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Invertible Toeplitz Operators Products on the Bergman Space of the Polydisk

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Abstract We prove a reverse Hölder inequality by using the cartesian product of dyadic rectangles and the dyadic cartesian product maximal function on Bergman space of polydisk. Next, we further describe when for which square integrable analytic functions f and g on the polydisk the densely defined products $T_fT_{\bar{g}}$ are bounded invertible Toeplitz operators.

Keywords Toeplitz operators; Bergman space; polydisk; reverse Hölder inequality.

MR(2010) Subject Classification 47B35; 47B47

1. Introduction

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Its boundary is the circle $\mathbb{T}$. The polydisk $\mathbb{D}^n$ and the torus $\mathbb{T}^n$ are the subsets of $\mathbb{C}^n$ which are cartesian product of n copies $\mathbb{D}$ and $\mathbb{T}$, respectively. Let $dA(z)$ be the normalized volume measure on $\mathbb{D}^n$. For $\lambda \in \mathbb{D}$, let φ_λ be the fractional linear transformation on $\mathbb{D}$ given by $\varphi_\lambda = \frac{\lambda - z}{1-\bar{\lambda} z}$. Each φ_λ is an automorphism on the disk, in fact, $\varphi_\lambda^{-1} = \varphi_\lambda$. For $w = (w_1, \dots, w_n) \in \mathbb{D}^n$ the mapping φ_w on the polydisk $\mathbb{D}^n$ given by $\varphi_w(z) = (\varphi_{w_1}(z_1), \dots, \varphi_{w_n}(z_n))$ is an automorphism on $\mathbb{D}^n$. The Bergman space $L_a^2(\mathbb{D}^n)$ is the subspace of $L^2(\mathbb{D}^n, dA)$ whose functions are holomorphic in $\mathbb{D}^n$. There is an orthogonal projection P from $L^2(\mathbb{D}^n, dA)$ onto $L_a^2(\mathbb{D}^n)$. The reproducing kernel in $L_a^2(\mathbb{D}^n)$ is given by

$$K_w(z) = \prod_{j=1}^n \frac{1}{(1-\overline{w}_j z_j)^2} = \prod_{j=1}^n [\sum_{k=0}^\infty (k+1)\overline{w_j}^k {z_j}^k]$$

for $z = (z_1, z_2, \dots, z_n), w = (w_1, w_2, \dots, w_n) \in \mathbb{D}^n$. Let $\varphi \in L^\infty(\mathbb{D}^n)$. The Toeplitz operators with symbol φ is the operator $T_\varphi : L_a^2(\mathbb{D}^n) \to L_a^2(\mathbb{D}^n)$ defined by

$$T_\varphi f = P(\varphi f) = \int_{\mathbb{D}^n} \varphi(w) f(w) \overline{K_z(w)} \mathrm{d}A(w).$$

The Berezin transform of a function $f \in L^2(\mathbb{D}^n, \mathrm{d}A)$ is defined on $\mathbb{D}^n$ by

$$\widetilde{f}(w) = \int_{\mathbb{D}^n} f(z) |k_w(z)|^2 \mathrm{d}A(z)$$

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for $w\in\mathbb{D}^n$, where $k_w(z)=\prod_{j=1}^n \frac{1-|w_j|^2}{(1-\overline{w_j}z_j)}$ are the normalized reproducing kernels for $L_a^2(\mathbb{D}^n)$.

The question for which f and g in $L_a^2(\mathbb{D}^n)$ the Toeplitz operator $T_fT_{\bar g}$ is bounded on $L_a^2(\mathbb{D}^n)$ was considered in [1]. The following result was proved in [1].

Theorem 1.1 *Let f and g be in $L_a^2(\mathbb{D}^n)$.*

(i) If the Toeplitz product $T_fT_{\bar g}$ is bounded on $L_a^2(\mathbb{D}^n)$, then

$$\sup_{w\in\mathbb{D}^n}\widetilde{|f|^2}(w)\widetilde{|g|^2}(w)<\infty,$$

(ii) If

$$\sup_{w\in\mathbb{D}^n}\widetilde{|f|^{2+\varepsilon}}(w)\widetilde{|g|^{2+\varepsilon}}(w)<\infty$$

for some $\varepsilon>0$, then operator $T_fT_{\bar g}$ is bounded on $L_a^2(\mathbb{D}^n)$.

In the above theorem the necessary condition is very close to being sufficient for boundedness. In [4], Stroethoff and Zheng proved the analogous results on $L_a^2(\mathbb{D})$ and made the conjecture that for f and g in $L_a^2(\mathbb{D})$ this product $T_fT_{\bar g}$ be bounded on $L_a^2(\mathbb{D})$ if and only if $\sup_{w\in\mathbb{D}}\widetilde{|f|^2}(w)\widetilde{|g|^2}(w)<\infty$. The authors [2] showed that if f and g are in $L_a^2(\mathbb{D})$, then the product $T_fT_{\bar g}$ is bounded and invertible on $L_a^2(\mathbb{D})$ if and only if $\sup_{w\in\mathbb{D}}\widetilde{|f|^2}(w)\widetilde{|g|^2}(w)<\infty$ and $\inf_{w\in\mathbb{D}}|f(w)g(w)|>0$. Next, in [3] the authors extended the results [2] to weighted Bergman space of unit disk.

