]=c_1, c_1^3=1, [a,c_1]=b^{3m_1}, [b,c_1]=a^{3^{n-1}}\rangle$. If $m_1=1$, then we get group (5). If $m_1=2$, then we get group (6). If $m_1=4$, then it reduces to the case of $m_1=1$.

Subcase 3.2 $x^h\neq 1$.

We have $h\neq 0$. Let $x_1=x^h$. Then $G=\langle a,b,c_1,x_1\mid a^{3^n}=1, b^3=x_1^{jh}, [a,b]=c_1, c_1^3=1, [a,c_1]=x_1^{mh}, [b,c_1]=a^{3^{n-1}}x_1, x_1^3=1, [x_1,a]=[x_1,b]=1\rangle$.

Assume $b^3=1$. If $mh=1$, then G is isomorphic to group (1). In fact, $\sigma: a\to a^2b,\ b\to b$ is an isomorphism from group (1) to G. If $mh=2$, then, letting $a_1=a, b_1=b^2$ and $c_2=c_1^2a^{-3^{n-1}}x^{-1}$, it reduces to the case of $mh=1$. If $mh=4$, then it reduces to the case of $mh=1$.

If $b^3\neq 1$, then $j\neq 0$. If $mh=1$, then, letting $j_1=jh$, we deduce that $j_1\equiv 1$ or $2\ (\mathrm{mod}\,3)$. If $j_1\equiv 1\ (\mathrm{mod}\,3)$, then $\sigma: a\to ab,\ b\to a^{2\times 3^{n-1}}b^2$ is an isomorphism from group (3) to G. If $j_1\equiv 2\ (\mathrm{mod}\,3)$, then $\sigma: a\to ab,\ b\to a^{2\times 3^{n-1}}b^2$ is an isomorphism from group (2) to G. If $mh=2$, then, letting $b_1=b^2$, $c_2=c_1^2a^{-3^{n-1}}x_1^{-1}$ and $j_1=2jh$, it reduces to the case $mh=1$. If $mh=4$, then it also reduces to the case of $mh=1$.

Subcase 4 $G/N\cong H_8$.

By an argument similar to that in Subcase 3, we get groups (1)–(3) and (7)–(9).

Those groups listed in the statement of the theorem are pairwise non-isomorphic, and satisfy all hypotheses. The details are omitted. □

4.2. Irregular $\mathcal{C}_3$-groups of order $\geq 3^7$ whose center is cyclic

Lemma 23 *Assume G is a $\mathcal{C}_3$-group of order p^n. Then G has a maximal subgroup M which is a $\mathcal{C}_2$-group.*

Proof Since G is a $\mathcal{C}_3$-group of order p^n, there exists $a \in G$ such that $o(a) = p^{n-3}$. Thus G has a subnormal series $\langle a\rangle < N < M < G$. Obviously, the maximal subgroup M of G is a $\mathcal{C}_2$-group.

In the following theorem, unless otherwise stated, the values of all parameters are $0, 1$ or 2.

Theorem 24 *Assume G is an irregular group of order 3^{n+3} whose center is cyclic, $n \geq 4$ and $G' \cong C_3 \times C_3$. Then G is a $\mathcal{C}_3$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b, c, x \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, x^3 = 1, [a, b] = [a, c] = [b, c] = [a, x] = 1, [b, x] = a^{3^{n-1}}, [c, x] = b\rangle$;

(2) $\langle a, b, c, x \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = a^{3^{n-1}}, x^3 = 1, [a, x] = b, [c, x] = a^{3^{n-1}}, [a, c] = [b, c] = [b, x] = 1\rangle$;

(3) $\langle a, b, c, x \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = a^{3^{n-1}}, x^3 = 1, [a, x] = bc, [c, x] = a^{3^{n-1}}, [a, c] = [b, c] = [b, x] = 1\rangle$;

(4) $\langle a, b, c, x \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = a^{3^{n-1}}, x^3 = 1, [a, x] = bc^2, [c, x] = a^{3^{n-1}}, [a, c] = [b, c] = [b, x] = 1\rangle$;

(5) $\langle a, b, c, x \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = a^{3^{n-1}}, x^3 = 1, [a, x] = 1, [b, x] = c, [c, x] = a^{3^{n-1}}, [a, c] = [b, c] = 1\rangle$.

Proof By Lemma 23, G has a maximal subgroup M such that $M \cong H_i$, where H_i is one of the groups listed in Theorem 4. Let $x \in G \setminus M$. Then $G = \langle M, x\rangle$.

Since $G' \cong C_3 \times C_3$, we get by Lemma 2 that for all $g_1, g_2 \in G$, $[g_1^3, g_2] = [g_1, g_2]^3[g_1, g_2, g_1]^3 [g_1, g_2, g_1, g_1] = 1$. Thus, $g_1^3 \in Z(G)$ for all $g \in G$; that is, $\mho_1(G) \leq Z(G)$. Thus $\langle x^3\rangle \leq Z(G)$. Assume $a \in M$ and $o(a) = 3^n$. Then $\langle a^3\rangle \leq Z(G)$. By hypothesis, $o(a) \geq o(x)$. Since $Z(G)$ is cyclic, $\langle x^3\rangle \leq \langle a^3\rangle$. Assume $x^9 = a^{9m}$, m is an integer. Let $x_1 = xa^{-m} \in G \setminus M$. By Lemma 16(3) we get $x_1^9 = (xa^{-m})^9 = x^9a^{-9m} = 1$. Similarly, $\langle x_1^3\rangle \leq \langle a^3\rangle$. Since $o(x_1) \leq 9$, we can assume $x_1^3 = a^{t3^{n-1}}$. Let $x_2 = x_1a^{-t3^{n-2}}$. Then ${x_2}^3 = \left(x_1a^{-t3^{n-2}}\right)^3 = x_1^3a^{-t3^{n-1}} = 1$. Thus $G = \langle M, x_2\rangle$. For convenience, we replace x_2 by x, so $G = \langle M, x\rangle$, where $x^3 = 1$. Since $G' \cong C_3 \times C_3$, we have $c(G) = 3$ by Lemma 15(3).

Case 1 $M \cong H_2, H_5, H_9, H_{10}$, H_{11} or H_{12}.

If $M \cong H_2, H_9$ or H_{11}, then, by Theorem 17, $\mho_1(M)$ are not cyclic. But $\mho_1(G) \leq Z(G)$, a contradiction. If $M \cong H_5$, then by Theorem 4 we have $M = \langle a, b, c \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [a, c] = [b, c] = 1\rangle$. Since $\langle c\rangle = M' \operatorname{char} M \trianglelefteq G$, $\langle c\rangle \trianglelefteq G$. Since $|\langle c\rangle| = 3$, $|\langle c\rangle| \leq Z(G)$. By Lemma 16(3), $a^9 \in Z(G)$. Thus $Z(G)$ is not cyclic, a contradiction. If $M \cong H_{10}$ or H_{12}, then, by Lemma 17, $M' \cong C_9$, which contradicts $G' \cong C_3 \times C_3$.

