Journal of Mathematical Research with Applications
Sept., 2012, Vol. 32, No. 5, pp. 615–625
DOI:10.3770/j.issn:2095-2651.2012.05.010
Http://jmre.dlut.edu.cn

Differences of Weighted Composition Operators from Mixed-Norm Spaces to Weighted-Type Spaces

Xianhao ZHAO, Yuxia LIANG*
Department of Mathematics, Tianjin University, Tianjin 300072*, P. R. China*

Abstract In this paper, we study the boundedness and estimate the essential norm of the differences of weighted composition operators from mixed-norm spaces to weighted-type spaces on the unit ball of C^N.

Keywords weighted composition operator; mixed-norm spaces; weighted-type space; essential norm; difference.

MR(2010) Subject Classification 30H05

1. Introduction

Throughout this paper, let $N \geq 1$ be a fixed integer. Let B_N denote the open unit ball of the complex N-dimensional Euclidean space C^N. Let $H(B_N)$ be the space of all holomorphic functions on B_N. The compact open topology on the space $H(B_N)$ will be denoted by *co*. Denote by $S(B_N)$ the collection of all the holomorphic self-maps of B_N. Let $d\sigma$ be the normalized Lebesgue measure on the boundary ∂B_N of B_N and let $d\nu$ denote the normalized Lebesgue measure on B_N. For $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ in C^N, we denote the inner product of z and w by $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$, and we write $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_N|^2}$.

A positive continuous function μ on $[0, 1)$ is called normal [14], if there exist three positive constants $0 \leq \delta < 1$, and $0 < a < b < \infty$, such that for $r \in [\delta, 1)$,

$$\frac{\mu(r)}{(1-r)^a} \downarrow 0, \quad \frac{\mu(r)}{(1-r)^b} \uparrow \infty,$$

as $r \to 1$. In the rest of this paper we always assume that μ is normal on $[0, 1)$, and from now on if we say that a function $\mu : B_N \to [0,\infty)$ is normal, we will also assume that it is radial on B_N, that is, $\mu(z) = \mu(|z|)$, $z \in B_N$.

Let $0 < p \leq \infty$, $0 < q \leq \infty$, and μ be normal on $[0, 1)$. Then f is said to belong to the mixed norm space $L(p, q, \mu)$ if f is measurable function on B_N and $\|f\|_{p,q,\mu} < \infty$, where

$$\|f\|_{p,q,\mu} = \Big\{\int_0^1 r^{2n-1}(1-r)^{-1}\mu^p(r)M_q^p(r, f)\mathrm{d}r\Big\}^{1/p}, \quad 0 < p < \infty,\ 0 < q \leq \infty,$$

$$\|f\|_{\infty,q,\mu} = \sup_{0\leq r<1} \mu(r)M_q(r, f), M_\infty(r, f) = \sup_{\zeta\in\partial B_N} |f(r\zeta)|,$$

Received December 25, 2010; Accepted August 31, 2011
Supported by the National Natural Science Foundation of China (Grant Nos. 10971153; 10671141).
* Corresponding author
E-mail address: zhaoxianhaozxh@126.com (Xianhao ZHAO); liangyx1986@126.com (Yuxia LIANG)

$$M_q(r,f)=\Big\{\int_{\partial B_N}|f(r\zeta)|^q\mathrm{d}\sigma(\zeta)\Big\}^{1/q},\quad 0<q<\infty.$$

If $0<p=q<1$, then $L(p,q,\mu)$ is just the space $L^p(\mu)=\{f$ is measurable function on $B_N:\int_{B_N}|f(z)|^p\frac{\mu^p(z)}{1-|z|}\mathrm{d}\nu(z)<\infty\}$.

Let $H(p,q,\mu)=L(p,q,\mu)\cap H(B_N)$. If $0<p=q<1$, then $H(p,q,\mu)$ is just the weighted Bergman space $L^p_a(\mu)$. In particular, $H(p,q,\mu)$ is Bergman space $L^p_a(\mu)$ if $0<p=q<\infty$ and $\mu(r)=(1-r)^{1/p}$. Otherwise, if $p=q=2$ and $\mu(r)=(1-r)^{\beta/2}(\beta<0)$, then $H(p,q,\mu(r))$ is the Dirichlet type space.

