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A Note on Twisted Hopf Algebras

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Abstract In this paper, we get some properties of the antipode of a twisted Hopf algebra. We proved that the graded global dimension of a twisted Hopf algebra coincides with the graded projective dimension of its trivial module k, which is also equal to the projective dimension of k.

Keywords Hopf algebras; twisted Hopf algebras; projective dimension; global dimension.

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1. Introduction

Twisted Hopf algebras were introduced by Li and Zhang in [6]. Its unique difference from a Hopf algebra is the comultiplication. On $A \otimes A$ we consider a multiplication different from the component-wise one, specially, the twisted multiplication by Lusztig's rule [6]. It includes some important and exciting examples such as the free algebra and polynomial algebra over the field k, the twisted Ringel-Hall algebra [8], Lusztig's free algebra and non-degenerate algebra [4], the positive part of the Drinfeld-Jimbo quantized enveloping algebra [2, 3] and Rosso's quantum shuffle algebra [11].

The antipode of Hopf algebras plays an important role in Hopf algebras. If a Hopf algebra H is commutative (or cocommutative), then S^2 is the identity map. We prove that this is also true for twisted Hopf algebras. Using the fundamental theorem of Hopf module, Lorenz-Lorenz proved that the global dimension of a Hopf algebra is exactly the projective dimension of the trivial module k (see [5, 2.4]). Using the fundamental theorem of Yetter-Drinfeld Hopf module [1, Theorem 1], we generalized Lorenz's conclusion to Yetter-Drinfeld Hopf algebras [13, Theorem 4.5]. Following [5] and [13], but with a different approach, we prove that it is also true for twisted Hopf algebras. The graded global dimension of a twisted Hopf algebra coincides with the graded projective dimension of the trivial module k, which equals the projective dimension of k.

The paper is organized as follows: In Section 2, we provide some background materials for twisted Hopf algebras. It is proved that if A is twist commutative or twist cocommutative, then $S^2 = \mathrm{id}$. In Section 3, we consider projective modules in a graded A-module category. We prove the main theorem in a different approach from those in [5] and [13]: Let A be a twisted Hopf

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algebra. The graded global dimensional of A is equal to the graded projective dimensional of left A-module k, which also equals the projective dimension of left A-module k.

In this paper, all tensor products are assumed to be over k. Let V and W be vector spaces. For any $a \in V \otimes W$, denote by $a = \sum_i v_i \otimes w_i$ the sum of linearly independent elements of $\{v_i\}$ and $\{w_i\}$. In the following, "$\star$" denotes the convolution product.

2. Twisted Hopf algebras

Let k be a field, c be a non-zero element in k, and I be a set. Denote by $\mathbb{Z}I$ the free abelian group with I as basis. An element in $\mathbb{Z}I$ is written as $x = (x_i)_{i \in I}$ with $x_i \in \mathbb{Z}$, where $x_i = 0$ for almost all $i \in I$. Let $\mathbb{N}_0$ denote the set of non-negative integers. Denote by $\mathbb{N}_0 I$ the subset $\{x = (x_i)_{i \in I} \in \mathbb{Z}I | x_i \in \mathbb{N}_0\}$.

An $\mathbb{N}_0 I$ graded algebra $A = (A, m, u)$ means an associative k-algebra with a direct decomposition of k-spaces $A = \oplus_{x \in \mathbb{N}_0 I} A_x$ with $A_0 = k$ such that $A_x A_y \subseteq A_{x+y}$, for $x, y \in \mathbb{N}_0 I$, where $m : A \otimes A \longrightarrow A$ is the multiplication and $u : k \longrightarrow A$ the unit of A.

A nonzero element $a \in A_x$ is said to be homogeneous of degree x, where x is called the degree of a, denoted $\deg(a) = |a| = x$.