In this paper, we will be concerned with the question for which f and g in $L_a^2(\mathbb{D}^n)$ the Toeplitz operator product $T_fT_{\bar g}$ is invertible on $L_a^2(\mathbb{D}^n)$. In [2] and [3] the proof made use of dyadic rectangles and dyadic maximal function. Our method is partially adapted from those in [2] and [3]. Let $d(Q)$ denote the distance between dyadic rectangles Q and unit circle $\partial\mathbb{D}$. The proof in [2] and [3] need two different methods in view of the distance $d(Q)>0$ and $d(Q)=0$. We need consider the question on the dyadic rectangles cartesian product $Q_1\times\cdots\times Q_n$. In our case, some $d(Q_k)=0$, for $k\in\alpha$, where $\alpha=\{\alpha_1,\dots,\alpha_m\}$ is a subset of $\{1,\dots,n\}$ with $\alpha_1<\cdots<\alpha_m$, and α runs over all subsets of $\{1,\dots,n\}$; the other $d(Q_k)>0$, $k\in\{1,\dots,n\}\backslash\alpha$. Therefore, the cases will be more complicated than on the unit disk. Our proof for Lemma 2.5 need use weighted condition (A_2). Thus, we cannot directly obtain the necessary and sufficient condition for $T_fT_{\bar g}$ to be bounded and invertible on $L_a^2(\mathbb{D}^n)$. The theorems that we get have more complicated forms. First, we are ready to prove the reverse Hölder inequality. By means of the inequality, we will get our main result.

2. A reversed Hölder inequallty

In this section we will prove a reverse Hölder inequality for $f(\cdot,\eta)$ in $L_a^2(\mathbb{D}^n)$ satisfying the following invariant weight condition:

$$\sup_{\eta\in\mathbb{D}^\alpha}\widetilde{|f|^2}(\cdot,\eta)\widetilde{|f|^{-2}}(\cdot,\eta)<\infty. \tag{A$_2$}$$

We will prove that the above condition implies that

$$\sup_{\eta\in\mathbb{D}^\alpha}\widetilde{|f|^{2+\varepsilon}}(\cdot,\eta)\widetilde{|f|^{-(2+\varepsilon)}}(\cdot,\eta)<\infty, \tag{A$_{2+\varepsilon}$}$$

for sufficiently small $\varepsilon > 0$. Where $\eta \in \mathbb{D}^\alpha = \mathbb{D}_{\alpha_1} \times \cdots \times \mathbb{D}_{\alpha_m}$, $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ is a nonempty subset of $\{1, \ldots, n\}$ with $\alpha_1 < \cdots < \alpha_m$, and α runs over all subsets of $\{1, \ldots, n\}$; $f(\cdot, \eta)$ denotes the function dependent on variables η. Meanwhile, we can get $f(\cdot, \eta) \in L_a^2(\mathbb{D}^\alpha)$ according to $f(\cdot, \eta) \in L_a^2(\mathbb{D}^n)$.

Theorem 2.1 *Suppose that $f(\cdot, \eta) \in L_a^2(\mathbb{D}^n)$ satisfies condition (A_2) with*

$$M = \sup_{\eta \in \mathbb{D}^\alpha} \widetilde{|f|^2}(\cdot, \eta) \widetilde{|f|^{-2}}(\cdot, \eta) < \infty.$$

Then there exist constants ε_M and C_M which depend on M such that

$$\widetilde{|f|^{2+\varepsilon}}(\cdot, \eta) \leq C_M \Big(\widetilde{|f|^2}(\cdot, \eta) \Big)^{(2+\varepsilon)/2}$$

for every $\eta \in \mathbb{D}^\alpha$ and $0 < \varepsilon < \varepsilon_M$. Where $\mathbb{D}^\alpha = \mathbb{D}_{\alpha_1} \times \cdots \times \mathbb{D}_{\alpha_m}$, $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ is a nonempty subset of $\{1, \ldots, n\}$ with $\alpha_1 < \cdots < \alpha_m$, and α runs over all subsets of $\{1, \ldots, n\}$.

As in [2], the proof of Theorem 2.1 will make use of the cartesian product of dyadic rectangles and the dyadic cartesian product maximal function. We first discuss the dyadic rectangles and prove some elementany properties related to these rectangles.

Dyadic rectangles Any set of the form

$$Q_{l_j, m_j, k_j} = \{ r_j e^{i\theta_j} : (m_j - 1) 2^{-l_j} \leq r_j < m_j 2^{-l_j} \text{ and } (k_j - 1) 2^{-l_j + 1} \pi \leq \theta_j < k_j 2^{-l_j + 1} \pi \},$$

where l_j, m_j, k_j are positive integers such that $m_j \leq 2^{l_j}$, $k_j \leq 2^{l_j}$, $j = 1, 2, \ldots, n$, is called a dyadic rectangle. The center of the above dyadic rectangle $Q_j = Q_{l_j, m_j, k_j}$ is the point $z_{Q_j} = (m_j - \frac{1}{2}) 2^{-l_j} e^{i v_j}$ with $v_j = (k_j - \frac{1}{2}) 2^{1 - l_j} \pi$. Let $|E|$ denote the normalized area of measurable set $E \in \mathbb{D}$. If $d(Q_j)$ denotes the distance between Q_j and $\partial \mathbb{D}$, then a calculation shows that

$$|Q_j| = 8 |z_{Q_j}| (1 - |z_{Q_j}| - d(Q_j))^2. \tag{2.1}$$

In view of the Lemma 2.2 of [2], we can obtain the following Lemma.

Lemma 2.2 *Let Q_j be a dyadic rectangle with center $w_j = z_{Q_j}$. For a subset $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ of $\{1, \ldots, n\}$ with $\alpha_1 < \cdots < \alpha_m$, there is a constant $C_1 > 0$ such that*

$$\prod_{j \in \alpha} |k_{w_j}(z_j)|^2 \geq C_1 \prod_{j \in \alpha} \frac{1}{(1 - |w_j|)^2}$$

for every $z_j \in Q_{\alpha_j}$, where α runs over all subsets of $\{1, \ldots, n\}$.

Let $D(w_j, s_j)$ denote the pseudohyperbolic disk with center $w_j \in \mathbb{D}_j$ and radius $0 < s_j < 1$, i.e.,

$$D(w_j, s_j) = \{ z_j \in \mathbb{D}_j : |\varphi_{w_j}(z_j)| < s_j \}.$$

Lemma 2.3 *Suppose that $f(\cdot, w) \in L_a^2(\mathbb{D}^n)$ satisfies the invariant weight condition (A_2) and let $0 < s_j < 1$. For a subset $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ of $\{1, \ldots, n\}$ with $\alpha_1 < \cdots < \alpha_m$, let $w = \{w_{\alpha_1}, \ldots, w_{\alpha_m}\}$. There is a constant $C_{s_j} > 0$ such that*

$$\frac{1}{C_{s_j}} \leq \frac{|f(\cdot, \xi)|}{|f(\cdot, w)|} \leq C_{s_j}$$

whenever $\xi \in D(w_{\alpha_1}, s_{\alpha_1}) \times \cdots \times D(w_{\alpha_m}, s_{\alpha_m})$, *where* α *runs over all subsets of* $\{1, \ldots, n\}$.