For $0<p,\ q<\infty$, $-1<\gamma<\infty$, let $\mu(r)=(1-r)^{\frac{\gamma+1}{p}}r^{-\frac{2n-1}{p}}$. It is easy to see that the mixed norm space $H(p,q,\mu)$ written as $H_{p,q,\gamma}$, consists of all $f\in H(B_N)$ such that

$$\|f\|_{H_{p,q,\gamma}}=\Big\{\int_0^1 M_q^p(r,f)(1-r)^\gamma\mathrm{d}r\Big\}^{1/p}<\infty. \tag{1}$$

For $\varphi\in S(B_N), u\in H(B_N)$, we define a weighted composition operator $W_{\varphi,u}$ by

$$W_{\varphi,u}(f)=u\cdot(f\circ\varphi)$$

for $f\in H(B_N)$. As for $u\equiv 1$, the weighted composition operator $W_{\varphi,1}$ is the usual composition operator, denoted by C_φ. When φ is the identity mapping I, the operator $W_{I,u}$ is also called the multiplication operator. Much effort has been expended on characterizing those analytic maps which induce bounded or compact composition operators. Readers interested in this topic can refer to the books [11] by Shapiro, [6] by Cowen and MacCluer, and [16, 17] by Zhu, which are excellent sources for the development of the theory of composition operators and function spaces. Composition operators have been investigated mainly in spaces of analytic functions to characterize the operator-theoretic behavior of C_φ in terms of the function-theoretic properties of the symbol φ. We can refer to the recent papers [10, 15, 18, 19] and their references therein.

For $0<\alpha<\infty$, let H^∞_α be the weighted Banach space of holomorphic functions f on B_N satisfying

$$\|f\|_{H^\infty_\alpha}=\sup_{z\in B_N}(1-|z|^2)^\alpha|f(z)|<\infty.$$

As we all know, H^∞_α is a Banach space under the norm $\|.\|_{H^\infty_\alpha}$.

Let ϕ be a positive continuous function on B_N (weight). The weighted-type space $H^\infty_\phi(B_N)=H^\infty_\phi$ consists of all $f\in H(B_N)$ such that

$$\|f\|_{H^\infty_\phi}=\sup_{z\in B_N}\phi(z)|f(z)|<\infty.$$

It is known that H^∞_ϕ is a Banach space. For related results on the weighted-type space we can refer to [2–5, 7, 14] and their references therein.

For any point $a\in B_N\setminus\{0\}$, we define

$$\varphi_a(z)=\frac{a-P_a(z)-s_aQ_a(z)}{1-\langle z,a\rangle},\quad z\in B_N,$$

where $s_a=\sqrt{1-|a|^2}$, P_a is the orthogonal projection from C^N onto the one-dimensional subspace $[a]$ generated by a, and $Q_a=I-P_a$ is the projection onto the orthogonal complement of

$[a]$, that is

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2}a,\ \ Q_a(z) = z - P_a(z), \quad z \in B_N.$$

When $a = 0$, we simply define $\varphi_a(z) = -z$. It is well known that each φ_a is a homeomorphism of the closed unit ball $\overline{B_N}$ onto $\overline{B_N}$. Then we define the pseudohyperbolic metric on B_N

$$\rho(a, z) = |\varphi_a(z)|.$$

We know that $\rho(a, z)$ is invariant under automorphisms [17].

For any two points z and w in B_N, let $\gamma(t) = (r_1(t), \dots, \gamma_N(t)) : [0, 1] \to B_N$ be a smooth curve to connect z and w. Define

$$l(\gamma) = \int_0^1 \sqrt{\langle B(\gamma(t))\gamma'(t), \gamma'(t)\rangle}\mathrm{d}t.$$

The infimum of the set consisting of all $l(\gamma)$ is denoted by $\beta(z, w)$, where γ is a smooth curve in B_N from z and w. We call β the Bergman metric on B_N. The relationship between the pseudohyperbolic metric and the Bergman metric is known as follows

$$\beta(z, w) = \frac{1}{2}\log\frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Some lemmas

In this section we present several lemmas which will be used in the proofs of the main results.