By definition [9, p.206], an $\mathbb{N}_0 I$-graded k-coalgebra $C = (C, \triangle, \epsilon)$ is a graded k-space $C = \oplus_{x \in \mathbb{N}_0 I} C_x$ with $C_0 = k$ and with k-linear maps $\triangle : C \longrightarrow C \otimes C$ and $\epsilon : C \longrightarrow k$ satisfying the following conditions:

(i) $\triangle$ is a coassociative comultiplication, i.e., $(\triangle \otimes \mathrm{id})\triangle = (\mathrm{id} \otimes \triangle)\triangle$;

(ii) ϵ is the projection onto $C_0 = k$, i.e., $\epsilon(C_x) = 0$ for $x \neq 0$ and $\epsilon(1) = 1$;

(iii) ϵ is a counit, i.e., $(\mathrm{id} \otimes \epsilon)\triangle = \mathrm{id} = (\epsilon \otimes \mathrm{id})\triangle$;

(iv) $\triangle$ respects the grading, i.e., $\triangle(C_z) \subseteq \oplus_{x+y=z} C_x \otimes C_y$.

Let $\chi : \mathbb{Z}I \times \mathbb{Z}I \longrightarrow \mathbb{Z}$ be a bilinear form (not necessarily symmetric), c be a non-zero element in k, and (A, m, u) be an $\mathbb{N}_0 I$-graded algebra. In [10], Ringel introduced a new multiplication m_χ on (A, m, u): for $a \in A_x$, $b \in A_y$, defined

$$m_\chi(a \otimes b) = c^{\chi(|a|,\ |b|)} ab.$$

Then there is a unique $\mathbb{N}_0 I$-graded, associative k-algebra structure on A with multiplication m_χ. Following Ringel, denote this new algebra by A_χ.

Dually, let $(C, \triangle, \epsilon)$ be an $\mathbb{N}_0 I$-graded coalgebra. Consider a new k-linear comultiplication $\triangle_\chi$ on homogeneous elements of C defined

$$\triangle_\chi(a) = \sum c^{\chi(|a_1|, |a_2|)} (a_1 \otimes a_2), \quad a \in C,$$

where $\triangle(a) = \sum a_1 \otimes a_2$.

Li and Zhang proved that $(C, \triangle\chi, \epsilon)$ is again an $\mathbb{N}_0 I$-graded coalgebra [6, Lemma 2.4], denoted by C_χ.

For a bilinear form $\chi : \mathbb{Z}I \times \mathbb{Z}I \longrightarrow \mathbb{Z}$, define a new bilinear form $\chi^T : \mathbb{Z}I \times \mathbb{Z}I \longrightarrow \mathbb{Z}$ by

$$\chi^T(x, y) = \chi(y, x).$$

Define the inverse of χ as $-\chi : \mathbb{Z}I \times \mathbb{Z}I \longrightarrow \mathbb{Z}$, i.e., $\chi(x,\ y) + (-\chi(x,y)) = 0$.

Combining $\mathbb{N}_0 I$-graded algebras and $\mathbb{N}_0 I$-graded coalgebras, one can get twisted Hopf algebras [6, p.719].

Definition 1 *Let k, c, I be as above, and $\chi : \mathbb{Z}I \times \mathbb{Z}I \longrightarrow \mathbb{Z}$ be an arbitrary bilinear form. If k-module A satisfies the following conditions*

(T1) $(A = \oplus_{x\in\mathbb{N}_0 I} A_x, m, u)$ is an $\mathbb{N}_0 I$-graded k-algebra and $(A, \triangle, \epsilon)$ is an $\mathbb{N}_0 I$-graded k-coalgebra.

(T2) The counit $\epsilon : A \longrightarrow k$ and comultiplication $\triangle : A \longrightarrow A \otimes A$ are algebra maps in the following sense

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \quad \triangle(ab) = \sum c^{\chi(|a_2|,|b_1|)} a_1 b_1 \otimes a_2 b_2. \tag{1}$$

(T3) There is a k-linear map $S : A \longrightarrow A$ such that

$$m(\mathrm{id} \otimes S)\triangle = u\epsilon = m(S \otimes \mathrm{id})\triangle.$$

Then $(A, m, u, \triangle, \epsilon)$ is called a (k, c, I, χ)-Hopf algebra or twisted Hopf algebra, S is called the antipode of A.