Case 2 $M \cong H_1$.

By Theorem 4, we have $M = \langle a, b, c \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = [a, c] = [b, c] = 1\rangle$. Obviously, $\langle a^3\rangle \leq Z(G)$. Since $Z(G)$ is cyclic, we have $[b, x] \neq 1, [c, x] \neq 1$ and $G' = \langle [b, x]\rangle \times$

$\langle [c,x]\rangle \cong C_3\times C_3$. Thus there exist integers m,n such that $[ab^mc^n,x]=[a,x][b,x]^m[c,x]^n=1$. Let $a_1=ab^mc^n$. Then $[a_1,x]=1$.

Since $G'\leq M$ and $G'\cong C_3\times C_3$, we have $[b,x]=a_1^{i3^{n-1}}b^jc^k$.

Subcase 2.1 If $k=0$, then $j=0$ by $[b,x,x,x]=1$. That is, $[b,x]=a_1^{i3^{n-1}}$, where $i\neq 0$. Assume $[c,x]=a_1^{r3^{n-1}}b^sc^t$. By $[c,x,x,x]=1$ we have $t=0$. Since $G'\cong C_3\times C_3$, $s\neq 0$. Let $b_1=a_1^{r3^{n-1}}b^s$. Then $[c,x]=b_1$. Let $a_2=a_1^{is}$. It is easy to deduce that $[b_1,x]=a_2^{3^{n-1}}$. It follows that $G=\langle a_2,b_1,c,x\mid a_2^{3^n}=1,b_1^3=1,c^3=1,x^3=1,[a_2,b_1]=[a_2,c]=[b_1,c]=[a_2,x]=1,[b_1,x]=a_2^{3^{n-1}},[c,x]=b_1\rangle$. This is group (1).

Subcase 2.2 If $k\neq 0$, letting $c_1=a_1^{i3^{n-1}}b^jc^k$, then $[b,x]=c_1$. Assume $[c_1,x]=a_1^{r3^{n-1}}b^sc_1^t$. Then $[c_1,x,x]\in Z(G)$ since $[c_1,x,x]\in G_3$. That is, $[c_1,x,x]=[b^sc_1^t,x]=[b,x]^s[c_1,x]^t=c_1^sa_1^{rt3^{n-1}}b^{st}c_1^{t^2}\in Z(G)$. Since $Z(G)$ is cyclic, $b^{st}c_1^{t^2}c_1^s\in\langle a^{3^{n-1}}\rangle$. It follows that $st\equiv 0 \pmod 3$, $s+t^2\equiv 0\pmod 3$. Then we have $s=0$, $t=0$. So $[c_1,x]=a_1^{r3^{n-1}}$, $r\neq 0$. Thus there exists m such that $[c_1^m,x]=a_1^{3^{n-1}}$. Let $b_1=c_1^m$ and $c_2=b$. Then $[b_1,x]={a_1}^{3^{n-1}},[c_2,x]=b_1^m$. This reduces to Subcase 2.1.

Case 3 $M\cong H_3$.

By Theorem 4, we have $M=\langle a,b,c\mid a^{3^n}=1,b^3=1,c^3=1,[a,b]=a^{3^{n-1}},[a,c]=[b,c]=1\rangle$. Since $\langle a^3\rangle\times\langle c\rangle=Z(M)\trianglelefteq G$ and $[c,x]^3=(cc^x)^3=1$, we have $[c,x]=a^{i3^{n-1}}c^j$. Since $[c,x,x,x]=1$, we get $j=0$. Thus $[c,x]=a^{i3^{n-1}}$. Since $Z(G)$ is cyclic, $i\neq 0$. Assume $[a,x]=a^{r3^{n-1}}b^sc^t$, $[b,x]=a^{u3^{n-1}}b^vc^w$. From $[b,x,x,x]=1$ we get $v=0$. Thus $G=\langle a,b,c,x\mid a^{3^n}=1,b^3=1,c^3=1,[a,b]=a^{3^{n-1}},[a,c]=[b,c]=1,x^3=1,[c,x]=a^{i3^{n-1}},[a,x]=a^{r3^{n-1}}b^sc^t,[b,x]=a^{u3^{n-1}}c^w\rangle$.

Since $i\neq 0$, there exists m_1 satisfying $u+im_1\equiv 0\pmod 3$. Let $b_1=bc^{m_1}$ such that $[b_1,x]=c^w$. Since $i\neq 0$, there exists m_2 satisfying $r+im_2\equiv 0 \pmod 3$. Let $a_1=ac^{m_2}$ such that $[a_1,x]=b_1^sc^{t_1}$. We observe that t_1 may be different from t. Then $G=\langle a_1,b_1,c,x\mid a_1^{3^n}=1,b_1^3=1,c^3=1,[a_1,b_1]=a_1^{3^{n-1}},[a_1,c]=[b_1,c]=1,x^3=1,[c,x]=a_1^{i3^{n-1}},[a_1,x]=b_1^sc^{t_1},[b_1,x]=c^w\rangle$.

If $w=0$, by considering all possible values of parameters s,t_1,i, we get groups (2), (3) and (4).

If $w\neq 0$, then $G'=\langle a^{3^{n-1}}\rangle\times\langle c\rangle$. Since $w\neq 0$, there exists m satisfying $t+wm\equiv 0\pmod 3$. Let $a_2=a_1b_1^m$ such that $[a_2,x]=1$. Then $G=\langle a_2,b_1,c,x\mid a_2^{3^n}=1,b_1^3=1,c^3=1,[a_2,b_1]=a_2^{3^{n-1}},[a_2,c]=[b_1,c]=1,x^3=1,[c,x]=a_2^{i3^{n-1}},[a_2,x]=1,[b_1,x]=c^w\rangle$, where $w,i\neq 0$.

If $i=2$, then, replacing x by x^2, it reduces to the case $i=1$. Thus $G=\langle a_2,b_1,c,x\mid a_2^{3^n}=1,b_1^3=1,c^3=1,[a_2,b_1]=a_2^{3^{n-1}},[a_2,c]=[b_1,c]=1,x^3=1,[c,x]=a_2^{3^{n-1}},[a_2,x]=1,[b_1,x]=c^{2w}\rangle$, where $w\neq 0$. If $2w\equiv 2\pmod 3$, then, letting $x_1=x^2$ and $a_3=a_2^2$, it reduces to the case $2w\equiv 1\pmod 3$. Thus we get group (5).

Case 4 $M\cong H_4$.