Lemma 1 *Assume that* $0 < p,\ q < \infty$, $-1 < \gamma < \infty$ *and* $f \in H_{p,q,\gamma}$. *Then there is a positive constant* C *independent of* f *such that*

$$|f(z)| \le C\frac{\|f\|_{H_{p,q,\gamma}}}{(1 - |z|^2)^{N/q+(\gamma+1)/p}}.$$

Proof By the monotonicity of the integral means, the following asymptotic relations

$$1 - r \asymp 1 - |z|, \quad r \in [(1 + |z|)/2, (3 + |z|)/4],$$

and Theorem 1.12 in [17], we have

$$\begin{aligned}\|f\|^p_{H_{p,q,\gamma}} &\ge \int_{(1+|z|)/2}^{(3+|z|)/4} M_q^p(r, f)(1 - r)^\gamma \mathrm{d}r \ge CM_q^p(\frac{1 + |z|}{2}, f)\int_{(1+|z|)/2}^{(3+|z|)/4}(1 - r)^\gamma \mathrm{d}r\\ &\ge CM_q^p(\frac{1 + |z|}{2}, f)(1 - |z|^2)^{\gamma+1} \ge C(1 - |z|^2)^{\gamma+1+(Np)/q}|f(z)|^p,\end{aligned}$$

from which the desired result follows. This completes the proof of this lemma. □

Lemma 2 *Assume that* $0 < p, q < \infty$, $-1 < \gamma < \infty$ *and* $f \in H_{p,q,\gamma}$. *Then*

$$|(1 - |z|^2)^{N/q+(\gamma+1)/p}f(z) - (1 - |w|^2)^{N/q+(\gamma+1)/p}f(w)| \le C\|f\|_{H_{p,q,\gamma}}\rho(z, w)$$

for all $z, w \in B_N$.

Proof By Lemma 1 we have that if $f \in H_{p,q,\gamma}(B_N)$, then $f \in H^\infty_{(1-|z|^2)^{N/q+(\gamma+1)/p}}$, and moreover $\|f\|_{H^\infty_{(1-|z|^2)^{N/q+(\gamma+1)/p}}} \le C\|f\|_{H_{p,q,\gamma}}$. By Lemma 3.2 in [7], for each $f \in H^\infty_{(1-|z|^2)^{N/q+(\gamma+1)/p}}(B_N)$ and $z, w \in B_N$ there is a $C > 0$ such that

$$\begin{aligned}&|(1-|z|^2)^{N/q+(\gamma+1)/p}f(z) - (1-|w|^2)^{N/q+(\gamma+1)/p}f(w)|\\ &\quad\le C\|f\|_{H^\infty_{(1-|z|^2)^{N/q+(\gamma+1)/p}}}\rho(z,w) \le C\|f\|_{H_{p,q,\gamma}}\rho(z,w).\end{aligned}$$

This completes the proof of this Lemma. □

The following lemma is the crucial criterion for compactness, whose proof is an easy modification of that of Proposition 3.11 of [6].

Lemma 3 *Assume that* $0 < p, q < \infty$, $-1 < \gamma < \infty$, $u, v \in H(B_N)$, $\varphi, \psi \in S(B_N)$ *and the operator* $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H^\infty_\phi$ *is bounded. Then the operator* $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H^\infty_\phi$ *is compact if and only if for every bounded sequence* $\{f_n\}$ *in* $H_{p,q,\gamma}$ *such that* $f_n \to 0$ *uniformly on every compact subset of* B_N *as* $n \to \infty$, *it follows that* $\|(W_{\varphi,u} - W_{\psi,v})f_n\|_{H^\infty_\phi} \to 0$, $n \to \infty$.

Lemma 4 *For each sequence* (w_j) *in* B_N *with* $|w_j| \to 1$ *as* $j \to \infty$, *there exists its subsequence* (η_k) *and functions* (f_{j_k}) *in* $H^\infty(B_N)$ *such that*

$$\sum_{k=1}^{\infty} |f_{j_k}(z)| \le 1,$$

for all $z \in B_N$ *and*

$$f_{j_k}(\eta_k) > 1 - \frac{1}{2^k}, \quad k \in \mathbb{N}.$$

Proof This Lemma can be found in Lemma 6 of [13].

Lemma 5 *For* $\beta > -1$ *and* $m > 1 + \beta$ *we have*

$$\int_0^1 \frac{(1-r)^\beta}{(1-\lambda r)^m}\mathrm{d}r \le C(1-\lambda)^{1+\beta-m}, \quad 0 < \lambda < 1.$$

3. The boundedness of $W_{\varphi,u} - W_{\psi,v}$

In this section we consider the boundedness of $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H^\infty_\phi$. For that purpose, consider the following three coditions:

$$\sup_{z\in B_N} \frac{\phi(z)|u(z)|}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}}\rho(\varphi(z),\psi(z)) < \infty, \tag{2}$$

$$\sup_{z\in B_N} \frac{\phi(z)|v(z)|}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\rho(\varphi(z),\psi(z)) < \infty, \tag{3}$$

$$\sup_{z\in B_N} \Big|\frac{\phi(z)u(z)}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} - \frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big| < \infty. \tag{4}$$

Theorem 1 *Assume that* $0 < p$, $q < \infty$, $-1 < \gamma < \infty$, $u, v \in H(B_N)$ *and* $\varphi, \psi \in S(B_N)$. *Then the following statements are equivalent.*

(i) $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H^\infty_\phi$ *is bounded;*

(ii) The conditions (2) and (4) hold;

(iii) The conditions (3) and (4) hold.

