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Gorenstein Homological Dimensions and Change of Rings

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Abstract In this paper, we shall be concerned with what happens of Gorenstein homological dimensions when certain modifications are made to a ring. The five structural operations addressed later are the formation of excellent extensions, localizations, Morita equivalences, polynomial extensions and power series extensions.

Keywords Gorenstein projective module; Gorenstein injective module; Gorenstein flat module; change of ring.

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1. Introduction

Unless stated otherwise, throughout this paper all rings are associative with identity, all modules are unitary modules. We denote by R-Mod (resp., Mod-R) the category of left (resp., right) R-modules. For any R-module M, pd_RM (resp., id_RM, fd_RM) denotes the projective (resp., injective, flat) dimension.

When R is two-sided Noetherian, Auslander and Bridger [1] introduced the G-dimension, $G\text{-}\dim_RM$, for every finitely generated R-module M. They proved the inequality $G\text{-}\dim_RM \leq \mathrm{pd}_RM$ with equality $G\text{-}\dim_RM = \mathrm{pd}_RM$ when pd_RM is finite. Over a general ring R, Enochs and Jenda defined in [4] a homological dimension, namely the Gorenstein projective dimension $\mathrm{Gpd}_R(-)$, for arbitrary (non-finite) modules. It is defined via resolution with (the so-called) Gorenstein projective modules. Avramov, Bachweitz, Martsinkovsky and Reiten proved that a finite module over a Noetherian ring is Gorenstein projective if and only if $G\text{-}\dim_RM = 0$ (see the remark following [2, Theorem 4.2.6]). Holm [8] gave homological descriptions of the Gorenstein dimensions over arbitrary rings. He proved that these dimensions are similar to the classical homological dimensions, that is, projective, injective and flat dimensions, respectively. In this paper, we consider the Gorenstein homological dimensions of excellent extensions, localizations, Morita equivalences, polynomial extensions and power series extensions.

Next we shall recall some notions and definitions which we need in the later sections.

A complete projective resolution is an exact sequence of projective R-modules

$$\mathbb{P}:\ \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

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such that $\mathrm{Hom}_R(\mathbb{P}, Q)$ is exact for every projective R-module Q. An R-module M is called Gorenstein projective (G-projective for short) if there exists a complete projective resolution $\mathbb{P}$ with $M \cong \mathrm{Im}(P_0 \to P^0)$. Every projective module is Gorenstein projective. The class of all Gorenstein projective R-modules is denoted by $\mathcal{GP}(R)$. Gorenstein injective (G-injective for short) modules are defined dually. The class of all such modules is denoted by $\mathcal{GI}(R)$. A complete flat resolution is an exact sequence of flat (left) R-modules

$$\mathbb{F}: \ \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that $I \otimes_R \mathbb{F}$ is exact for every injective right R-module I. An R-module M is called Gorenstein flat (G-flat for short) if there exists a complete flat resolution $\mathbb{F}$ with $M \cong \mathrm{Im}(F_0 \to F^0)$. The class of all Gorenstein flat R-modules is denoted by $\mathcal{GF}(R)$.

The Gorenstein projective dimension, $\mathrm{Gpd}_R M$, of an R-module M is defined by declaring that $\mathrm{Gpd}_R M \leq n$ ($n > 0$) if and only if M has a Gorenstein projective resolution of length n. Similarly, one defines the Gorenstein injective dimension, $\mathrm{Gid}_R M$, the Gorenstein flat dimension, $\mathrm{Gfd}_R M$, of M.

2. Excellent extensions

Let R and S be rings, $R \subseteq S$.

(1) The ring S is called right R-projective in case for any right S-module M_S with an S-submodule N_S, $N_R | M_R$ implies $N_S | M_S$. For example, every $n \times n$ matrix ring $M_n(R)$ is right R-projective.

(2) The ring extension $S \geq R$ is called a finite normalizing extension in case there is a finite subset $\{s_1, \ldots, s_n\}$ of S such that $S = \sum_{i=1}^n s_i R$ and $s_i R = R s_i$ for $i = 1, \ldots, n$.

(3) A finite normalizing extension $S \geq R$ is called an excellent extension in case condition (1) is satisfied and ${}_R S$, S_R are free modules with a common basis $\{s_1, \ldots, s_n\}$. The class of excellent extensions includes $n \times n$ matrix rings and crossed products $R * G$ where G is a finite group with $|G|^{-1} \in R$.

Proposition 2.1 *Assume that $S \geq R$ is an excellent extension. Then the following are equivalent for any right S-module M:*

(a) M_R is G-projective;

(b) $(M \otimes_R S)_R$ is G-projective;

(c) $(M \otimes_R S)_S$ is G-projective;

(d) M_S is G-projective.

Proof By condition, there exists a positive integer t such that $R_R | S_R^t$ and $S_R | R_R^t$.

(a) $\Rightarrow$ (c). Since M_R is G-projective, there is a complete projective resolution of right R-modules

$$\mathbb{P}: \ \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Im}(P_0 \to P^0)$. So $\mathbb{F} \otimes_R S: \cdots \to P_1 \otimes_R S \to P_0 \otimes_R S \to P^0 \otimes_R S \to P^1 \otimes_R S \to \cdots$

is exact such that $M\otimes_R S\cong \mathrm{Im}(P_0\otimes_R S\to P^0\otimes_R S)$ and $P_i\otimes_R S$, $P^i\otimes_R S$ are projective right S-modules for each i. Let $\bar{Q}$ be any projective right S-module. Then $\bar{Q}$ is a projective right R-module, and hence $\mathrm{Hom}_S(\mathbb{P}\otimes_R S,\bar{Q})\cong \mathrm{Hom}_R(\mathbb{P},\bar{Q})$ is exact. Thus $(M\otimes_R S)_S$ is G-projective.