The condition (T3) is equivalent to the following:

(T3)' There is a k-linear map $S : A \longrightarrow A$ such that in the convolution algebra $\mathrm{Hom}_k(A, A)$, we have

$$S \star \mathrm{id} = \mathrm{id} \star S = u\epsilon. \tag{2}$$

Remark Let A be a twisted Hopf algebra. By [6, Theorem 2.10], we have

(i) S is an $\mathbb{N}_0 I$-graded map, i.e., $|S(a)| = |a|$ for $a \in A$.

(ii) $S : A \longrightarrow A_{\chi^T}$ is an algebra anti-homomorphism, i.e., S satisfies

$$S(ab) = c^{\chi(|b|,|a|)} S(b)S(a), \quad a, b \in A. \tag{3}$$

(iii) S is a coalgebra anti-homomorphism, i.e., S satisfies

$$\triangle(S(a)) = \sum c^{\chi(|a_1|,|a_2|)} S(a_2) \otimes S(a_1), \quad a \in A. \tag{4}$$

Comparing (2) with [12, Proposition 4.0.1], we may say that the antipode S is an algebra and coalgebra anti-morphism under the "twisted".

Lemma 2 *Let A be a twisted Hopf algebra. Then*

$$\epsilon(a)c^{\chi(|a|,|b|)} = \epsilon(a)c^{-\chi(|a|,|b|)} = \epsilon(a)c^{\chi^T(|a|,|b|)} = \epsilon(a), \quad \forall a, b \in A. \tag{5}$$

Proof If $a \notin A_0$, i.e., $|a| \neq 0$, then $\epsilon(a) = 0$, the left side and right side of (5) are zero. If $a \in A_0$, then $\chi(|a|, |b|) = 0$, hence $c^{\chi(|a|,|b|)} = 1$, and the first equality holds. Similarly, one proves other equalities.

Corollary 3 *Let A be a twisted Hopf algebra, $a \in A$ and $\triangle(a) = \sum a_1 \otimes a_2$. Then*

$$\epsilon(a) \sum c^{\chi(|a_1|,|a_2|)} = \epsilon(a) \sum c^{-\chi(|a_1|,|a_2|)} = \epsilon(a) \sum c^{\chi^T(|a_1|,|a_2|)} = \epsilon(a). \tag{6}$$

Proof We only prove the first equation. If $a \notin A_0$, it holds as in Lemma 2. If $a \in A_0$, then $\triangle(a) = \sum a_1 \otimes a_2 = a(1 \otimes 1)$, and $|a_1| = |a_2| = 0$. Thus $\sum \chi(|a_1|, |a_2|) = \chi(0,0) = 0$. We have $\sum c^{\chi(|a_1|,|a_2|)} = \sum c^{-\chi(|a_1|,|a_2|)} = \sum c^{\chi^T(|a_1|,|a_2|)}$. This completes the proof. □

In a parallel manner with Sweedler's [12, Proposition 4.0.1], we have the following conclusion:

Proposition 4 *Let $A = (A, m, u, \triangle, \epsilon)$ be a twisted Hopf algebra. Then for $a \in A$, the following are equivalent:*

(S1) $\sum S(a_2)a_1 = u\epsilon(a)$;

(S2) $\sum a_2 S(a_1) = u\epsilon(a)$;

(S3) $S^2 = \mathrm{id}$.