By Theorem 4, we have $M=\langle a,b,c\mid a^{3^n}=1,b^3=1,c^3=1,[b,c]=a^{3^{n-1}},[a,b]=[a,c]=1\rangle$. Since $\langle a\rangle=Z(M)\trianglelefteq G$, we can assume $[a,x]=a^{i3^{n-1}}$. Furthermore, let $[b,x]=a^{r3^{n-1}}b^sc^t$, $[c,x]=a^{u3^{n-1}}b^vc^w$.

By the symmetry of b and c, we may assume $t\neq 0$ without loss of generality.

Let $c_1^t = a^{r3^{n-1}}b^sc^t$. Then $[b,x]=c_1^t$, $t\neq 0$. Hence $[c_1,x]\in G_3\le Z(G)$. It follows from $[c,x,x]=1$ that $v=0$ and $w=0$. Thus $G=\langle a,b,c_1,x \mid a^{3^n}=1, b^3=1, c_1^3=1, [b,c_1]=a^{3^{n-1}}, x^3=1, [a,x]=a^{i3^{n-1}}, [b,x]=c_1^t, [c_1,x]=a^{u3^{n-1}}, [a,c_1]=[b,a]=1\rangle$, where $t\neq 0$. If $[c_1,x]=1$, then $\langle a,x,c_1\rangle$ is isomorphic to H_2 or H_3. This reduces to Cases 1 or 3. If $[c_1,x]\neq 1$, letting $x_1=xb^u$, then $[c_1,x_1]=1$. Thus the maximal subgroup $\langle a,x_1,c_1\rangle$ is isomorphic to H_2 or H_3, This reduces to Cases 1 or 3 again.

Case 5 $M\cong H_6$.

By Theorem 4, we have $M=\langle a,b \mid a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [a,c]=a^{3^{n-1}}, [b,c]=1\rangle$. Since $M'=\langle a^{3^{n-1}}\rangle\times\langle c\rangle\trianglelefteq G$, we have $[c,x]=a^{i3^{n-1}}c^j$. Since $[c,x,x,x]=1$, we have $j=0$. Since $G'=\langle a^{3^{n-1}}\rangle\times\langle c\rangle$, we have $[b,x]=a^{r3^{n-1}}c^s$, $[a,x]=a^{u3^{n-1}}c^v$ and m is an integer satisfying $m+u\equiv 0 \pmod 3$. Let $x_1=xc^m$. Then $[a,x_1]=c^v$. Let l be an integer satisfying $l+v\equiv 0 \pmod 3$ and $x_2=x_1b^l$. Then $[a,x_2]=1$. By calculation, we have $x_2^3\in\langle a^{i3^{n-1}}\rangle$. Let $x_2^3=a^{m3^{n-1}}$ and $x_3=x_2a^{-m3^{n-2}}$. Then $x_3^3=1$.

If $[c,x_3]=1$, then $\langle a,c,x_3\rangle\cong H_3$. Thus the problem reduces to Case 3. If $[c,x_3]\neq 1$, then $G=\langle a,b,c,x_3 \mid a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [a,c]=a^{3^{n-1}}, [b,c]=1, x_3^3=1, [a,x_3]=1, [b,x_3]=a^{r3^{n-1}}c^s, [c,x_3]=a^{i3^{n-1}}\rangle$, where $i\neq 0$. Let $a_1=a^ix_3$. Then $[a_1,c]=1$. Since $\langle a_1,c,x_3\rangle$ is a maximal subgroup of G isomorphic to H_4, this reduces to Case 4.

Case 6 $M\cong H_7$ or H_8.

If $M\cong H_7$, then by Theorem 4 we have $M=\langle a,b \mid a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [b,c]=a^{3^{n-1}}, [a,c]=1\rangle$. Since $M'=\langle a^{3^{n-1}}\rangle\times\langle c\rangle\trianglelefteq G$, we have $[c,x]=a^{i3^{n-1}}c^j$. By $[c,x,x,x]=1$ we get $j=0$. Since $G'=\langle a^{3^{n-1}}\rangle\times\langle c\rangle$, we have $[b,x]=a^{r3^{n-1}}c^s$, $[a,x]=a^{u3^{n-1}}c^v$. Let m be an integer satisfying $m+v\equiv 0 \pmod 3$ and $x_1=xb^m$. Then $[a,x_1]=a^{u3^{n-1}}$. Since $[a^{x_1},b^{x_1}]=[a,bc^s]=[a,c^s][a,b]=c=c^{x_1}=ca^{i3^{n-1}}$, we have $i=0$, that is, $[c,x_1]=1$. Thus $\langle a,c,x_1\rangle\cong H_3$ or are abelian. This reduces to Cases 1, 2 or 3.

If $M\cong H_8$, then a similar argument likewise reduces to Cases 1, 2 or 3.

Those groups listed in the statement of the theorem are pairwise non-isomorphic, and satisfy all hypotheses. The details are omitted. □

Theorem 25 *Assume G is an irregular group of order 3^{n+3} whose center is cyclic, $n\ge 4$ and $G'\cong C_3\times C_3\times C_3$. Then G is a $\mathcal{C}_3$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups:*

(1) $\langle a,b,c,x | a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [a,c]=a^{3^{n-1}}, [b,c]=1, x^3=1, [a,x]=b, [b,x]=1, [c,x]=1\rangle$;

(2) $\langle a,b,c,x | a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [a,c]=a^{3^{n-1}}, [b,c]=1, x^3=1, [a,x]=b, [b,x]=a^{3^{n-1}}, [c,x]=1\rangle$;

(3) $\langle a,b,c,x | a^{3^n}=1, b^3=1, [a,b]=c, c^3=1, [a,c]=a^{3^{n-1}}, [b,c]=1, x^3=1, [a,x]=b, [b,x]=a^{2\times 3^{n-1}}, [c,x]=1\rangle$.

Proof By Lemma 23, G has a maximal subgroup M which is isomorphic to $M\cong H_i$, where H_i is one of the group listed in Theorem 4. Let $x\in G\setminus M$. Then $G=\langle M,x\rangle$. Obviously, $G'\le M$.

Since $G' \cong C_3 \times C_3 \times C_3$, $G' \leq \Omega_1(M)$.

Case 1 $M \cong H_2, H_4, H_5, H_7, H_8, H_9, H_{10}, H_{11}$ or H_{12}.

If $M \cong H_2, H_4, H_7, H_8, H_9$ or H_{11}, then, by Lemma 17, $\Omega_1(H_2) \cong \Omega_1(H_9) \cong \Omega_1(H_{11}) \cong C_3 \times C_3$, $\Omega_1(H_4) \cong \Omega_1(H_7) \cong \Omega_1(H_8) \cong M_3(1,1,1)$. Thus $G' \nleq \Omega_1(M)$, a contradiction.

If $M \cong H_5$, then, by Theorem 4, we have $M = \langle a, b, c \mid a^{3^n} = 1, b^3 = 1, [a,b] = c, c^3 = 1, [a,c] = [b,c] = 1\rangle$. Since $\langle c\rangle = M' \trianglelefteq G$, $\langle c\rangle \leq Z(G)$. By Theorem 16, $\langle a^9\rangle \leq Z(G)$. Thus $Z(G)$ is not cyclic, a contradiction.