(c) $\Rightarrow$ (d). Since M_S is isomorphic to a summand of $(M\otimes_R S)_S$, we have M_S is G-projective.

(d) $\Rightarrow$ (a). Since M_S is G-projective, there exists a complete projective resolution of right S-modules

$$\bar{\mathbb{P}}:\ \cdots\longrightarrow \bar{P}_1\longrightarrow \bar{P}_0\longrightarrow \bar{P}^0\longrightarrow \bar{P}^1\longrightarrow\cdots$$

with $M\cong \mathrm{Im}(\bar{P}_0\to\bar{P}^0)$. Let Q be any projective right R-module. Then $\mathrm{Hom}_R(S,Q)|Q^t$ for some $t\in\mathbb{N}$. Thus $\mathrm{Hom}_R(S,Q)$ is a projective right R-module, and so $\mathrm{Hom}_R(S,Q)$ is a projective right S-module. Since $\bar{P}_i$, $\bar{P}^i$ are projective right R-modules for every i, and

$$\mathrm{Hom}_R(\bar{\mathbb{P}},Q)\cong \mathrm{Hom}_R(\bar{\mathbb{P}}\otimes_S S,Q)\cong \mathrm{Hom}_S(\bar{\mathbb{P}},\mathrm{Hom}_R(S,Q))$$

is exact, it follows that M_R is G-projective.

(c) $\Leftrightarrow$ (b). By (a) $\Leftrightarrow$ (d). $\square$

Proposition 2.2 *Assume that $S\geq R$ is an excellent extension. Then the following are equivalent for any right S-module M:*

(a) M_R is G-injective;

(b) $Hom_R(S,M)_R$ is G-injective;

(c) $Hom_R(S,M)_S$ is G-injective;

(d) M_S is G-injective.

Proof (a) $\Rightarrow$ (c). Since M_R is G-injective, there is a complete injective resolution of right R-modules

$$\mathbb{E}:\ \cdots\longrightarrow E_1\longrightarrow E_0\longrightarrow E^0\longrightarrow E^1\longrightarrow\cdots$$

with $M\cong \mathrm{Im}(E_0\to E^0)$. Then

$$\mathrm{Hom}_R(S,\mathbb{E}):\cdots\to \mathrm{Hom}_R(S,E_1)\to \mathrm{Hom}_R(S,E_0)\to \mathrm{Hom}_R(S,E^0)\to \mathrm{Hom}_R(S,E^1)\to\cdots$$

is exact with $\mathrm{Hom}_R(S,M)\cong \mathrm{Im}(\mathrm{Hom}_R(S,E_0)\to \mathrm{Hom}_R(S,E^0))$ and $\mathrm{Hom}_R(S,E_i)$, $\mathrm{Hom}_R(S,E^i)$ are injective right S-modules for each i. Let $\bar{I}$ be any injective right S-module. Then $\bar{I}$ is an injective right R-module, and hence $\mathrm{Hom}_S(\bar{I},\mathrm{Hom}_R(S,\mathbb{E}))\cong \mathrm{Hom}_R(\bar{I},\mathbb{E})$ is exact. It follows that $\mathrm{Hom}_R(S,M)_S$ is G-injective.

(c) $\Rightarrow$ (d). Since M_S is isomorphic to a summand of $\mathrm{Hom}_R(S,M)_S$, M_S is G-injective.

(d) $\Rightarrow$ (a). Since M_S is G-injective, there exists a complete injective resolution of right S-modules

$$\bar{\mathbb{E}}:\ \cdots\longrightarrow \bar{E}_1\longrightarrow \bar{E}_0\longrightarrow \bar{E}^0\longrightarrow \bar{E}^1\longrightarrow\cdots$$

with $M\cong \mathrm{Im}(\bar{E}_0\to\bar{E}^0)$. Let I be any injective right R-module. Then $I\otimes_R S$ is an injective right S-module. Since $\bar{E}_i$, $\bar{E}^i$ are injective right R-modules for every i, and

$$\mathrm{Hom}_R(I,\bar{\mathbb{E}})\cong \mathrm{Hom}_R(I,\mathrm{Hom}_S(S,\bar{\mathbb{E}}))\cong \mathrm{Hom}_S(I\otimes_R S,\bar{\mathbb{E}})$$

is exact, we see that M_R is G-injective.

(c) $\Leftrightarrow$ (b). By (a) $\Leftrightarrow$ (d). $\square$

Proposition 2.3 *Assume that $S \geq R$ is an excellent extension. Then the following are equivalent for any right S-module M:*

(a) M_R is G-projective;

(b) $(M \otimes_R S)_R$ is G-projective;

(c) $(M \otimes_R S)_S$ is G-projective;

(d) M_S is G-projective.

Proof (a) $\Rightarrow$ (c). Since ${}_RM$ is G-flat, there exists a complete flat resolution of left R-modules

$$\mathbb{F}: \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Im}(F_0 \to F^0)$. Then $S \otimes_R \mathbb{F} : \cdots \to S \otimes_R F_1 \to S \otimes_R F_0 \to S \otimes_R F^0 \to S \otimes_R F^1 \to \cdots$ is exact such that $S \otimes_R M \cong \mathrm{Im}(S \otimes_R F_0 \to S \otimes_R F^0)$ and $S \otimes_R F_i$, $S \otimes_R F^i$ are flat left S-modules for each i. Let $\bar{I}$ be any injective right S-module. Then $\bar{I}$ is an injective right R-module, and hence $\bar{I} \otimes_S (S \otimes_R \mathbb{F}) \cong \bar{I} \otimes_R \mathbb{F}$ is exact. Thus ${}_S(S \otimes_R M)$ is G-flat.