Proof Note that S is the convolution inverse of identity by (T3)′. We will show that S^2 is the right (or left) convolution inverse of S, and so it is equal to id.

$$\begin{aligned}(S \star S^2)(a) &= \sum S(a_1)S^2(a_2) = \sum c^{-\chi(|a_1|,\ |a_2|)} S(S(a_2)a_1) \quad \text{by (3)}\\ &= \sum c^{-\chi(|a_1|,\ |a_2|)} S(u\epsilon(a)) \quad \text{by (S1) and Remarks (1)}\\ &= S(u\epsilon(a) \sum c^{-\chi(|a_1|,|a_2|)}) = S(u\epsilon(a)) \quad \text{by (6)}\\ &= u\epsilon(a)\end{aligned}$$

This shows $(S1) \Longrightarrow (S3)$. Next we prove $(S3) \Longrightarrow (S2)$.

$$\begin{aligned}u\epsilon(a) &= \mathrm{id} \star S(a) = \sum a_1 S(a_2)\\ &= \sum S^2(a_1)S(a_2) = \sum c^{-\chi(|a_1|,|a_2|)} S(a_2 S(a_1)).\end{aligned}$$

If $a \notin A_0$, then $0 = u\epsilon(a) = \sum c^{-\chi(|a_1|,|a_2|)} S(a_2 S(a_1))$. Recall that $\sum c^{-\chi(|a_1|,|a_2|)} S(a_2 S(a_1))$ is the sum of linearly independent elements, thus $S(\sum a_2 S(a_1)) = 0$. So $\sum a_2 S(a_1) = 0 = u\epsilon(a)$. If $a \in A_0$, then $c^{-\chi(|a_1|,|a_2|)} = 1$. Thus $u\epsilon(a) = \sum a_2 S(a_1)$. We have shown $(S1) \Longrightarrow (S3) \Longrightarrow (S2)$.

Similarly, one can prove $(S2) \Longrightarrow (S3) \Longrightarrow (S1)$ and the proof of Proposition 4 is completed. □

Recall that A is commutative if $ab = ba$. Dually, A is cocommutative if $\triangle(a) = \sum a_2 \otimes a_1$. The next corollary follows directly from Proposition 4.

Corollary 5 *If A is commutative or cocommutative, we have* $S^2 = \mathrm{id}$.

We call A twisted commutative if $ab = c^{\chi(|a|,|b|)} ba$. Dually, A is cocommutative if $\triangle(a) = \sum c^{\chi(|a_1|,|a_2|)} a_2 \otimes a_1$. By the properties of counit ϵ, we have the following conclusion:

Corollary 6 *A is twist commutative or twisted cocommutative. Then* $S^2 = \mathrm{id}$.

Proof If A is twisted commutative, we have $u\epsilon(a) = \sum a_1 S(a_2) = \sum c^{\chi(|a_1|,|a_2|)} S(a_2)a_1$. Assume $a \notin A_0$, then $0 = u\epsilon(a) = \sum c^{\chi(|a_1|,|a_2|)} S(a_2)a_1$. Thus $\sum S(a_2)a_1 = 0$. So $\sum S(a_2)a_1 = 0 = u\epsilon(a)$. If $a \in A_0$, then $c^{\chi(|a_1|,|a_2|)} = 1$. Thus $u\epsilon(a) = \sum S(a_2)a_1$. In a word, we have $u\epsilon(a) = \sum S(a_2)a_1$. By Proposition 4 (S1), we prove the conclusion.

2. The global dimension of twisted Hopf algebras

A graded right A-module M is a right A-module with a decomposition $M=\oplus_{x\in\mathbb{N}_0 I}M_x$ such that $M_xA_y\subseteq M_{xy}$. We denote the module as $M\otimes A\longrightarrow M:\ m\otimes a\longrightarrow ma$ for any $m\in M,\ a\in A$. Denote the graded A-module category as A-gr. For graded A-modules M and N, we define the morphism in A-gr as:

$$\mathrm{Hom}_{A\text{-gr}}(M,N)=\{f\in\mathrm{Hom}_A(M,N)|f(M_x)\subseteq N_x,\ \forall x\in\mathbb{N}_0 I\}.$$

Note that $|f|=x$.