If $M \cong H_{10}$ or H_{12}, then by Lemma 17, $M' \cong C_9$, which contradicts $G' \cong C_3 \times C_3 \times C_3$.

Case 2 $M \cong H_1$.

By Theorem 4 we have $M = \langle a, b, c \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a,b] = [a,c] = [b,c] = 1\rangle$. By Lemma 17, $\Omega_1(H_1) \cong C_3 \times C_3 \times C_3$. Obviously, $G' = \Omega_1(M) = \langle a^{3^{n-1}}\rangle \times \langle b\rangle \times \langle c\rangle$. By Theorem 16, $\langle a^9\rangle \leq Z(G)$. Since $Z(G)$ is cyclic, we have $[b,x] \neq 1, [c,x] \neq 1$. Assume $[b,x] = a^{i3^{n-1}} b^j c^k$.

Subcase 2.1 $k = 0$.

By $[b,x,x,x,x] = 1$ we get $j = 0$. That is, $[b,x] = a^{i3^{n-1}}, i \neq 0$. Assume $[c,x] = a^{r3^{n-1}} b^s c^t$. By $[c,x,x,x,x] = 1$ we get $t = 0$. Since $Z(G)$ is cyclic, $s \neq 0$. Let $b_1 = a^{r3^{n-1}} b^s$. Then $[c,x] = b_1$. Assume $[a,x] = a^{u3^{n-1}} b_1^v c^w$. Since $G' \cong C_3 \times C_3 \times C_3$, we get $w \neq 0$. Since $[a,x,x] = [a^{u3^{n-1}} b_1^v c^w, x] = [b_1,x]^v [c,x]^w = a^{v3^{n-1}} b_1^w$, $[a,x,x,x] = [a^{v3^{n-1}} b_1^w, x] = [b_1,x]^w = a^{w3^{n-1}} \neq 1$. It follows that $[a,x^3] = [a,x]^3 [a,x,x]^3 [a,x,x,x] = [a,x,x,x] \neq 1$. On the other hand, $x^3 \in M$ and M is abelian, so $[a,x^3] = 1$, a contradiction.

Subcase 2.2 $k \neq 0$.

Let $c_1 = a^{i3^{n-1}} b^j c^k$. Then $[b,x] = c_1$. Assume $[c_1,x] = a^{r3^{n-1}} b^s c_1^t$. If $s = 0$, then $t = 0$ by $[c_1,x,x,x,x] = 1$. Replacing c_1 by b, and b by c_1, this reduces to subcase 2.1. If $s \neq 0$, then, from $[c_1,x,x] = [b^s c_1^t, x] = [b,x]^s [c_1,x]^t = c_1^s a^{rt3^{n-1}} b^{st} c_1^{t^2}$, we have $[c_1,x,x,x] = [b^{st} c_1^{s+t^2}, x] = [b,x]^{st} [c_1,x]^{s+t^2} = c_1^{st} a^{r(s+t^2)3^{n-1}}\ b^{s(s+t^2)} c_1^{t(s+t^2)}$. Since $[c_1,x,x,x] \in Z(G)$, $st \equiv 0 \pmod 3$ and $s + t^2 \equiv 0 \pmod 3$, a contradiction.

Case 3 $M \cong H_3$.

By Theorem 4 we have $M = \langle a, b, c \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a,b] = a^{3^{n-1}}, [a,c] = [b,c] = 1\rangle$. Since $\langle a^3\rangle \times \langle c\rangle = Z(M) \trianglelefteq G$, we can assume $[c,x] = a^{i3^{n-1}} c^j$. By $[c,x,x,x,x] = 1$ we get $j = 0$. That is, $[c,x] = a^{i3^{n-1}}$. Since $Z(G)$ is cyclic, $3 \nmid i$. Since $G' = \Omega_1(M) = \langle a^{3^{n-1}}\rangle \times \langle b\rangle \times \langle c\rangle$, we have $[a,x] = a^{r3^{n-1}} b^s c^t$ and $[b,x] = a^{u3^{n-1}} b^v c^w$. Since $[b,x,x,x,x] = 1$, we have $v = 0$. Thus we have $a^{3^n} = 1, b^3 = 1, c^3 = 1, [a,b] = a^{3^{n-1}}, [a,c] = [b,c] = 1, [c,x] = a^{i3^{n-1}}, [a,x] = a^{r3^{n-1}} b^s c^t, [b,x] = a^{u3^{n-1}} c^w$.

Since $3 \nmid i$, there exists l satisfying $li + u \equiv 0 \pmod 3$. Let $b_1 = bc^l$. Since $3 \nmid i$, there exists m satisfying $mi + r \equiv 0 \pmod 3$. Let $a_1 = ac^m$. Since $G' \cong C_3 \times C_3 \times C_3$, we have $s \neq 0, w \neq 0$. Since $(w,3) = 1$ there exists m_1 satisfying $m_1 w + t \equiv 0 \pmod 3$. Let $a_2 = a_1 {b_1}^{m_1}$. We have $a_2^{3^n} = 1, b_1^3 = 1, c^3 = 1, [a_2, b_1] = a_2^{3^{n-1}}, [a_2, c] = [b_1, c] = 1, [c,x] = a_2^{i3^{n-1}}, [a_2,x] = b_1^s, [b_1,x] = c^w$.

Since $x^3 \in M$, we have $x^3 = a_2^{l_1} b_1^{l_2} c^{l_3}$, where $1 \leq l_1 \leq 3^n, 1 \leq l_2, l_3 \leq 3$. Since $[x^3, x] = 1$, $l_2 = 0$. Furthermore, $[a_2, x^3] = 1$. On the other hand, $[a_2, x^3] = [a_2, x]^3[a_2, x, x]^3[a_2, x, x, x] = [a_2, x, x, x] \neq 1$, a contradiction.

Case 4 $M \cong H_6$.