(c) $\Rightarrow$ (d). Since ${}_SM$ is isomorphic to a summand of ${}_S(S \otimes_R M)$, we see that ${}_SM$ is G-flat.

(d) $\Rightarrow$ (a). Since ${}_SM$ is G-flat, there exists a complete flat resolution of left S-modules

$$\bar{\mathbb{F}}: \cdots \longrightarrow \bar{F}_1 \longrightarrow \bar{F}_0 \longrightarrow \bar{F}^0 \longrightarrow \bar{F}^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Im}(\bar{F}_0 \to \bar{F}^0)$. Let I be any injective right R-module. Then $I \otimes_R S$ is an injective right S-module. Since $\bar{F}_i$, $\bar{F}^i$ are flat left R-modules for every i, and $I \otimes_R \bar{\mathbb{F}} \cong I \otimes_R S \otimes_S \bar{\mathbb{F}}$ is exact, which means that ${}_RM$ is G-flat.

(c) $\Leftrightarrow$ (b). By (a) $\Leftrightarrow$ (d). $\square$

Corollary 2.4 *Assume that $S \geq R$ is an excellent extension. Then*

(1) $\mathrm{Gpd}_S M = \mathrm{Gpd}_S(M \otimes_R S) = \mathrm{Gpd}_R M$ for each right S-module M;

(2) $\mathrm{Gid}_S M = \mathrm{Gid}_S \mathrm{Hom}_R(S, M) = \mathrm{Gid}_R M$ for each right S-module M;

(3) $\mathrm{Gfd}_S M = \mathrm{Gfd}_S(S \otimes_R M) = \mathrm{Gfd}_R M$ for each left S-module M.

Corollary 2.5 *Let $R*G$ be a crossed product, where G is a finite group with $|G|^{-1} \in R$. Then*

(1) $\mathrm{Gpd}_{R*G} M = \mathrm{Gpd}_{R*G}(M \otimes_R (R*G)) = \mathrm{Gpd}_R M$ for each right $R*G$-module M;

(2) $\mathrm{Gid}_{R*G} M = \mathrm{Gid}_{R*G} \mathrm{Hom}_R(R*G, M) = \mathrm{Gid}_R M$ for each right $R*G$-module M;

(3) $\mathrm{Gfd}_{R*G} M = \mathrm{Gfd}_{R*G}((R*G) \otimes_R M) = \mathrm{Gfd}_R M$ for each left $R*G$-module M.

Corollary 2.6 *Let R be a ring and n any positive integer. Then*

(1) $\mathrm{Gpd}_{M_n(R)} M = \mathrm{Gpd}_{M_n(R)}(M \otimes_R M_n(R)) = \mathrm{Gpd}_R M$ for each $M \in$ Mod-$M_n(R)$;

(2) $\mathrm{Gid}_{M_n(R)} M = \mathrm{Gid}_{M_n(R)} \mathrm{Hom}_R(M_n(R), M) = \mathrm{Gid}_R M$ for each $M \in$ Mod-$M_n(R)$;

(3) $\mathrm{Gfd}_{M_n(R)} M = \mathrm{Gfd}_{M_n(R)}(M_n(R) \otimes_R M) = \mathrm{Gfd}_R M$ for each $M \in M_n(R)$-Mod.

Let R be graded by a finite group G. The smash product $R\sharp G^{\cdot}$ is a free right and left R-module with basis $\{p_a : a \in G\}$ and multiplication determined by $(rp_a)(rp_b) = rs_{a^{-1}}p_b$, where $s_{a^{-1}}$ is the ab^{-1} component of s.

Corollary 2.7 *Let $R\sharp G^{\cdot}$ be a smash product with $|G|^{-1} \in R$. Then*

(1) $\mathrm{Gpd}_{R\sharp G^{\cdot}} M = \mathrm{Gpd}_R M$ *for each right* $(R\sharp G^{\cdot})$*-module* M*;*

(2) $\mathrm{Gid}_{R\sharp G^{\cdot}} M = \mathrm{Gid}_R M$ *for each right* $(R\sharp G^{\cdot})$*-module* M*;*

(3) $\mathrm{Gfd}_{R\sharp G^{\cdot}} M = \mathrm{Gfd}_R M$ *for each left* $(R\sharp G^{\cdot})$*-module* M*.*

Proof Note that $(R\sharp G^{\cdot}) * G \cong M_n(R)$ by [10, Theorem 4.1]. $\square$

3. Quotients and localizations

Let n be a positive integer. A ring R is said to be left (resp., right) n-perfect if every flat left (resp., right) R-module has projective dimension less than or equal to n. Let R be a commutative ring, S a multiplicatively closed subset of R. Then $S^{-1}R = (R \times S)/\sim = \{a/s | a \in R, s \in S\}$ is a ring and $S^{-1}M = (M \times S)/\sim = \{x/s | x \in M, s \in S\}$ is an $S^{-1}R$-module.