The graded A-module structure of $\mathrm{Hom}(M,N)$ is

$$(af)(m)=\sum c^{\chi(|a_2|,|m|)}a_1(f(S(a_2)m)),\quad \forall f\in\mathrm{Hom}(M,N),a\in A,m\in M. \tag{7}$$

Let A be a twisted Hopf algebra, and M,N be graded left A-modules. Then $M\otimes N$ is a graded left A-module with

$$a(m\otimes n)=\sum c^{\chi(|a_2|,|m|)}a_1m\otimes a_2n,\forall a\in A,m\in M,n\in N. \tag{8}$$

Lemma 7 *If A is a twisted Hopf algebra, M is a left A-module, then we have*

$$\epsilon(a)c^{\chi(|a|,|m|)}=\epsilon(a)c^{-\chi(|a|,|m|)}=\epsilon(a)c^{\chi^T(|a|,|m|)}=\epsilon(a),\quad \forall a\in A,m\in M. \tag{9}$$

Proof It is similar to the proof of Lemma 2.

Proposition 8 *Let A be a twisted Hopf algebra, P,N,W be graded left A-modules. Then $\mathrm{Hom}_{A\text{-gr}}(P\otimes_k N,W)\cong\mathrm{Hom}_{A\text{-gr}}(P,\mathrm{Hom}_k(N,W))$ as vector space, where $P\otimes N$ is viewed as a left A-module via (8), $\mathrm{Hom}_k(N,W)$ as a left A-module via (7).*

Proof It is obvious if $A=k$. Here, we assume that $A\neq k$. Let

$$\phi:\ \mathrm{Hom}_{A\text{-gr}}(P\otimes N,\ W)\longrightarrow\mathrm{Hom}_{A\text{-gr}}(P,\mathrm{Hom}(N,W)),$$
$$g\longmapsto\phi(g),$$

where $\phi(g)(p)(n)=g(p\otimes n)$. And

$$\psi:\ \mathrm{Hom}_{A\text{-gr}}(P,\ \mathrm{Hom}(N,W))\longrightarrow\mathrm{Hom}_{A\text{-gr}}(P\otimes N,W)$$
$$f\longmapsto\psi(f),$$

where $\psi(f)(p\otimes n)=f(p)(n)$. Note that ϕ and ψ are the usual bijections of the Hom-Tensor adjunction.

First, we check ϕ is a graded A-module map. Let g be a graded A-module map. For any $a\in A$, $|a|\neq 0$, $p\in P$, $n\in N$, we have

$$\begin{aligned}(a\phi(g)(p))(n)&=\sum c^{\chi(|a_2|,|p|)}a_1(\phi(g)(p)(S(a_2)n))\\&=\sum c^{\chi(|a_2|,|p|)}a_1(g(p\otimes S(a_2)n))\\&=\sum c^{\chi(|a_2|,|p|)}g(a_1(p\otimes S(a_2)n))\\&=\sum c^{\chi(|a_3|,|p|)+\chi(|a_2|,|p|)}g(a_1p\otimes a_2S(a_3)n)\quad\text{by (8)}\\&=\sum c^{\chi(|a_2|,|p|)}g(a_1p\otimes\epsilon(a_2)n)\end{aligned}$$

$$= \sum g(a_1 p \otimes \epsilon(a_2)n) \quad \text{by (9)}$$
$$= g(ap \otimes n) = (\phi(g)(ap))(n).$$

Thus ϕ is a graded A-module isomorphism.