By Theorem 4 we have $M = \langle a, b \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [a, c] = a^{3^{n-1}}, [b, c] = 1\rangle$. By Lemma 16(4) we get $x^9 \in Z(G), a^9 \in Z(G)$. Since $Z(G)$ is cyclic and $o(a) = \exp G$, we can assume $x^9 = a^{9m}$, where m is an integer. By Lemma 16(3), G is 9-abelian. Replacing x by xa^{-m}, we get $x^9 = 1$. By Lemma 17, $M' = \langle a^{3^{n-1}}\rangle \times \langle c\rangle \trianglelefteq G$, $\Omega_1(M) = \langle a^{3^{n-1}}\rangle \times \langle b\rangle \times \langle c\rangle \trianglelefteq G$. Since $G' \leq \Omega_1(M)$, $G' = \langle a^{3^{n-1}}\rangle \times \langle b\rangle \times \langle c\rangle$. We consider the quotient group $\overline{G} = G/\langle a^{3^{n-1}}\rangle$. Then $\langle \bar{c}\rangle = \overline{M'} \leq Z(\overline{G})$. Thus we can assume $[c, x] = a^{i3^{n-1}}$. We consider the quotient group $\overline{G} = G/M'$. Then $\langle \bar{b}\rangle = \overline{\Omega_1(M)} \leq Z(\overline{G})$. Thus we can assume $[b, x] = a^{r3^{n-1}} c^s$. Furthermore, we assume $[a, x] = a^{u3^{n-1}} b^v c^w$. Thus we have $a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [a, c] = a^{3^{n-1}}, [b, c] = 1, [a, x] = a^{u3^{n-1}} b^v c^w, [b, x] = a^{r3^{n-1}} c^s, [c, x] = a^{i3^{n-1}}, x^9 = 1$.

Let m_1 satisfy $m_1 + u \equiv 0 \pmod 3$, $x_1 = xc^{m_1}$, l satisfy $l + w \equiv 0 \pmod 3$, $x_2 = x_1 b^l$. Since $G' = \langle a^{3^{n-1}}\rangle \times \langle b\rangle \times \langle c\rangle$, we have $v \neq 0$. Since $[a^{x_2}, b^{x_2}] = c^{x_2}$ we get $i = s$. Thus $a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [a, c] = a^{3^{n-1}}, [b, c] = 1, [a, x_2] = b^v, [b, x_2] = a^{r3^{n-1}} c^i, [c, x_2] = a^{i3^{n-1}}, x_2^9 = 1$.

Subcase 4.1 $[c, x_2] \neq 1$. That is, $i \neq 0$.

Since $x_2^9 = 1$, ${x_2}^3 \in \Omega_1(H_6)$. We have ${x_2}^3 = a^{m_1 3^{n-1}} b^{m_2} c^{m_3}$. Since $[a, {x_2}^3] = [a, x_2]^3[a, x_2, x_2]^3[a, x_2, x_2, x_2] = [a, x_2, x_2, x_2] = [b^v, x_2, x_2] = [c^{vi}, x_2] = a^{vi^2 3^{n-1}} = a^{v3^{n-1}}$ and $[a, a^{m_1 3^{n-1}} b^{m_2} c^{m_3}] = [a, b^{m_2} c^{m_3}] = [a, c^{m_3}][a, b^{m_2}] = [a, c]^{m_3}[a, b]^{m_2} = a^{m_3 3^{n-1}} c^{m_2}$, we get $3 \mid m_2$ and $m_3 = v$. Therefore $1 = [{x_2}^3, x_2] = [c^v, x_2] = a^{iv3^{n-1}} \neq 1$, a contradiction.

Subcase 4.2 $[c, x_2] = 1$.

Since $[a, {x_2}^3] = [a, x_2]^3[a, x_2, x_2]^3[a, x_2, x_2, x_2] = 1$ and $[b, {x_2}^3] = [b, x_2]^3 = 1$, ${x_2}^3 \in Z(G)$. Since $Z(G)$ is cyclic, we have ${x_2}^3 = a^{m3^{n-1}}$. Replacing x_2 by $x_2 a^{-m3^{n-2}}$, we get ${x_2}^3 = 1$. Thus $G = \langle a, x_2 \mid a^{3^n} = 1, b^3 = 1, [a, b] = c, c^3 = 1, [a, c] = a^{3^{n-1}}, [b, c] = 1, {x_2}^3 = 1, [a, x_2] = b^v, [b, x_2] = a^{r3^{n-1}}, [c, x_2] = 1\rangle$. If $v = 1$ and $r = 0$, then we get group (1). If $v = 1$ and $r = 1$, then we get group (2). If $v = 1$ and $r = 2$, then we get group (3). If $v = 2$, then, replacing x_2 by ${x_2}^2$, there exists m satisfying $m + 2r \equiv 0 \pmod 3$. Replacing x_2 by $x_2 c^m$ reduces to the case $v = 1$.

Those groups listed in the statement of the theorem are paiewise non-isomorphic, and satisfy all hypotheses. The details are omitted. □

Theorem 26 *Assume G is an irregular group of order 3^{n+3} whose center is cyclic, $n \geq 4$ and $G' \cong C_9 \times C_3$. Then G is a $\mathcal{C}_3$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b, c, x | a^{3^n} = 1, x^3 = 1, [a, x] = a^{3^{n-2}} b^2, b^3 = 1, [a, b] = a^{3^{n-1}}, [b, x] = c, c^3 = 1, [c, x] = a^{3^{n-1}}, [a, c] = [b, c] = 1\rangle$;

(2) $\langle a, b, c, x | a^{3^n} = 1, x^3 = 1, [a, x] = a^{3^{n-2}} c^2, c^3 = 1, [c, x] = b, b^3 = 1, [b, x] = a^{3^{n-1}}, [a, b] = [a, c] = [b, c] = 1\rangle$;

(3) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}},[b,x]=1\rangle$;
(4) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}},[b,x]=a^{3^{n-1}}\rangle$;
(5) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}},[b,x]=a^{2\times 3^{n-1}}\rangle$;
(6) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}}b,[b,x]=1\rangle$;
(7) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}}b,[b,x]=a^{3^{n-1}}\rangle$;
(8) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}}b,[b,x]=a^{2\times 3^{n-1}}\rangle$;
(9) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}}b^2,[b,x]=1\rangle$;
(10) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}}b^2,[b,x]=a^{3^{n-1}}\rangle$;
(11) $\langle a,b,x|a^{3^n}=1,b^3=1,[a,b]=x^3,x^9=1,[a,x]=a^{3^{n-2}}b^2,[b,x]=a^{2\times 3^{n-1}}\rangle$.

Proof By Lemma 23, G has a maximal subgroup M which is isomorphic to H_i, where H_i is one of the groups listed in Theorem 4. Let $x\in G\backslash M$. Then $G=\langle M,x\rangle$. Assume $a\in M$ and $o(a)=3^n$. Then $\langle a^9\rangle\le Z(G)$, $\langle x^9\rangle\le Z(G)$, $o(a)\ge o(x)$. Since $Z(G)$ is cyclic, $\langle x^9\rangle\le\langle a^9\rangle$. Thus we have $x^9=a^{9m}$. Obviously, $xa^{-m}\in G\setminus M$. Let $x_1=xa^{-m}$. Then $x_1^9=(xa^{-m})^9=x^9a^{-9m}=1$. We have $G=\langle M,x_1\rangle$. For convenience, we replace x_1 by x, so $G=\langle M,x\rangle, x^9=1$. Obviously, $G'\le M$. Since $G'\cong C_9\times C_3$, $G'\le\Omega_2(M)$.