Proposition 3.1 *Let R be a commutative n-perfect ring and S a multiplicatively closed subset of R. If B is a G-projective R-module, then $S^{-1}B$ is a G-projective $S^{-1}R$-module.*

Proof Since B is a G-projective R-module, there is a complete projective resolution of R-modules

$$\mathbb{P}: \ \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $B \cong \mathrm{Im}(P_0 \to P^0)$. Then $S^{-1}\mathbb{P} : \cdots \to S^{-1}P_1 \to S^{-1}P_0 \to S^{-1}P^0 \to S^{-1}P^1 \to \cdots$ is exact such that $S^{-1}B \cong \mathrm{Im}(S^{-1}P_0 \to S^{-1}P^0)$ and $S^{-1}P_i$, $S^{-1}P^i$ are projective $S^{-1}R$-modules for each i. Let $\bar{Q}$ be any projective $S^{-1}R$-module. Then $\bar{Q}$ is a flat R-module, and so $\mathrm{pd}_R\bar{Q} \le n$. Thus

$$\mathrm{Hom}_{S^{-1}R}(S^{-1}\mathbb{P}, \bar{Q}) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}R \otimes_R \mathbb{P}, \bar{Q}) \cong \mathrm{Hom}_R(\mathbb{P}, \bar{Q})$$

is exact, which gives that $S^{-1}B$ is a G-projective $S^{-1}R$-module. $\square$

Corollary 3.2 *Let R be a commutative n-perfect ring and S a multiplicatively closed subset of R. Then $\mathrm{Gpd}_{S^{-1}R}S^{-1}B \le \mathrm{Gpd}_R B$ for any R-module B.*

Proposition 3.3 *Let R be a commutative n-perfect ring, S a multiplicatively closed subset of R. Then*

(a) *If B is a G-injective R-module, then $\mathrm{Hom}_R(S^{-1}R, B)$ is a G-injective $S^{-1}R$-module;*

(b) *$\mathrm{Hom}_R(S^{-1}R, B)$ is a G-injective R-module if and only if $\mathrm{Hom}_R(S^{-1}R, B)$ is a G-injective $S^{-1}R$-module.*

Proof (a) Since B is a G-injective R-module, there exists a complete injective resolution of R-modules

$$\mathbb{E}: \ \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

with $B \cong \mathrm{Im}(E_0 \to E^0)$. Since $S^{-1}R$ is a flat R-module, $\mathrm{pd}_R S^{-1}R \le n$, and so

$$\mathrm{Hom}_R(S^{-1}R, \mathbb{E}): \ \cdots \longrightarrow \mathrm{Hom}_R(S^{-1}R, E_0) \longrightarrow \mathrm{Hom}_R(S^{-1}R, E^0) \longrightarrow \cdots$$

is exact by [3, Lemma 2.2], such that $\mathrm{Hom}_R(S^{-1}R,B)\cong \mathrm{Im}(\mathrm{Hom}_R(S^{-1}R,E_0)\to \mathrm{Hom}_R(S^{-1}R,E^0))$ and $\mathrm{Hom}_R(S^{-1}R,E_i)$, $\mathrm{Hom}_R(S^{-1}R,E^i)$ are injective R-modules for each i. Let $\bar{I}$ be any injective $S^{-1}R$-module. Then $\bar{I}$ is an injective R-module, and so

$$\mathrm{Hom}_{S^{-1}R}(\bar{I},\mathrm{Hom}_R(S^{-1}R,\mathbb{E}))\cong \mathrm{Hom}_R(\bar{I},\mathbb{E})$$

is exact. Thus $\mathrm{Hom}_R(S^{-1}R,B)$ is a G-injective $S^{-1}R$-module.

(b) " $\Rightarrow$ ". By $\mathrm{Hom}_R(S^{-1}R,\mathrm{Hom}_R(S^{-1}R,B))\cong \mathrm{Hom}_R(S^{-1}R,B)$ and (a).

" $\Leftarrow$ ". Since $\mathrm{Hom}_R(S^{-1}R,B)$ is a G-injective $S^{-1}R$-module, there is a complete injective resolution of $S^{-1}R$-modules

$$\bar{\mathbb{E}}:\ \cdots\longrightarrow \bar{E}_1\longrightarrow \bar{E}_0\longrightarrow \bar{E}^0\longrightarrow \bar{E}^1\longrightarrow\cdots$$

with $\mathrm{Hom}_R(S^{-1}R,B)\cong \mathrm{Im}(\bar{E}_0\to\bar{E}^0)$. Let I be any injective R-module. Then $S^{-1}I$ is an injective $S^{-1}R$-module by [5, Proposition 3.3.2]. Since $\bar{E}_i$, $\bar{E}^i$ are injective R-modules for each i, and

$$\mathrm{Hom}_R(I,\bar{\mathbb{E}})\cong \mathrm{Hom}_R(I,\mathrm{Hom}_{S^{-1}R}(S^{-1}R,\bar{\mathbb{E}}))\cong \mathrm{Hom}_{S^{-1}R}(S^{-1}I,\bar{\mathbb{E}})$$

is exact, $\mathrm{Hom}_R(S^{-1}R,B)$ is a G-injective R-module. $\square$

Corollary 3.4 *Let R be a commutative n-perfect ring, S a multiplicatively closed subset of R. Then*

(1) $\mathrm{Gid}_{S^{-1}R}\mathrm{Hom}_R(S^{-1}R,B)\le \mathrm{Gid}_R B$ for any R-module B;

(2) $\mathrm{Gid}_R\mathrm{Hom}_R(S^{-1}R,B)=\mathrm{Gid}_{S^{-1}R}\mathrm{Hom}_R(S^{-1}R,B)$ for any R-module B.

Theorem 3.5 *Let R be a commutative ring and S a multiplicatively closed subset of R. Let $B\in R$-Mod and $\bar{A}\in S^{-1}R$-Mod. Then*

(a) If B is a G-flat R-module, then $S^{-1}B$ is a G-flat R-module;

(b) If B is a G-flat R-module, then $S^{-1}B$ is a G-flat $S^{-1}R$-module;

(c) $\bar{A}$ is a G-flat R-module if and only if $\bar{A}$ is a G-flat $S^{-1}R$-module.