Proof Let u be in $D(0, s_{\alpha_1}) \times \cdots \times D(0, s_{\alpha_m})$. Since $f(\cdot, \xi) \in L_a^2(\mathbb{D}^\alpha)$, we have $f(u) = \langle f, K_u \rangle$. Applying the Cauchy-Schwarz inequality, for each $u \in D(0, s_{\alpha_1}) \times \cdots \times D(0, s_{\alpha_m})$, we obtain

$$|f(\cdot, \xi)| \le \|f\|_2 \|K_u\|_2 = \|f\|_2 \prod_{j\in\alpha} \frac{1}{1 - |u_j|^2} \le \prod_{j\in\alpha} \frac{\|f\|_2}{1 - {s_j}^2}.$$

Now if $\xi_j \in D(w_{\alpha_j}, s_j)$, let $|u_j| = |\varphi_{w_{\alpha_j}}(\xi_j)| < s_j$. Then for some $u_j \in D(0, s_j)$, $u_j = \varphi_{w_{\alpha_j}}(\xi_j)$, i.e., $\varphi_{w_{\alpha_j}}(u_j) = \varphi_{w_{\alpha_j}} \circ \varphi_{w_{\alpha_j}}(\xi_j) = \xi_j$. Replacing f by $f \circ \varphi_w$ in the above inequality gives

$$|f(\cdot, \xi)| = |(f \circ \varphi_w)(\cdot, u)| \le \prod_{j\in\alpha} \frac{\|f \circ \varphi_w\|_2}{1 - s_j^2} = \prod_{j\in\alpha} \frac{1}{1 - s_j^2} \widetilde{|f|^2}(\cdot, w)^{\frac{1}{2}}.$$

Since $f(\cdot, w) \in L_a^2(\mathbb{D}^\alpha)$, by the Cauchy-Schwarz inequality, we can get

$$|f(\cdot, w)|^{-1} = |(f^{-1} \circ \varphi_w)(\cdot, 0, \ldots, 0)| \le \int_{\mathbb{D}^\alpha} |f^{-1} \circ \varphi_w|(\cdot, z) \mathrm{d}A(z) \le \|f^{-1} \circ \varphi_w\|_2 = \widetilde{|f^{-1}|^2}(\cdot, w)^{1/2}.$$

Combining these inequalities and the invariant weight condition (A_2), we have

$$\frac{f(\cdot, \xi)}{f(\cdot, w)} \le \prod_{j\in\alpha} \frac{1}{1 - s_j^2} \widetilde{|f|^2}(\cdot, w)^{\frac{1}{2}} \widetilde{|f|^{-2}}(\cdot, w)^{\frac{1}{2}} \le C_{s_j},$$

for all $\xi \in D(w_{\alpha_1}, s_{\alpha_1}) \times \cdots \times D(w_{\alpha_m}, s_{\alpha_m})$. Replacing f by f^{-1} gives the other inequality. $\square$

Lemma 2.4 ([3]) *There exists* $0 < R_j < 1$ *such that*

$$Q_j \subset D(z_{Q_j}, R)$$

for every dyadic rectangle in $\mathbb{D}$ *that has positive distance to* $\partial\mathbb{D}$.

Lemma 2.5 *If* $f(\cdot, \eta) \in L_a^2(\mathbb{D}^n)$ *satisfies the invariant weight condition* (A_2), *then there is a constant* $C(M)$ *depending on* M *such that*

$$\Big(\frac{1}{|Q|} \int_Q |f|^2(\cdot, \eta) \mathrm{d}A(\eta)\Big) \Big(\frac{1}{|Q|} \int_Q |f|^{-2}(\cdot, \eta) \mathrm{d}A(\eta)\Big) \le C(M)$$

for every $\eta \in Q$, *where* $Q = Q_{\zeta_1} \times Q_{\zeta_2} \times \cdots \times Q_{\zeta_m}$ *is the cartesian product of dyadic rectangles,* $\zeta = \{\zeta_1, \ldots, \zeta_m\}$ *is a nonempty subset of* $\{1, \ldots, n\}$ *with* $\zeta_1 < \cdots < \zeta_m$, *and* ζ *runs over all subsets of* $\{1, \ldots, n\}$.

Proof Now suppose that $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ is a subset of $\{\zeta_1, \zeta_2, \ldots, \zeta_m\}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_t$, $\beta = \{\zeta_1, \zeta_2, \ldots, \zeta_m\} \setminus \alpha = \{\beta_1, \beta_2, \ldots, \beta_{m-t}\}$. Next assume that

$$\begin{cases} d(Q_j) = 0, & \text{if } j \in \alpha; \\ d(Q_j) > 0, & \text{if } j \in \beta. \end{cases}$$