Case 1 $M\cong H_2,H_4,H_5$, H_7, H_8, H_9 or H_{11}.

If $M\cong H_2$, by Theorem 4 we have $M=\langle a,b\mid a^{3^n}=1,b^{3^2}=1,[a,b]=1\rangle$. If $o([b,x])\le 3$, then $[b^3,x]=[b,x]^3=1$. Thus $\langle b^3\rangle\in Z(G)$ and so $Z(G)$ is not cyclic, a contradiction. If $o([b,x])=9$, then we have $[b,x]=a^{i3^{n-2}}b^j$. Since $[b,x,x,x,x]=1$, we get $j=0$. It follows from $x^3\in M$ that $[b,x^3]=1$. On the other hand, $[b,x^3]=[b,x]^3=a^{i3^{n-1}}\ne 1$, a contradiction. If $M\cong H_9$ or H_{11}, then a contradiction arises by a similar argument.

If $M\cong H_4$, then, by Theorem 4 we have $M=\langle a,b,c\mid a^{3^n}=1,b^3=1,c^3=1,[b,c]=a^{3^{n-1}},[a,b]=[a,c]=1\rangle$. Thus $[c,x]^3=(c^{-1}c^x)^3=1$ and $[b,x]^3=(b^{-1}b^x)^3=1$. By Lemma 17, $Z(M)=\langle a\rangle$. Obviously, $\langle a\rangle\trianglelefteq G$. It follows from $x^3\in M$ that $[a,x^3]=1$. On the other hand, since $G'\cong C_9\times C_3$, we have $[a,x]=a^{i3^{n-2}}$. Then $G'=\langle[a,x],[b,x],[c,x],[a,b],[a,c],[b,c],G_3\rangle$. By Lemma 16, $3\nmid i$. Thus $[a,x^3]=[a,x]^3=a^{i3^{n-1}}\ne 1$, a contradiction.

If $M\cong H_5$, then, by Theorem 4 we have $M=\langle a,b,c\mid a^{3^n}=1,b^3=1,[a,b]=c,c^3=1,[a,c]=[b,c]=1\rangle$. Since $\langle c\rangle=(M)'\trianglelefteq G$, $\langle c\rangle\le Z(G)$. By Theorem 16, $\langle a^9\rangle\le Z(G)$. Thus $Z(G)$ is not cyclic, a contradiction.

If $M\cong H_7$, then, by Theorem 4 we have $M=\langle a,b,c\mid a^{3^n}=1,b^3=1,[a,b]=c,c^3=1,[b,c]=a^{3^{n-1}},[a,c]=1\rangle$. By Lemma 17, $M'=\langle a^{3^{n-1}}\rangle\times\langle c\rangle\trianglelefteq G$. Since G is 9-abelian, $\Omega_2(M)=\langle a^{3^{n-2}},b,c\rangle$. By $G'\le\Omega_2(M)$, $G'=\langle a^{3^{n-2}}\rangle\times\langle c\rangle$. We consider the quotient group $G/\langle a^{3^{n-1}}\rangle$. Then $\langle\bar{c}\rangle=\overline{M'}\le\overline{Z(G)}$. Thus we can assume $[c,x]=a^{i3^{n-1}}$. Since $[b,x]^3=(bb^x)^3=1$, we get $o([b,x])\le 3$. Since $G'\cong C_9\times C_3$, we have $[a,x]=a^{r3^{n-2}}c^s$, where $3\nmid r$. Thus $[a,x^3]=[a,x]^3=a^{r3^{n-1}}$. On the other hand, it follows from $x^9=1$ that $x^3\in\Omega_1(M)$. Thus we have $x^3=a^{m_13^{n-1}}b^{m_2}c^{m_3}$. It follows that $[a,x^3]=[a,a^{m_13^{n-1}}b^{m_2}c^{m_3}]=[a,b^{m_2}]=[a,b]^{m_2}[a,b,b]^{m_2(m_2-1)/2}=c^{m_2}a^{-3^{n-1}m_2(m_2-1)/2}$, a contradiction. If $M\cong H_8$, then a contradiction arises by a similar argument.

Case 2 $M \cong H_3$.

By Theorem 4 we have $M = \langle a, b, c \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = a^{3^{n-1}}, [a, c] = [b, c] = 1\rangle$. By Lemma 17, $Z(M) = \langle a^3\rangle \times \langle c\rangle \trianglelefteq G$ and $\Omega_1(M) = \langle a^{3^{n-1}}\rangle \times \langle b\rangle \times \langle c\rangle \trianglelefteq G$. We consider the quotient group $\overline{G} = G/\langle a^3\rangle$. Then $\langle \bar{c}\rangle = \overline{M'} \le Z(\overline{G})$. By $[c, x]^3 = (c^{-1}c^x)^3 = c^{-3}(c^x)^3 = 1$, we can assume $[c, x] = a^{i3^{n-1}}$. Since $Z(G)$ is cyclic, $[c, x] \neq 1$. That is, $i \neq 0$. We consider the quotient group $\overline{G} = G/\langle a^{3^{n-1}}\rangle \times \langle c\rangle$. Then $\langle \bar{b}\rangle = \overline{\Omega_1(M)} \le Z(\overline{G})$. Thus we have that $[b, x] = a^{u3^{n-1}}c^w$ and $[a, x] = a^{r3^{n-2}}b^s c^t$, $0 \le r \le 8$. Since $G' \cong C_9 \times C_3$, $3 \nmid r$. Thus $a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = a^{3^{n-1}}, [a, c] = [b, c] = 1, x^9 = 1, [c, x] = a^{i3^{n-1}}, [b, x] = a^{u3^{n-1}}c^w, [a, x] = a^{r3^{n-2}}b^s c^t$, where $i \neq 0$, $0 \le r \le 8$, $3 \nmid r$.

Since $i \neq 0$, there exists m_1 satisfying $u+im_1 \equiv 0 \pmod 3$. Let $b_1 = bc^{m_1}$. Since $G' \cong C_9\times C_3$, $w \neq 0$. Thus there exists m_2 satisfying $t - m_1 s + wm_2 \equiv 0 \pmod 3$. Let $a_1 = ab^{m_2}$, $c_1 = c^w$ and $a_2 = a_1^{iw}$. We have $[a_2, x] = a_2^{r_1 3^{n-2}} a^{r_2 3^{n-1}} b_1^{s_1}$. Then we have $1 \le r_1 \le 2, 0 \le r_2 \le 2$. Let $x_1 = xb_1^{-r_2}$. Replacing a by a_2, b by b_1, c by c_1 and x by x_1, we have $a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = a^{3^{n-1}}, [a, c] = [b, c] = 1, x^9 = 1, [c, x] = a^{3^{n-1}}, [b, x] = c, [a, x] = a^{r_1 3^{n-2}} b^{s_1}$, where $1 \le r_1 \le 2$.