Proof (a) Since B is a G-flat R-module, there exists a complete flat resolution of R-modules

$$\mathbb{F}:\ \cdots\longrightarrow F_1\longrightarrow F_0\longrightarrow F^0\longrightarrow F^1\longrightarrow\cdots$$

with $B\cong \mathrm{Im}(F_0\to F^0)$. Then $S^{-1}\mathbb{F}:\cdots\to S^{-1}F_1\to S^{-1}F_0\to S^{-1}F^0\to S^{-1}F^1\to\cdots$ is exact such that $S^{-1}B\cong\mathrm{Im}(S^{-1}F_0\to S^{-1}F^0)$ and $S^{-1}F_i$, $S^{-1}F^i$ are flat R-modules for each i. Let I be any injective R-module. Then $S^{-1}I$ is an injective R-module. So $I\otimes_R S^{-1}\mathbb{F}\cong S^{-1}I\otimes_R\mathbb{F}$ is exact by [11, Proposition 5.17], it follows that $S^{-1}B$ is a G-flat R-module.

(b) Since B is a G-flat R-module, there is a complete flat resolution of R-modules

$$\mathbb{F}:\ \cdots\longrightarrow F_1\longrightarrow F_0\longrightarrow F^0\longrightarrow F^1\longrightarrow\cdots$$

with $B\cong \mathrm{Im}(F_0\to F^0)$. Then $S^{-1}\mathbb{F}:\cdots\to S^{-1}F_1\to S^{-1}F_0\to S^{-1}F^0\to S^{-1}F^1\to\cdots$ is exact such that $S^{-1}B\cong\mathrm{Im}(S^{-1}F_0\to S^{-1}F^0)$ and $S^{-1}F_i$, $S^{-1}F^i$ are flat $S^{-1}R$-modules for every i. Let $\bar{I}$ be any injective $S^{-1}R$-module. Then $\bar{I}$ is an injective R-module, and so $\bar{I}\otimes_{S^{-1}R}S^{-1}\mathbb{F}\cong\bar{I}\otimes_R\mathbb{F}$ is exact by [11, Proposition 5.17]. Thus $S^{-1}B$ is a G-flat $S^{-1}R$-module.

(c) " $\Rightarrow$ ". By $S^{-1}\bar{A} \cong \bar{A}$ and (b).

" $\Leftarrow$ ". Since $\bar{A}$ is a G-flat $S^{-1}R$-module, there exists a complete flat resolution of $S^{-1}R$-modules

$$\bar{\mathbb{F}}: \ \cdots \longrightarrow \bar{F}_1 \longrightarrow \bar{F}_0 \longrightarrow \bar{F}^0 \longrightarrow \bar{F}^1 \longrightarrow \cdots$$

with $\bar{A} \cong \mathrm{Im}(\bar{F}_0 \to \bar{F}^0)$. Let I be any injective R-module. Then $S^{-1}I$ is an injective $S^{-1}R$-module. Since $\bar{F}_i$, $\bar{F}^i$ are flat R-modules for each i, and $I \otimes_R \bar{\mathbb{F}} \cong S^{-1}I \otimes_{S^{-1}R} \bar{\mathbb{F}}$ is exact by [11, Proposition 5.17], which implies that $\bar{A}$ is a G-flat R-module. □

Corollary 3.6 *Let R be a commutative ring and S a multiplicatively closed subset of R. Then*

(1) $\mathrm{Gfd}_{S^{-1}R}S^{-1}B \leq \mathrm{Gfd}_R B$ for any R-module B;

(2) $\mathrm{Gfd}_R \bar{A} = \mathrm{Gfd}_{S^{-1}R}\bar{A}$ for any $S^{-1}R$-module $\bar{A}$.

4. Computational consideration

Proposition 4.1 *Suppose $L : S$-Mod$\longrightarrow R$-Mod is an equivalence and $B \in R$-Mod. Then*

(1) B is a G-projective left S-module if and only if $L(B)$ is a G-projective left R-module;

(2) B is a G-injective left S-module if and only if $L(B)$ is a G-injective left R-module;

(3) B is a G-flat left S-module if and only if $L(B)$ is a G-flat left R-module.

Proof (1) " $\Rightarrow$ ". Since ${}_SB$ is G-projective, there exists a complete projective resolution of left S-modules

$$\mathbb{P}: \ \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $B \cong \mathrm{Im}(P_0 \to P^0)$. Then $L(\mathbb{P}) : \cdots \to L(P_1) \to L(P_0) \to L(P^0) \to L(P^1) \to \cdots$ is exact such that $L(B) \cong \mathrm{Im}(L(P_0) \to L(P^0))$ and $L(P_i)$, $L(P^i)$ are projective left R-modules for each i. Let Q be any projective left R-module. Then there is a projective left S-module P such that $L(P) = Q$, and so

$$\mathrm{Hom}_R(L(\mathbb{P}), Q) = \mathrm{Hom}_R(L(\mathbb{P}), L(P)) \cong L(\mathrm{Hom}_S(\mathbb{P}, P))$$

is exact. Thus $L(B)$ is a G-projective left R-module.