First we fix $j \in \beta$, $d(Q_j) > 0$. By Lemma 2.4, $Q_j \subset D(z_{Q_j}, R_j)$, where $0 < R_j < 1$. By Lemma 2.3, there exists a positive constant C such that

$$\frac{1}{C} |f(\cdot, z_{Q_{\beta_1}}, \ldots, z_{Q_{\beta_{m-t}}})| < |f(\cdot, \xi)| < C |f(\cdot, z_{Q_{\beta_1}}, \ldots, z_{Q_{\beta_{m-t}}})|$$

for all $\xi \in Q_{\beta_1} \times Q_{\beta_2} \times \cdots \times Q_{\beta_{m-t}}$. Therefore for $f(\cdot,\eta) \in L^2_a(\mathbb{D}^\alpha)$, we have

$$\begin{aligned}
\mathbb{I} &= \frac{1}{|Q|}\int_Q |f|^2(\cdot,\eta)\mathrm{d}A(\eta)\\
&= \frac{1}{|Q_1|\times|Q_2|\times\cdots\times|Q_m|}\int_{Q_1}\int_{Q_2}\cdots\int_{Q_m}|f(\cdot,\xi_1,\xi_2,\ldots,\xi_m)|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_m\\
&= \frac{1}{\prod_{j=1}^m|Q_j|}\int_{\prod\limits_{h\in\beta}Q_h}\int_{\prod\limits_{t\in\alpha}Q_t}|f(\cdot,\xi_{\alpha_1},\ldots,\xi_{\alpha_t},\eta_{\beta_1},\ldots,\eta_{\beta_{m-t}})|^2\mathrm{d}\xi_{\alpha_1}\ldots\mathrm{d}\xi_{\alpha_t}\mathrm{d}\eta_{\beta_1}\ldots\mathrm{d}\eta_{\beta_{m-t}}\\
&\le \frac{C}{|Q_{\alpha_1}|\times|Q_{\alpha_2}|\times\cdots|Q_{\alpha_t}|}\int_{\prod\limits_{h\in\alpha}Q_h}|f(\cdot,\xi_{\alpha_1},\ldots,\xi_{\alpha_t},z_{Q_{\beta_1}},\ldots,z_{Q_{\beta_{m-t}}})|^2\mathrm{d}\xi_{\alpha_1}\cdots\mathrm{d}\xi_{\alpha_t}.
\end{aligned}$$

Next assume that $j \in \alpha$ and $d(Q_j) = 0$, then we have $|z_{Q_j}| \ge \frac{1}{2}$, and it follows from (2.1) that

$$|Q_j| = 8|z_{Q_j}|(1-|z_{Q_j}|-d(Q_j))^2 = 8|z_{Q_j}|(1-|z_{Q_j}|)^2 \ge 4(1-|z_{Q_j}|)^2.$$

Using Lemma 2.2 gives

$$\begin{aligned}
\mathbb{J} &= \frac{C}{|Q_{\alpha_1}|\times|Q_{\alpha_2}|\times\cdots|Q_{\alpha_m}|}\int_{\prod\limits_{h\in\alpha}Q_h}|f(\cdot,\xi_{\alpha_1},\ldots,\xi_{\alpha_t},z_{Q_{\beta_1}},\ldots,z_{Q_{\beta_{m-t}}})|^2\mathrm{d}\xi_{\alpha_1}\cdots\mathrm{d}\xi_{\alpha_t}\\
&\le \frac{C_1}{\prod\limits_{j\in\alpha}(|1-|z_{Q_j}|)^2}\int_{\prod\limits_{h\in\alpha}Q_h}|f(\cdot,\xi_{\alpha_1},\ldots,\xi_{\alpha_t},z_{Q_{\beta_1}},\ldots,z_{Q_{\beta_{m-t}}})|^2\mathrm{d}\xi_{\alpha_1}\cdots\mathrm{d}\xi_{\alpha_t}\\
&\le C_2\int_{\prod\limits_{h\in\alpha}Q_h}|f(\cdot,\xi_{\alpha_1},\ldots,\xi_{\alpha_t},z_{Q_{\beta_1}},\ldots,z_{Q_{\beta_{m-t}}})|^2\prod_{j\in\alpha}|k_{z_{Q_j}}(\xi_{\alpha_j})|^2\mathrm{d}\xi_{\alpha_1}\ldots\mathrm{d}\xi_{\alpha_t}\\
&\le C_2\int_{\prod\limits_{h\in\alpha}\mathbb{D}_h}|f(\cdot,\xi_{\alpha_1},\ldots,\xi_{\alpha_t},z_{Q_{\beta_1}},\ldots,z_{Q_{\beta_{m-t}}})|^2\prod_{j\in\alpha}|k_{z_{Q_j}}(\xi_{\alpha_j})|^2\mathrm{d}\xi_{\alpha_1}\ldots\mathrm{d}\xi_{\alpha_t}\\
&= C_2\widetilde{|f|^2}(\cdot,z_{Q_{\alpha_1}},z_{Q_{\alpha_2}},\ldots,z_{Q_{\alpha_t}}).
\end{aligned}$$

A similar inequality holds for f^{-1}. Thus we have

$$\begin{aligned}
&\Big(\frac{1}{|Q|}\int_Q|f|^2(\cdot,\eta)\mathrm{d}A\Big)\Big(\frac{1}{|Q|}\int_Q|f|^{-2}(\cdot,\eta)\mathrm{d}A\Big)\\
&\quad\le C_2^2[\widetilde{|f|^2}(\cdot,z_{Q_{\alpha_1}},\ldots,z_{Q_{\alpha_t}})][\widetilde{|f|^{-2}}(\cdot,z_{Q_{\alpha_1}},\ldots,z_{Q_{\alpha_t}})] \le MC_2^2. \quad \square
\end{aligned}$$

Lemma 2.6 *Suppose that $f(\cdot,\eta) \in L^2_a(\mathbb{D}^n)$ satisfies the invariant weighted condition (A_2). For every η, $w \in \mathbb{D}^\alpha$, let $d\mu_w = |f\circ\varphi_w|^2(\cdot,\eta)\mathrm{d}A(\eta)$. If $0<\gamma<1$, then there exists a $0<\delta(\gamma,M)<1$ such that*

$$\mu_w(E) \le \delta(\gamma,M)\mu_w(Q^\alpha)$$

whenever E is a subset of $Q^\alpha = Q_{\alpha_1}\times\cdots\times Q_{\alpha_m}$ with $|E| \le \gamma|Q^\alpha|$, where $\alpha = \{\alpha_1,\ldots,\alpha_m\}$ is a nonempty subset of $\{1,\ldots,n\}$ with $\alpha_1<\cdots<\alpha_m$, and $\delta(\gamma,M)$ depends on γ and M.