Since $x^3 \in \Omega_1(M)$, we have that $x^3 = a^{m_1 3^{n-1}} b^{m_2} c^{m_3}$. It follows from $[x^3, x] = 1$ that $[a^{m_1 3^{n-1}} b^{m_2} c^{m_3}, x] = [b^{m_2} c^{m_3}, x] = c^{m_2} a^{m_3 3^{n-2}} = 1$. Thus $m_2 = m_3 = 0$. So $[a, x^3] = 1$. Since $[a, x^3] = [a, x]^3 [a, x, x, x] = a^{r_1 3^{n-1}} a^{s_1 3^{n-1}} = 1$, $r_1 + s_1 \equiv 0 \pmod 3$. Replacing x by $xa^{-m_1 3^{n-2}}$, we get $x^3 = 1$. Thus $G = \langle a, x | a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = a^{3^{n-1}}, [a, c] = [b, c] = 1, x^3 = 1, [c, x] = a^{3^{n-1}}, [b, x] = c, [a, x] = a^{r_1 3^{n-2}} b^{-r_1}\rangle$, where $1 \le r_1 \le 2$. If $r_1 = 1$, we get group (1). If $r_1 = 2$, then, replacing x by x^2, b by bc^2 and c by c^2, we get group (1) again.

Case 3 $M \cong H_1$.

By Theorem 4 we have $M = \langle a, b, c \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = [a, c] = [b, c] = 1\rangle$. Since $Z(G)$ is cyclic, $[b, x] \neq 1, [c, x] \neq 1$. It follows from $[b, x]^3 = (b^{-1}b^x)^3 = 1$ and $[c, x]^3 = (c^{-1}c^x)^3 = 1$ that $o([b, x]) = o([c, x]) = 3$. Let $[b, x] = a^{i3^{n-1}} b^j c^k$.

Subcase 3.1 $k = 0$.

Since $[b, x, x, x, x] = 1$, we get $j = 0$. Thus $[b, x] = a^{i3^{n-1}}$, $i \neq 0$. Assume $[c, x] = a^{r3^{n-1}} b^s c^t$. Since $[b, x, x, x, x] = 1$, we get $t = 0$. If $s = 0$, letting m be an integer satisfying $im+r \equiv 0 \pmod 3$, and replacing c by $b^m c$, we obtain $[c, x] = 1$. It follows that $Z(G)$ is not cyclic, a contradiction. Thus $s \neq 0$. So $a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = [a, c] = [b, c] = 1, [b, x] = a^{i3^{n-1}}, [c, x] = a^{r3^{n-1}} b^s, x^9 = 1$, where $i \neq 0, s \neq 0$.

Replacing b by $a^{r3^{n-1}} b^s$ and a by a^{si}, we have $a^{3^n} = 1, b^3 = 1, c^3 = 1, [a, b] = [a, c] = [b, c] = 1, [b, x] = a^{3^{n-1}}, [c, x] = b, x^9 = 1$.

Since $G' = C_9 \times C_3$, we have $[a, x] = a^{u3^{n-2}} b^v c^w$, where $0 \le u \le 8, 3 \nmid u$. If $w = 0$, then $[a, x^3] = [a, x]^3 [a, x, x]^3 \neq 1$. On the other hand, since M is abelian, $[a, x^3] = 1$, a contradiction. Thus $w \neq 0$. We have $[a, x^3] = [a, x]^3 [a, x, x]^3 [a, x, x, x] = a^{u3^{n-1}} a^{w3^{n-1}}$. It follows from $[a, x^3] = 1$ that $w + u \equiv 0 \pmod 3$. By $x^3 \in Z(G)$, we can assume that $x^3 = a^{l3^{n-1}}, l \neq 0$. Replacing x by $xa^{-l3^{n-2}}$ and a by ac^v, we get $G = \langle a, x \mid a^{3^n} = 1, b^3 = 1, c^3 = 1, x^3 = 1, [a, b] = [a, c] = [b, c] = 1, [a, x] = a^{u3^{n-2}} c^{-u}, [b, x] = a^{3^{n-1}}, [c, x] = b\rangle$, where $u \neq 0$.

If $u = 1$, then we get group (2). If $u = 2$, then, replacing x by x^2, b by b^2 and c by c^2, it reduces to the case of $u = 1$.

Subcase 3.2 $k \neq 0$.

Replacing c by $a^{i3^{n-1}}b^jc^k$, we get $[b,x] = c$. Assume $[c,x] = a^{r3^{n-1}}b^sc^t$. Since $G' \cong C_9 \times C_3 \leq \Omega_2(G)$, we get $s = 0$. By $[c,x,x,x,x] = 1$, we get $t = 0$. Replacing c by b and b by c, it reduces to Subcase 3.1.

Case 4 $M \cong H_6$.

By Theorem 4 we have $M = \langle a, b \mid a^{3^n} = 1, b^3 = 1, [a,b] = c, c^3 = 1, [a,c] = a^{3^{n-1}}, [b,c] = 1\rangle$. By Lemma 17, $M' = \langle a^{3^{n-1}}\rangle \times \langle c\rangle \trianglelefteq G$, and $\Omega_1(M) = \langle a^{3^{n-1}}\rangle \times \langle b\rangle \times \langle c\rangle \trianglelefteq G$. We consider the quotient group $G/\langle a^{3^{n-1}}\rangle$. Then $\langle \bar{c}\rangle = \overline{M'} \trianglelefteq \overline{G}$. It follows that $\langle \bar{c}\rangle \leq Z(\overline{G})$. Assume $[c,x] = a^{u3^{n-1}}$. We consider $G/\langle a^{3^{n-1}}\rangle \times \langle c\rangle$. Then $\langle \bar{b}\rangle = \overline{\Omega_1(M)} \trianglelefteq \overline{G}$. So $\langle \bar{b}\rangle \leq Z(\overline{G})$. Assume $[b,x] = a^{r3^{n-1}}c^t$. Since $G' \cong C_9 \times C_3$, we can assume $[a,x] = a^{i3^{n-2}}b^jc^k$, where $1 \leq i \leq 8, 3 \nmid i$. Thus $a^{3^n} = 1, b^3 = 1, [a,b] = c, c^3 = 1, [a,c] = a^{3^{n-1}}, [b,c] = 1, [c,x] = a^{u3^{n-1}}, [a,x] = a^{i3^{n-2}}b^jc^k, [b,x] = a^{r3^{n-1}}c^t, x^9 = 1$.

Replacing x by xb^{-k}, we get $[a,x] = a^{i3^{n-2}}b^j$. Thus $G' = \langle a^{i3^{n-2}}b^j\rangle \times \langle c\rangle$. It follows from $[a^x, b^x] = c^x$ that $u = t$. Thus there exists m satisfying $3m + i \equiv 1$ or $2 \pmod 3$. Replacing x by xc^m which forces $i = 1$ or 2, we have $a^{3^n} = 1, b^3 = 1, [a,b] = c, c^3 = 1, [a,c] = a^{3^{n-1}}, [b,c] = 1, [c,x] = a^{u3^{n-1}}, [a,x] = a^{i3^{n-2}}b^j, [b,x] = a^{r3^{n-1}}c^u, x^9 = 1$, where $i = 1$ or 2.