Next, let f be a graded A-module map. We show that $\psi(f)$ is also a graded A-module map.

$$\begin{aligned}
\psi(f)(a(p \otimes n)) &= \sum \psi(f)(c^{\chi(|a_2|,|p|)} a_1 p \otimes a_2 n) \\
&= \sum f(c^{\chi(|a_2|,|p|)} a_1 p)(a_2 n) \\
&= \sum c^{\chi(|a_2|,|p|)} (a_1 f(p))(a_2 n) \\
&= \sum c^{\chi(|a_3|,|p|)+\chi(|a_2|,|p|)} a_1 [f(p)(S(a_2)a_3 n)] \quad \text{by (7)} \\
&= \sum c^{\chi(|a_2|,|p|)} a_1 [f(p)(\epsilon(a_2)n)] \\
&= \sum a_1 [f(p)(\epsilon(a_2)n)] \quad \text{by (9)} \\
&= \sum a[f(p)(n)] = a(\psi(f)(p \otimes n)).
\end{aligned}$$

Note that the proposition means that if the left A-module P is projective, then $P \otimes N$ is projective for any N. In fact, functor $\mathrm{Hom}_A(P, \mathrm{Hom}_k(N, -)) = \mathrm{Hom}_A(P, -) \circ \mathrm{Hom}(N, -)$ is exact since functor $\mathrm{Hom}_A(P, -)$ and $\mathrm{Hom}(N, -)$ are exact. So $\mathrm{Hom}_A(P \otimes_k N, W)$ is exact by Proposition 8. Thus $P \otimes N$ is projective.

Graded projective module and projective dimension of graded module appeared in [7, 2.2]. For later use, we write them as definitions.

Definition 9 *Let $P \in A$-gr. P is called gr-projective if P is a projective A-module.*

Definition 10 *A projective resolution of M in A-gr is an exact sequence of A-module*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\phi} M \longrightarrow 0,$$

in which all P_n are projective in A-gr, and d_i and ϕ are morphisms in A-gr.

If a projective resolution of M in A-gr exists, we define the projective dimension of M as follows:

Definition 11 *The projective dimension of graded A-module M is defined to be the smallest number d for which there is an exact sequence*

$$0 \longrightarrow P_d \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

with projective objects in A-gr. We denote this by $\mathrm{gr.p.dim}\, M = d$. *If no such projective resolution in A-gr exists, then we define* $\mathrm{gr.p.dim}\, M = \infty$.

Definition 12 *Let A be a twisted Hopf algebras. The global dimension of A is defined to be the supremum of projective dimensions of graded A-modules. We denote this by* $\mathrm{gr.gl.dim}\, A = \sup\{\mathrm{gr.p.dim}\, M \mid \forall M \in A\text{-gr}\}$.

Following [5] and [13], we have the main theorem:

Theorem 13 *Let A be a twisted Hopf algebra. Then*

$$\text{gr.gl.dim}\, A = \text{gr.p.dim}_A k = \text{p.dim}_A k.$$

Proof To prove the conclusion, we only need to prove $\text{gr.p.dim}_A N \leq \text{gr.p.dim}_A k$ for any graded left A-module N. If $\text{gr.p.dim}_A k = \infty$, it holds. Now we assume that $\text{gr.p.dim}_A k < \infty$. Suppose that we have a projective resolution of ${}_A k$ in the category of graded left A-module:

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow k \longrightarrow 0.$$

We can get a new exact sequence by applying the functor $- \otimes N$:

$$0 \longrightarrow P_n \otimes N \longrightarrow P_{n-1} \otimes N \longrightarrow \cdots \longrightarrow P_0 \otimes N \longrightarrow k \otimes N \cong N \longrightarrow 0.$$

Since P_i are graded projective left A-module, by Proposition 8, we know $P_i \otimes N$ are also graded projective left A-modules with module structure (8). We get $0 \longrightarrow P_n \otimes N \longrightarrow P_{n-1} \otimes N \longrightarrow \cdots \longrightarrow P_0 \otimes N \longrightarrow_A k \otimes N \cong N \longrightarrow 0$ is a graded projective resolution of N, and therefore $\text{gr.p.dim}_A N \leq \text{gr.p.dim}_A k$.

Thus the graded global dimension of A is the graded projective dimension of ${}_A k$, i.e., $\text{gr.gl.dim}\, A = \text{gr.p.dim}_A k$. Note that $\text{gr.p.dim}_A k = \text{p.dim}_A k$ for any graded algebras [7, 2.3.3]. □

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