Proof Suppose that $\widetilde{|f|^2}(\cdot,\eta)\widetilde{|f|^{-2}}(\cdot,\eta) \le M$, for all $\eta \in \mathbb{D}^\alpha$. Let E be a subset of Q^α with $|E| \le \gamma|Q^\alpha|$. Applying the inequality of Cauchy-Schwarz and Lemma 2.5, we have

$$(|Q^\alpha|-|E|)^2 = \Big(\int_{Q^\alpha\setminus E}|f\circ\varphi_w||f\circ\varphi_w|^{-1}(\cdot,\eta)\mathrm{d}A(\eta)\Big)^2$$

$$
\begin{aligned}
&\le \Big(\int_{Q^\alpha\setminus E}|f\circ\varphi_w|^2(\cdot,\eta)\mathrm{d}A(\eta)\Big)\Big(\int_{Q^\alpha\setminus E}|f\circ\varphi_w|^{-2}(\cdot,\eta)\mathrm{d}A(\eta)\Big)\\
&\le \Big(\int_{Q^\alpha\setminus E}|f\circ\varphi_w|^2(\cdot,\eta)\mathrm{d}A(\eta)\Big)\Big(\int_{Q^\alpha}|f\circ\varphi_w|^{-2}(\cdot,\eta)\mathrm{d}A(\eta)\Big)\\
&\le \Big(\int_{Q^\alpha\setminus E}|f\circ\varphi_w|^2(\cdot,\eta)\mathrm{d}A(\eta)\Big)C(M)|Q^\alpha|^2\Big(\int_{Q^\alpha}|f\circ\varphi_w|^{-2}(\cdot,\eta)\mathrm{d}A(\eta)|\Big)^{-1}\\
&= C(M)|Q^\alpha|^2\Big[1-\frac{\mu_w(E)}{\mu_w(Q^\alpha)}\Big].
\end{aligned}
$$

It follows that

$$
\frac{\mu_w(E)}{\mu_w(Q^\alpha)}\le 1-\frac{1}{C(M)}\Big(1-\frac{|E|}{|Q^\alpha|}\Big)^2\le\delta(\gamma,M),
$$

if we put $\delta(\gamma,M)=1-(1-\gamma)^2/C(M)$. $\Box$

The dyadic cartesian product maximal function The dyadic cartesian product maximal operator M^Δ is defined by

$$
(M^\Delta f)(w)=\sup_{w\in Q}\frac{1}{|Q|}\int_Q|f|\mathrm{d}A
$$

where the supremum is over all the cartesian product of dyadic rectangles $Q=Q_1\times Q_2\times\cdots Q_n$ that contain w. The maximal function is greater than the dyadic cartesian products maximal function, so the dyadic cartesian product maximal function of any continuous function is finite on $\mathbb{D}^n$. In particular, if $f\in L^2_a(\mathbb{D}^n)$ satisfies the invariant condition (A_2), then the dyadic cartesian product maximal function $M^\Delta|f|^2$ is always finite. This can also be seen directly as follows. Given a point $w=(w_1,\dots,w_n)\in\mathbb{D}^n$, there is a number $R=\{R_1,\dots,R_n\}$, $0<R_j<1$, $1\le j\le n$ such that all but a finite number of the cartesian product of dyadic rectangles containing the point w lie inside the closed disk $\bar{D}(0,R)=\{z=(z_1,\dots,z_n):|z_i|\le R_i\}$. If $f\in L^2_a(\mathbb{D}^n)$ and Q is a cartesian product of dyadic rectangle containing w inside the disk $\bar{D}(0,R)$, then

$$
\frac{1}{|Q|}\int_Q|f(z)|^2\mathrm{d}A(z)\le\max\{|f(z)|^2:|z_j|\le R_j\}.
$$

If $Q_{\{1\}},\dots,Q_{\{s\}}$ are the cartesian product of dyadic rectangles containing w not contained in the disk $\bar{D}(0,R)$, then

$$
M^\Delta|f|^2(w)\le\max\{|f(z)|^2:|z_j|\le R_j\}+\max_{1\le i\le s}\frac{1}{|Q_{\{i\}}|}\int_{Q_{\{i\}}}|f(z)|^2\mathrm{d}A(z)<\infty.
$$

This proves that the dyadic cartesian product maximal function of $|f|^2$ is finite on $\mathbb{D}^n$.

The principal fact about the dyadic cartesian product maximal function is the Calderon-Zygmund decomposition formulated in the next theorem. We will need the notion of "doubling" of the dyadic rectangles in its proof. Suppose that $l_j\ge1$ and m_j, k_j are positive integers such that $m_j,k_j\le 2^{l_j}$. The double of $Q_j=Q_{l_j,m_j,k_j}$ denoted by $2Q_j$, is defined by

$$
2Q_j=Q_{l_j-1,[(m_j+1)/2],[(k_j+1)/2]},
$$

where $[k]$ denotes the greatest integer less than or equal to k. An elementary calculation shows that

$$|2Q_j|/|Q_j| \leq 8,$$

for every dyadic rectangle Q_j in the unit disk.

Calderon-Zygmund decomposition theorem *Let f be locally integrable on $\mathbb{D}^n$, and $t > 0$. Suppose that $\Omega = \{z \in \mathbb{D}^n : M^{\Delta} f(z) > t\}$ is not to $\mathbb{D}^n$. Then Ω may be written as disjoint union of cartesian product of dyadic rectangles $Q_{\{i\}}$ with*

$$t < \frac{1}{|Q_{\{i\}}|} \int_{Q_{\{i\}}} |f| \mathrm{d}A < 8^n t.$$

Proof Suppose that $w \in \Omega$, then $M^{\Delta} f(w) > t$. Then there exists a cartesian product of dyadic rectangles Q containing w such that