Subcase 4.1 $u = 0$

Assume $x^3 = a^{m_13^{n-1}}b^{m_2}c^{m_3}$. Then $[a,x^3] = [a, a^{m_13^{n-1}}b^{m_2}c^{m_3}] = c^{m_2}a^{m_33^{n-1}}$. On the other hand, $[a,x^3] = [a,x]^3[a,x,x]^3[a,x,x,x] = a^{i3^{n-1}}$. Thus $m_3 = i, m_2 = 0$. That is, $x^3 = a^{m_13^{n-1}}c^i$. Let $c_1 = a^{im_13^{n-1}}c$, $b_1 = bc_1^{im_1}$, $x_1 = xb_1^{ijm_1}$, $x_2 = x_1^i$, $x_3 = x_2{c_1}^{-jr}$ and $r_1 = ir$. Thus $G = \langle a, x_3 \mid a^{3^n} = 1, b_1^3 = 1, [a,b_1] = x_3^3, x_3^9 = 1, [a,x_3] = a^{3^{n-2}}b_1^{j_1}, [b_1,x_3] = a^{r_13^{n-1}}\rangle$, where $j_1, r_1 = 0, 1$ or 2, respectively. By considering all possible values of parameters j_1 and r_1, we get groups (3)–(11).

Subcase 4.2 $u \neq 0$.

It follws from $[a,x,x,x] = [b^j,x,x] = [c^{ju},x] = a^{ju^23^{n-1}}$ that $[a,x^3] = [a,x]^3[a,x,x]^3[a,x,x,x] = a^{i3^{n-1}}a^{ju^23^{n-1}} = a^{(i+j)3^{n-1}}$. On the other hand, since $x^3 \in \Omega_1(H_6)$, we have $x^3 = a^{m_13^{n-1}}b^{m_2}c^{m_3}$. Thus $a^{(i+j)3^{n-1}} = [a,x^3] = [a, a^{m_13^{n-1}}b^{m_2}c^{m_3}] = c^{m_2}a^{m_33^{n-1}}$. It follows that $m_2 = 0, m_3 = i+j$.

Since $[x^3,x] = [a^{m_13^{n-1}}c^{i+j}, x] = 1$, $i + j \equiv 0 \pmod 3$. It follows that $x^3 = a^{m_13^{n-1}}$. Replacing x by $xa^{-m_13^{n-2}}$, we get $x^3 = 1$. Since $i + j \equiv 0 \pmod 3$, we get $j \neq 0$. Thus $G = \langle a, x \mid a^{3^n} = 1, b^3 = 1, [a,b] = c, c^3 = 1, [a,c] = a^{3^{n-1}}, [b,c] = 1, x^3 = 1, [a,x] = a^{i3^{n-2}}b^j, [b,x] = a^{r3^{n-1}}c^u, [c,x] = a^{u3^{n-1}}\rangle$, where $u, j \neq 0$, $i + j \equiv 0 \pmod 3), 1 \leq i \leq 8, 3 \nmid i$.

Let m be an integer satisfying $1 - um \equiv 0 \pmod 3$. Then $[ax^m, b] = [a,b]^{x^m}[x^m,b] \equiv 1 \pmod{\langle a^{3^{n-1}}\rangle}$, $[ax^m, c] = [a,c][x,c]^m = a^{(1-um)3^{n-1}} = 1$. Thus the maximal subgroup $\langle ax^m, b, c\rangle$ of G is a $\mathcal{C}_2$-group generated by three elements. It follows that $\langle ax^m, b, c\rangle$ is one of H_1, H_3 or H_4. It reduces to one of Cases 1, 2 or 3.

Case 5 $M \cong H_{10}$ or H_{12}.

If $M \cong H_{10}$, then, by Theorem 4 we have $M = \langle a, b \mid a^{3^n} = 1, b^{3^2} = 1, [a, b] = a^{3^{n-2}} \rangle$. Since $M' = \langle a^{3^{n-2}} \rangle \leq G'$, we observe that $G' \cong C_9 \times C_3 \cong \langle a^{3^{n-2}} \rangle \times \langle b^3 \rangle$. Assume $[a, x] = a^{i3^{n-2}} b^{3j}$, $[b, x] = a^{r3^{n-2}} b^{3s}$ and $1 \leq i, r \leq 9$. Replacing x by xb^{-i}, we get $[a, x] = b^{3j}$. If the maximal subgroup $\langle a, x, b^3 \rangle$ of G is a $\mathcal{C}_2$-group generated by three elements, then $\langle a, x, b^3 \rangle \cong H_1, H_3$ or H_4. It reduces to one of Cases 1, 2 or 3. If $\langle a, x, b^3 \rangle$ is generated by two elements, then $j \neq 0$. Since $o(x^3) \leq 3$, we have $x^3 = a^{u3^{n-1}} b^{3v}$. Thus $[b, x^3] = 1$. Since $[a, x^3] = [a, x]^3 [a, x, x]^3 [a, x, x, x] = 1$, $x^3 \in Z(G)$. Since $Z(G)$ is cyclic, we have $x^3 = a^{m3^{n-1}}$. Replacing x by $xa^{-m3^{n-2}}$, we get $o(x) = 3$. Thus $\langle a, x, b^3 \rangle \cong H_6$. This reduces to Case 4. If $M \cong H_{12}$, then it reduces to one of Cases 1, 2, 3 or 4 by an argument similar to that for $M \cong H_{10}$.

Those groups listed in the statement of the theorem are pairwise non-isomorphic, and satisfy all hypotheses. The details are omitted. □

4.3. Irregular $\mathcal{C}_3$-groups of order less than 3^7

Since all 3-groups of order less than 3^7 can be found in the SmallGroups database, we learn the following using Magma [3, 4].

Theorem 27 *There are no irregular $\mathcal{C}_3$-groups of order 3^4.*

Theorem 28 *G is an irregular $\mathcal{C}_3$-groups of order 3^5 if and only if G is isomorphic to one of the following groups in the SmallGroups database:*

3,4,5,6,7,8,9,13,14,15,17,18,25,26, 27,28,29,30,51,52,53,54,55,56,57,58,59, or 60.

Theorem 29 *G is an irregular $\mathcal{C}_3$-groups of order 3^6 if and only if G is isomorphic to one of the following groups in the SmallGroups database:*

4,5,6,7,8,13,14,15,16,17,18,19,20,21,27,28,29,67,70,71,74,75,77,80,82,83, 86,90,95,96,97,98,99, 100,101,253,254,261,262,263,264,284,285,388,389,390.

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