" $\Leftarrow$ ". Since ${}_RL(B)$ is G-injective, there is a complete projective resolution of left R-modules

$$L(\mathbb{P}): \ \cdots \longrightarrow L(P_1) \longrightarrow L(P_0) \longrightarrow L(P^0) \longrightarrow L(P^1) \longrightarrow \cdots$$

with $L(B) \cong \mathrm{Im}(L(P_0) \to L(P^0))$. Then $\mathbb{P} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ is exact such that $B \cong \mathrm{Im}(P_0 \to P^0)$ since L is an isomorphism of group, and P_i, P^i are projective left S-modules for every i. Let Q be any projective left S-module. Then $L(\mathrm{Hom}_S(\mathbb{P}, Q)) = \mathrm{Hom}_R(L(\mathbb{P}), L(Q))$ is exact since $L(Q)$ is a projective left R-module. So $\mathrm{Hom}_S(\mathbb{P}, Q)$ is exact since L is an isomorphism of group, which gives that B is a G-projective left S-module.

(2) and (3) can be proved similarly to the proof of (1). □

Corollary 4.2 *Suppose $L : S$-Mod$\longrightarrow R$-Mod is an equivalence. Then*

(1) $\mathrm{Gpd}_R L(B) = \mathrm{Gpd}_S B$ for all $B \in S$-Mod;

(2) $\mathrm{Gid}_R L(B) = \mathrm{Gid}_S B$ for all $B \in S$-Mod;

(3) $\mathrm{Gfd}_R L(B) = \mathrm{Gfd}_S B$ *for all* $B \in S$*-Mod.*

Corollary 4.3 *Let* R *be a ring and* $e \in R$ *a non-zero idempotent. If* $ReR = R$*, then*

(1) $\mathrm{Gpd}_R M = \mathrm{Gpd}_{eRe}(eR \otimes_R M)$ *for all* $M \in R$*-Mod;*

(2) $\mathrm{Gpd}_{eRe} M = \mathrm{Gpd}_R(Re \otimes_{eRe} M)$ *for all* $M \in eRe$*-Mod;*

(3) $\mathrm{Gid}_R M = \mathrm{Gid}_{eRe}(eR \otimes_R M)$ *for all* $M \in R$*-Mod;*

(4) $\mathrm{Gid}_{eRe} M = \mathrm{Gid}_R(Re \otimes_{eRe} M)$ *for all* $M \in eRe$*-Mod;*

(5) $\mathrm{Gfd}_R M = \mathrm{Gfd}_{eRe}(eR \otimes_R M)$ *for all* $M \in R$*-Mod;*

(6) $\mathrm{Gfd}_{eRe} M = \mathrm{Gfd}_R(Re \otimes_{eRe} M)$ *for all* $M \in eRe$*-Mod.*

Corollary 4.4 *Let* R *be a ring and* $n \geq 1$*. Then*

(1) $\mathrm{Gpd}_R M = \mathrm{Gpd}_{M_n(R)}(M_n(R)e_{ii} \otimes_R M)$ *for all* $M \in R$*-Mod;*

(2) $\mathrm{Gpd}_{M_n(R)} M = \mathrm{Gpd}_R(e_{ii}M_n(R) \otimes_{M_n(R)} M)$ *for all* $M \in M_n(R)$*-Mod;*

(3) $\mathrm{Gid}_R M = \mathrm{Gid}_{M_n(R)}(M_n(R)e_{ii} \otimes_R M)$ *for all* $M \in R$*-Mod;*

(4) $\mathrm{Gid}_{M_n(R)} M = \mathrm{Gid}_R(e_{ii}M_n(R) \otimes_{M_n(R)} M)$ *for all* $M \in M_n(R)$*-Mod;*

(5) $\mathrm{Gfd}_R M = \mathrm{Gfd}_{M_n(R)}(M_n(R)e_{ii} \otimes_R M)$ *for all* $M \in R$*-Mod;*

(6) $\mathrm{Gfd}_{M_n(R)} M = \mathrm{Gfd}_R(e_{ii}M_n(R) \otimes_{M_n(R)} M)$ *for all* $M \in M_n(R)$*-Mod,*

where e_{ii} *are matrix units for* $i = 1, \ldots, n$*.*

Proposition 4.5 ${}_RM$ *is G-projective if and only if* ${}_{R[x]}M[x]$ *is G-projective.*

Proof "$\Rightarrow$". Since ${}_RM$ is G-projective, there exists a complete projective resolution of left R-modules

$$\mathbb{P}:\ \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Im}(P_0 \to P^0)$. Then $R[x] \otimes_R \mathbb{P}: \cdots \to P_1[x] \to P_0[x] \to P^0[x] \to P^1[x] \to \cdots$ is exact such that $M[x] \cong \mathrm{Im}(P_0[x] \to P^0[x])$ and $P_i[x]$, $P^i[x]$ are projective left $R[x]$-modules by [11, Proposition 5.11] for each i. Let $\bar{Q}$ be any projective left $R[x]$-module. Then $\bar{Q}$ is a projective left R-module, and so $\mathrm{Hom}_{R[x]}(R[x] \otimes_R \mathbb{P}, \bar{Q}) \cong \mathrm{Hom}_R(\mathbb{P}, \bar{Q})$ is exact. Thus $M[x]$ is a G-projective left $R[x]$-module.