$$\frac{1}{|Q|} \int_Q |f| \mathrm{d}A > t.$$

Now, if $z \in Q$, then

$$M^{\triangle} f(z) \geq \frac{1}{|Q|} \int_Q |f| \mathrm{d}A > t.$$

It follows $z \in \Omega$. Thus proves that $Q \subset \Omega$. We may assume that the $Q_{\{i\}}$ are the maximal cartesian product of dyadic rectangles. It follows that $\Omega = \bigcup_i Q_{\{i\}}$. Since $Q = Q_{\{i\}}$ is not equal to $\mathbb{D}^n$, by maximality its double $2Q$ is not contained in Ω. This means that $2Q$ contains a point z which is not in Ω. Since $M^{\Delta} f(z) > t$, we obtain

$$\frac{1}{|2Q|} \int_{2Q} |f| \mathrm{d}A \leq M^{\triangle} f(z) \leq t.$$

Hence, it follows that

$$\frac{1}{|Q|} \int_Q |f| \mathrm{d}A \leq \frac{1}{|Q|} \int_{2Q} |f| \mathrm{d}A \leq \frac{t|2Q|}{|Q|} \leq 8^n t. \quad \square$$

Before we prove the reverse Hölder inequality, we need one more preliminary result for the dyadic maximal function:

Proposition 2.7 *If $f(\cdot, \eta) \in L_a^2(\mathbb{D}^n)$, $\mathbb{D}^\alpha = \mathbb{D}_{\alpha_1} \times \cdots \times \mathbb{D}_{\alpha_m}$, then*

(i) $|f|^2(\cdot, \eta) \leq M^{\Delta} |f|^2(\cdot, \eta)$ on $\mathbb{D}^\alpha$, and

(ii) $\|f\|_2^2 \leq M^{\Delta} |f|^2(\cdot, 0, \ldots, 0) \leq 2^m \|f\|_2^2$.

Here $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ is a nonempty subset of $\{1, \ldots, n\}$ with $\alpha_1 < \cdots < \alpha_m$.

Proof (i) We will prove that if g is continuous on $\mathbb{D}^\alpha$, then $|g(\cdot, w)| \leq M^{\Delta} g(\cdot, w)$ for every $w \in \mathbb{D}^\alpha$. Fix $w \in \mathbb{D}^\alpha$. Let $Q_{\{0\}} = Q_{01} \times \cdots \times Q_{0m}$ be any cartesian product of dyadic rectangles containing w such that $Q_{0j} \subset \mathbb{D}_j$. Since function g is uniformly continuous on $Q_{\{0\}}$, given $\varepsilon > 0$, there is a $\delta > 0$, such that $|g(\cdot, z) - g(\cdot, w)| < \varepsilon$, whenever $z, w \in Q_{\{0\}}$ are such that $|z - w| = \max_{1 \leq j \leq m} |z_j - w_j| < \delta$. Subdivide $Q_{\{0\}}$ a number of times, then there exists a cartesian product of dyadic rectangles Q containing w with diameter less than δ. Then

$$|g(\cdot, w)| \leq |g(\cdot, z)| + |g(\cdot, z) - g(\cdot, w)| \leq |g(\cdot, z)| + \varepsilon$$

for all $z \in Q$. This implies that

$$|g(\cdot,w)| \le \frac{1}{|Q|}\int_Q |g(\cdot,z)|\mathrm{d}A(z)+\varepsilon \le M^{\Delta}g(\cdot,w)+\varepsilon.$$

Therefore, we can obtain

$$|g(\cdot,w)| \le M^{\Delta}g(\cdot,w).$$

(ii) Since $\mathbb{D}^\alpha$ is a cartesian product dyadic rectangles, we have

$$M^{\Delta}|f|^2(\cdot,0,\dots,0) \ge \frac{1}{|\mathbb{D}|^\alpha}\int_{\mathbb{D}^\alpha}|f|^2(\cdot,\eta)\mathrm{d}A(\eta)=\|f\|_2^2,$$

for $f(\cdot,\eta)\in L_a^2(\mathbb{D}^\alpha)$. If Q_j is a dyadic rectangle other than $\mathbb{D}$ containing 0, then $Q_j \subset D_j(0,1/2)$, $j\in\alpha$. For each $z\in\mathbb{D}^\alpha$, we have $f(\cdot,z)=\langle f,K_z\rangle$ and the inequality of Cauchy-Schwarz implies

$$|f(\cdot,z)|^2 \le \|f\|^2\|K_z\|^2=\prod_{j=1}^m \frac{1}{(1-|z_j|^2)}\|f\|_2^2$$

for all $z\in D_1(0,1/2)\times\cdots\times D_m(0,1/2)$. Since $Q_j\subset D_j(0,1/2)$, it follows that

$$\frac{1}{|Q|}\int_Q |f|^2\mathrm{d}A \le (4/3)^m\|f\|_2^2 \le 2^m\|f\|_2^2. \quad \square$$

We are now ready to prove the reversed Hölder inequality contained in Theorem 2.1.

Proof of Theorem 2.1 First we prove that for some constant $C_M>0$,

$$\int_{\mathbb{D}^\alpha}|f(\cdot,z)|^{2+\varepsilon}\mathrm{d}A(z) \le C_M\Big(\int_{\mathbb{D}^\alpha}|f|^2\mathrm{d}A(z)\Big)^{(2+\varepsilon)/2}.$$

For each integer $k\ge 0$, set

$$E_k=\{z\in\mathbb{D}^\alpha: M^{\Delta}|f|^2(\cdot,z)>2^{3mk+m}\|f\|_2^2\}.$$

Since $M^{\Delta}|f|^2(\cdot,0,\dots,0)\le 2^m\|f\|_2^2\le 2^{3mk+m}\|f\|_2^2$, it follows from proposition 2.7 (ii) that for every positive integer k set E_k is not equal to $\mathbb{D}^\alpha$. Fix $k\ge 1$. By the Calderon-Zygmund decomposition theorem, $E_k=\bigcup_i Q_{\{i\}}$, where $Q_{\{i\}}$ are disjoint cartesian product of dyadic rectangles in E_k that satisfy

$$2^{3mk+m}\|f\|_2^2 < \frac{1}{|Q_{\{i\}}|}\int_{Q_{\{i\}}}|f|(\cdot,z)\mathrm{d}A(z) < 8^m\times 2^{3mk+m}\|f\|_2^2,$$

thus

$$|Q_{\{i\}}| \le 2^{-3mk-m}\|f\|_2^{-2}\int_{Q_{\{i\}}}|f|\mathrm{d}A, \text{ and } \int_{Q_{\{i\}}}|f|\mathrm{d}A < 8^m\times 2^{3mk+m}\|f\|_2^2|Q_{\{i\}}|.$$

Let Q be a maximal cartesian product of dyadic rectangle in E_{k-1}. Summing over all such $Q_{\{i\}}\subset Q$ gives that

$$|E_k\bigcap Q|=\sum_{i:Q_{\{i\}}\subset Q}|Q_{\{i\}}| \le 2^{-3mk-m}\|f\|_2^{-2}\int_Q|f|^2\mathrm{d}A,$$

since the $Q_{\{i\}}$ are disjoint and their union is E_k. On the other hand, by maximality the double $2Q$ is not contained in E_{k-1}, and as in the proof of the Calderon-Zygmund decomposition theorem it follows that

$$\int_Q |f|^2 \mathrm{d}A \leq 2^{3m(k-1)} 8^m \|f\|_2^2 |Q|.$$

Hence

$$|E_k \cap Q| \leq 1/2^m |Q|.$$

By Lemma 2.6 there exists a $0 < \delta < 1$ such that

$$\mu(E_k \cap Q) \leq \delta\mu(Q),$$

where $\mathrm{d}\mu = |f|^2(\cdot, z)\mathrm{d}A(z)$. Taking the union over all maximal cartesian product of dyadic rectangles Q in E_{k-1} gives

$$\mu(E_k) \leq \delta\mu(E_{k-1}),$$

and therefore

$$\mu(E_k) \leq \delta^k \mu(E_0) \leq \delta^k \|f\|_2^2.$$

Using Proposition 2.7, we have

$$\begin{aligned}
\int_{\mathbb{D}^\alpha} |f|^{2+\varepsilon}(\cdot, z)\mathrm{d}A(z) &\leq \int_{\mathbb{D}^\alpha} (M^\Delta |f|^2)^{\varepsilon/2} |f|^2 \mathrm{d}A(z) \\
&= \int_{\{M^\Delta |f|^2 \leq 2^m \|f\|_2^2\}} (M^\Delta |f|^2)^{\varepsilon/2} |f|^2 \mathrm{d}A + \sum_{k=0}^{\infty} \int_{E_k \setminus E_{k+1}} (M^\Delta |f|^2)^{\varepsilon/2} |f|^2 \mathrm{d}A \\
&\leq 2^m \|f\|_2^\varepsilon \|f\|_2^2 + \sum_{k=0}^{\infty} 2^{(3m(k+1)+1)\varepsilon/2} \|f\|_2^\varepsilon \mu(E_k) \\
&\leq 2^m \|f\|_2^{2+\varepsilon} + \sum_{k=0}^{\infty} 2^{(3m(k+1)+1)\varepsilon/2} \delta^k \|f\|_2^{2+\varepsilon} \\
&\leq (2^m + \frac{2^{(3m+1)\varepsilon/2}}{1 - 2^{3m\varepsilon/2}\delta}) \|f\|_2^{2+\varepsilon},
\end{aligned}$$

if $2^{3m\varepsilon/2}\delta < 1$. We let $2^{3m\varepsilon_M/2}\delta = 2\delta/(1+\delta)$, then it follows that $\varepsilon_M = \ln(2/(1+\delta))/(3m\ln\sqrt{2})$. If $0 < \varepsilon < \varepsilon_M$, then $2^{3m\varepsilon/2}\delta < 2\delta/(1+\delta) < 1$, so that

$$\frac{2^{(3m+1)\varepsilon/2}}{1 - 2^{3m\varepsilon/2}\delta} < \frac{(2/(1+\delta))^{(1/3m+1)}}{1 - 2\delta/(1+\delta)} \leq \frac{2^{(1/3m+2)}}{1-\delta}.$$

So, if $C_M = 2^m + \frac{2^{(1/3m+2)}}{1-\delta}$, then for $0 < \varepsilon < \varepsilon_M$ we have shown that

$$\int_{\mathbb{D}^\alpha} |f|^{2+\varepsilon}(\cdot, z)\mathrm{d}A \leq C_M \Big(\int_{\mathbb{D}^\alpha} |f|^2(\cdot, z)\mathrm{d}A \Big)^{(2+\varepsilon)/2}.$$

For a fixed $w \in \mathbb{D}^\alpha$, by Möbius invariance of Berezin transform we also have

$$M = \sup_{z \in \mathbb{D}^\alpha} |\widetilde{f \circ \varphi_w}|^2(\cdot, z) |\widetilde{f \circ \varphi_w}|^{-2}(\cdot, z).$$

Applying the above argument to the function $|f \circ \varphi_w|^2$, we obtain

$$\int_{\mathbb{D}^\alpha} |f \circ \varphi_w|^{2+\varepsilon}(\cdot, z)\mathrm{d}A(z) \leq C_M \Big(\int_{\mathbb{D}^\alpha} |f \circ \varphi_w|^2(\cdot, z)\mathrm{d}A(z) \Big)^{(2+\varepsilon)/2},$$

that is,

$$\widetilde{|f|^{2+\varepsilon}}(\cdot, w) \le C_M \Big(\widetilde{|f|^2}(\cdot, w)\Big)^{(2+\varepsilon)/2}. \ \square$$

3. Invertible Toeplitz product

In this section, we will completely characterize the bounded invertible Toeplitz products $T_fT_{\bar g}$ on $L_a^2(\mathbb{D}^n)$. We have the fellowing result:

Theorem 3.1 *Let $f(\cdot,\eta)$ and $g(\cdot,\eta)$ be in $L_a^2(\mathbb{D}^n)$. Then $T_fT_{\bar g}(\cdot,\eta)$ is bounded and invertible on $L_a^2(\mathbb{D}^\alpha)$ if and only if*

$$\sup\{\widetilde{|f|^2}(\cdot, w)\widetilde{|g|^2}(\cdot, w) : w \in \mathbb{D}^\alpha\} < \infty$$

and

$$\inf\{|f(\cdot, w)||g(\cdot, w)| : w \in \mathbb{D}^\alpha\} > 0.$$

Here $\mathbb{D}^\alpha = \mathbb{D}_{\alpha_1} \times \cdots \times \mathbb{D}_{\alpha_m}$, $\alpha = \{\alpha_1, \dots, \alpha_m\}$ is a nonempty subset of $\{1, \dots, n\}$ with $\alpha_1 < \cdots < \alpha_m$, and α runs over all subsets of $\{1, \dots, n\}$.