"$\Leftarrow$". Since ${}_{R[x]}M[x]$ is G-projective, there is a complete projective resolution of left $R[x]$-modules

$$\bar{\mathbb{P}}:\ \cdots \longrightarrow \bar{P}_1 \longrightarrow \bar{P}_0 \longrightarrow \bar{P}^0 \longrightarrow \bar{P}^1 \longrightarrow \cdots$$

with $M[x] \cong \mathrm{Im}(\bar{P}_0 \to \bar{P}^0)$. Let Q be any projective left R-module. Then $Q[x]$ is a projective left $R[x]$-module, and hence $\mathrm{Hom}_R(\bar{\mathbb{P}}, Q[x]) \cong \mathrm{Hom}_{R[x]}(R[x] \otimes_R \bar{\mathbb{P}}, Q[x]) \cong \mathrm{Hom}_{R[x]}(\bar{\mathbb{P}}, Q[x])$ is exact, which implies that $\mathrm{Hom}_R(\bar{\mathbb{P}}, Q)$ is exact since Q is isomorphic to a summand of $Q[x]$. Thus $M[x]$ is a G-projective left R-module, and so M is a G-projective left R-module by [8, Theorem 2.5]. $\square$

Proposition 4.6 *Let* R *be right coherent. Then* ${}_RM$ *is G-flat if and only if* ${}_{R[x]}M[x]$ *is G-flat.*

Proof "$\Rightarrow$". Since ${}_RM$ is G-flat, there exists a complete flat resolution of left R-modules

$$\mathbb{F}:\ \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Im}(F_0 \to F^0)$. Then $R[x] \otimes_R \mathbb{F} : \cdots \to F_1[x] \to F_0[x] \to F^0[x] \to F^1[x] \to \cdots$ is exact such that $M[x] \cong \mathrm{Im}(F_0[x] \to F^0[x])$ and $F_i[x]$, $F^i[x]$ are flat left $R[x]$-modules for each i. Let $\bar{E}$ be any injective right $R[x]$-module. Then $\mathrm{Ext}^1_R(H, \bar{E}) \cong \mathrm{Ext}^1_R(H, \mathrm{Hom}_{R[x]}(R[x], \bar{E})) \cong \mathrm{Ext}^1_{R[x]}(H[x], \bar{E}) = 0$ by [13, pg.258, 9.21] for any right R-module H. Thus $\bar{E}$ is an injective right R-module, and hence $\bar{E} \otimes_{R[x]} R[x] \otimes_R \mathbb{F} \cong \bar{E} \otimes_R \mathbb{F}$ is exact, which gives that $M[x]$ is a G-flat left $R[x]$-module.

" $\Leftarrow$ " Since ${}_{R[x]}M[x]$ is G-flat, there exists a complete flat resolution of left $R[x]$-modules

$$\bar{\mathbb{F}} : \ \cdots \longrightarrow \bar{F}_1 \longrightarrow \bar{F}_0 \longrightarrow \bar{F}^0 \longrightarrow \bar{F}^1 \longrightarrow \cdots$$

with $M[x] \cong \mathrm{Im}(\bar{F}_0 \to \bar{F}^0)$. Let E be any injective right R-module. Then $\mathrm{Hom}_R(R[x], E)$ is an injective right $R[x]$-module, and hence

$$\begin{aligned}(\mathrm{Hom}_R(R[x], E) \otimes_R \bar{\mathbb{F}})^+ &\cong \mathrm{Hom}_R(\bar{\mathbb{F}}, \mathrm{Hom}_{R[x]}(R[x], \mathrm{Hom}_R(R[x], E)^+)) \\ &\cong \mathrm{Hom}_{R[x]}(\bar{\mathbb{F}}, \mathrm{Hom}_R(R[x], E)^+) \\ &\cong (\mathrm{Hom}_R(R[x], E) \otimes_{R[x]} \bar{\mathbb{F}})^+\end{aligned}$$

is exact. So $\mathrm{Hom}_R(R[x], E) \otimes_R \bar{\mathbb{F}}$ is exact, which means that $E \otimes_R \bar{\mathbb{F}}$ is exact since E is isomorphic to a summand of $\mathrm{Hom}_R(R[x], E)$. Thus $M[x]$ is a G-flat left R-module, and hence M is a G-flat left R-module by [8, Theorem 3.7]. $\square$

Proposition 4.7 *M_R is G-injective if and only if $M[[x^{-1}]]_{R[x]}$ is G-injective.*

Proof " $\Rightarrow$ " Since M_R is G-injective, there exists a complete injective resolution of right R-modules

$$\mathbb{E} : \ \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Im}(E_0 \to E^0)$. Then $\mathrm{Hom}_R(R[x], \mathbb{E}) : \cdots \to \mathrm{Hom}_R(R[x], E_0) \to \mathrm{Hom}_R(R[x], E^0) \to \cdots$ is exact with $\mathrm{Hom}_R(R[x], M) \cong \mathrm{Im}(\mathrm{Hom}_R(R[x], E_0) \to \mathrm{Hom}_R(R[x], E^0))$ and $\mathrm{Hom}_R(R[x], E_i)$, $\mathrm{Hom}_R(R[x], E^i)$ are injective right $R[x]$-modules for every i. Let $\bar{J}$ be any injective right $R[x]$-module. Then $\bar{J}$ is an injective right R-module, and hence $\mathrm{Hom}_{R[x]}(\bar{J}, \mathrm{Hom}_R(R[x], M)) \cong \mathrm{Hom}_R(\bar{J}, M)$ is exact, which implies that $M[[x^{-1}]] \cong \mathrm{Hom}_R(R[x], M)$ is a G-injective right $R[x]$-module by [12, Lemma 1.2].

" $\Leftarrow$ " Since $M[[x^{-1}]]_{R[x]}$ is G-injective, there is a complete injective resolution of right $R[x]$-modules

$$\bar{\mathbb{E}} : \ \cdots \longrightarrow \bar{E}_1 \longrightarrow \bar{E}_0 \longrightarrow \bar{E}^0 \longrightarrow \bar{E}^1 \longrightarrow \cdots$$

with $M[[x^{-1}]] \cong \mathrm{Im}(\bar{E}_0 \to \bar{E}^0)$. Let E be any injective right R-module. Then $\mathrm{Hom}_R(R[x], E)$ is an injective right $R[x]$-module. So $\mathrm{Hom}_R(\mathrm{Hom}_R(R[x], E), \bar{\mathbb{E}}) \cong \mathrm{Hom}_{R[x]}(\mathrm{Hom}_R(R[x], E), \bar{\mathbb{E}})$ is exact, and hence $\mathrm{Hom}_R(E, \bar{\mathbb{E}})$ is exact. Thus $M[[x^{-1}]]$ is a G-injective right R-module, and so M is a G-injective right R-module by [8, Theorem 2.6]. $\square$

Corollary 4.8 *Let R be a ring. Then*

(1) $Gpd_R M = Gpd_{R[x]} M[x]$ for any left R-module M;

(2) $Gid_R M = Gid_{R[x]} M[x^{-1}]$ for any right R-module M.

If R is right coherent, then

(3) $\mathrm{Gfd}_R M = \mathrm{Gfd}_{R[x]} M[x]$ for any left R-module M.

Proof (1) It suffices to prove the one dimension is finite if and only if so is the other.

Assume $\mathrm{Gpd}_R M \leq n$, there exists a $\mathcal{GP}(R)$-resolution $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$. So $0 \to G_n[x] \to G_{n-1}[x] \to \cdots \to G_0[x] \to M[x] \to 0$ is a $\mathcal{GP}(R[x])$-resolution of $M[x]$ by Proposition 4.5, and hence $\mathrm{Gpd}_{R[x]} M[x] \leq n$.

Assume $\mathrm{Gpd}_{R[x]} M[x] \leq n$, there is a $\mathcal{GP}(R[x])$-resolution $0 \to \bar{G}_n \to \bar{G}_{n-1} \to \cdots \to \bar{G}_0 \to M[x] \to 0$. If we consider the terms only as R-modules, then we have a $\mathcal{GP}(R)$-resolution of $\coprod M$ ($\aleph_0$ copies of M) of length n. Thus $\mathrm{Gpd}_R M \leq n$ by [8, Proposition 2.19].

(2) and (3) can be proved by analogy with the proof of (1). $\square$

Proposition 4.9 *Let R be left n-perfect. If M is a G-projective left R-module, then $M[[x]]$ is a G-projective left $R[[x]]$-module.*

Proof Since M is a G-projective left R-module, there is a complete projective resolution of left R-modules

$$\mathbb{P}: \ \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Im}(P_0 \to P^0)$. Then $R[[x]] \otimes_R \mathbb{P}: \cdots \to P_1[[x]] \to P_0[[x]] \to P^0[[x]] \to P^1[[x]] \to \cdots$ is exact since $R[[x]]$ is a faithfully flat R-module, such that $M[[x]] \cong \mathrm{Im}(P_0[[x]] \to P^0[[x]])$ and $P_i[[x]]$, $P^i[[x]]$ are projective left $R[x]$-modules since

$$\mathrm{Ext}^1_{R[[x]]}(P[[x]], -) \cong \mathrm{Hom}_R(P, \mathrm{Ext}^1_{R[[x]]}(R[[x]], -)) = 0$$

by [13, pg.258, 9.20] for any projective left R-module P. Let $\bar{Q}$ be any projective left $R[[x]]$-module. Then $\bar{Q}$ is a flat left R-module, and so $\mathrm{pd}_R \bar{Q} \leq n$. Thus $\mathrm{Hom}_{R[[x]]}(R[[x]] \otimes_R \mathbb{P}, \bar{Q}) \cong \mathrm{Hom}_R(\mathbb{P}, Q)$ is exact, which gives that $M[[x]]$ is a G-projective left $R[[x]]$-module. $\square$

Proposition 4.10 *If M is a G-flat left R-module, then $M[[x]]$ is a G-flat left $R[[x]]$-module.*

Proof Since M is a G-flat left R-module, there exists a complete flat resolution of left R-modules

$$\mathbb{F}: \ \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Im}(F_0 \to F^0)$. Then $R[[x]] \otimes_R \mathbb{F}: \cdots \to F_1[[x]] \to F_0[[x]] \to F^0[[x]] \to F^1[[x]] \to \cdots$ is exact such that $M[[x]] \cong \mathrm{Im}(F_0[[x]] \to F^0[[x]])$ and $F_i[[x]]$, $F^i[[x]]$ are flat left $R[[x]]$-modules by [5, pg.43, Exercise 9] for each i. Let $\bar{E}$ be any injective right $R[[x]]$-module and let $B \to C \to 0$ be exact in Mod-R. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_{R[[x]]}(C[[x]], \bar{E}) & \longrightarrow & \mathrm{Hom}_{R[[x]]}(B[[x]], \bar{E}) & \longrightarrow & 0 \\
\cong\big\downarrow & & \cong\big\downarrow & & \\
\mathrm{Hom}_R(C, \bar{E}) & \longrightarrow & \mathrm{Hom}_R(B, \bar{E}) & &
\end{array}$$

with the upper row exact. Then $\mathrm{Hom}_R(C, \bar{E}) \to \mathrm{Hom}_R(B, \bar{E}) \to 0$ is exact, and so $\bar{E}$ is an injective right R-module, which implies that $\bar{E} \otimes_{R[[x]]} R[[x]] \otimes_R \mathbb{F} \cong \bar{E} \otimes_R \mathbb{F}$ is exact. Thus $M[[x]]$ is a G-flat left $R[[x]]$-module. $\square$

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