Proof "$\Rightarrow$". Suppose that $T_fT_{\bar g}(\cdot,\eta)$ is bounded and invertible on $L_a^2(\mathbb{D}^\alpha)$. By Theorem 1.1 there exists a constant M such that

$$\widetilde{|f|^2}(\cdot, w)\widetilde{|g|^2}(\cdot, w) \le M, \tag{3.2}$$

for all $w \in \mathbb{D}^\alpha$. Note that

$$T_fT_{\bar g}k_w = \overline{g(\cdot, w)} f k_w.$$

Thus

$$\|T_fT_{\bar g}k_w\|_2^2 = |g(\cdot, w)|^2 \widetilde{|f|^2}(\cdot, w),$$

so the invertibility of $T_fT_{\bar g}$ yields

$$|g(\cdot, w)|^2 \widetilde{|f|^2}(\cdot, w) \ge \delta_1 > 0, \tag{3.3}$$

for some constant δ_1 and for all $w \in \mathbb{D}^\alpha$. Since $T_gT_{\bar f} = (T_fT_{\bar g})^*$ is bounded and invertible, there is also a constant δ_2 such that

$$|f(\cdot, w)|^2 \widetilde{|g|^2}(\cdot, w) \ge \delta_2 > 0, \tag{3.4}$$

for all $w \in \mathbb{D}^\alpha$. Put $\delta = \delta_1\delta_2$, then it follows from (3.2), (3.3) and (3.4) that

$$\delta \le |f(\cdot, w)|^2 |g(\cdot, w)|^2 \widetilde{|f|^2}(\cdot, w)\widetilde{|g|^2}(\cdot, w) \le M|f(\cdot, w)|^2 |g(\cdot, w)|^2,$$

and thus

$$|f(\cdot, w)||g(\cdot, w)| \ge \frac{\delta^{1/2}}{M^{1/2}},$$

for all $w \in \mathbb{D}^\alpha$.

"$\Leftarrow$". Suppose that

$$M = \sup\{\widetilde{|f|^2}(\cdot, w)\widetilde{|g|^2}(\cdot, w) : w \in \mathbb{D}^\alpha\} < \infty,\ \eta = \inf|f(\cdot, w)||g(\cdot, w)| > 0.$$

Since $f, g \in L_a^2(\mathbb{D}^\alpha)$, we have

$$|f(\cdot, w)|^2 = |f \circ \varphi_w(\cdot, 0, \ldots, 0)|^2 \leq \int_{\mathbb{D}^\alpha} |f \circ \varphi_w|^2(\cdot, z) \mathrm{d}A(z) = \widetilde{|f|^2}(\cdot, w),$$

for all $w \in \mathbb{D}^\alpha$. Thus, $|f(\cdot, w)||g(\cdot, w)| \leq M^{1/2}$, for all $w \in \mathbb{D}^\alpha$. So, fg is a bounded function on $\mathbb{D}^\alpha$ and f and g cannot have zeros in $\mathbb{D}^\alpha$. Since $|g(\cdot, z)|^2 \geq \eta^2 |f(\cdot, z)|^{-2}$, for all $z \in \mathbb{D}^\alpha$, we have

$$\widetilde{|g|^2}(\cdot, w) \geq \eta^2 \widetilde{|f|^{-2}}(\cdot, w),$$

for all $w \in \mathbb{D}^\alpha$. Consequently,

$$M \geq \widetilde{|f|^2}(\cdot, w) \widetilde{|g|^2}(\cdot, w) \geq \eta^2 \widetilde{|f|^2}(\cdot, w) \widetilde{|f|^{-2}}(\cdot, w),$$

so that

$$\widetilde{|f|^2}(\cdot, w) \widetilde{|f|^{-2}}(\cdot, w) \leq M/\eta^2,$$

for all $w \in \mathbb{D}^\alpha$. This means that f satisfies the condition (A_2). By the reverse Hölder inequality, for some $\varepsilon > 0$,

$$\sup_{w \in \mathbb{D}^\alpha} \widetilde{|f|^{2+\varepsilon}}(\cdot, w) \widetilde{|f|^{-(2+\varepsilon)}}(\cdot, w) < \infty,$$

for all $w \in \mathbb{D}^\alpha$. By Theorem 1.1, $T_f T_{\overline{f^{-1}}}$ is bounded on $L_a^2(\mathbb{D}^\alpha)$. Since fg is bounded on $(\mathbb{D}^\alpha)$, the operator $T_{\overline{fg}}$ is bounded on $L_a^2(\mathbb{D}^\alpha)$. It follows that $T_f T_{\bar{g}} = T_f T_{\overline{f^{-1}}} T_{\overline{fg}}$ is bounded on $L_a^2(\mathbb{D}^\alpha)$. The function $\psi = 1/(f\bar{g})$ is bounded on $\mathbb{D}^\alpha$, so that the operator T_ψ is bounded on $L_a^2(\mathbb{D}^\alpha)$. Using that

$$T_f T_{\bar{g}} T_\psi = I = T_\psi T_f T_{\bar{g}},$$

we conclude that $T_f T_{\bar{g}}$ is invertible on $L_a^2(\mathbb{D}^\alpha)$. $\square$

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