Proof First, we prove the implication (i)⇒(ii). Assume that $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H_\phi^\infty$ is bounded. Fix $w \in B_N$, consider the function f_w defined by

$$f_w(z) = \frac{(1-|\varphi(w)|^2)^{b-(\gamma+1)/p}}{(1-\langle z, \varphi(w)\rangle)^{N/q+b}} \cdot \frac{\langle \varphi_{\psi(w)}(z), \varphi_{\psi(w)}(\varphi(w))\rangle}{|\varphi_{\psi(w)}(\varphi(w))|} \tag{5}$$

for $z \in B_N$ and $b > (\gamma+1)/p$. In fact,

$$M_q(r, f_w(z)) = \Big(\int_{\partial B_N} |f_w(r\zeta)|^q \mathrm{d}\sigma(\zeta)\Big)^{1/q} \le \Big(\int_{\partial B_N} \frac{(1-|\varphi(w)|^2)^{(b-(\gamma+1)/p)q}}{|1-\langle r\zeta, \varphi(w)\rangle|^{(N/q+b)q}} \mathrm{d}\sigma(\zeta)\Big)^{1/q}.$$

By Theorem 1.12 in [17] we get

$$M_p(r, f_w(z)) \le \frac{(1-|\varphi(w)|^2)^{b-(\gamma+1)/p}}{(1-r|\varphi(w)|^2)^b}.$$

Using Lemma 5 and the above inequality gives

$$\begin{aligned} \|f_w\|^q_{H_{p,q,\gamma}} &= \int_0^1 M_q^p(r, f_w)(1-r)^\gamma \mathrm{d}r \\ &\le C\int_0^1 \frac{(1-|\varphi(w)|^2)^{pb-(\gamma+1)}}{(1-r|\varphi(w)|^2)^{pb}}(1-r)^\gamma \mathrm{d}r \\ &= C(1-|\varphi(w)|^2)^{pb-(\gamma+1)} \int_0^1 \frac{(1-r)^\gamma}{(1-r|\varphi(w)|^2)^{pb}} \mathrm{d}r \\ &\le C(1-|\varphi(w)|^2)^{pb-(\gamma+1)}(1-|\varphi(w)|^2)^{(\gamma+1)-pb} = C. \end{aligned} \tag{6}$$

Therefore, $f_w \in H_{p,q,\gamma}$, and $\sup_{w\in B_N} \|f_w\|_{H_{p,q,\gamma}} \le C$.

Note that

$$f_w(\varphi(w)) = \frac{\rho(\varphi(w), \psi(w))}{(1-|\varphi(w)|^2)^{N/q+(\gamma+1)/p}}, \quad f_w(\psi(w)) = 0. \tag{7}$$

By the boundeness of $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H_\phi^\infty$ and using (7), we have

$$\begin{aligned} \infty > C &\ge \|(W_{\varphi,u} - W_{\psi,v})f_w\|_{H_\phi^\infty} \\ &= \sup_{z\in B_N} \phi(z)|f_w(\varphi(z))u(z) - f_w(\psi(z))v(z)| \\ &\ge \phi(w)|f_w(\varphi(w))u(w) - f_w(\psi(w))v(w)| \\ &= \frac{\phi(w)|u(w)|\rho(\varphi(w), \psi(w))}{(1-|\varphi(w)|^2)^{N/q+(\gamma+1)/p}} \end{aligned} \tag{8}$$

for any $w \in B_N$. Since $w \in B_N$ is an arbitrary element, from (8) we can obtain (2).

Next we prove (4). For given $w \in B_N$ we consider the function

$$g_w(z) = \frac{(1-|\psi(w)|^2)^{b-(\gamma+1)/p}}{(1-\langle z, \psi(w)\rangle)^{N/q+b}}, \tag{9}$$

where $z \in B_N$, $b > (\gamma+1)/p$. By the proof similar to (6), we obtain $g_w \in H_{p,q,\gamma}(B_N)$ with $\|g_w\|_{H_{p,q,\gamma}} \le C$. Note

$$g_w(\psi(w)) = \frac{1}{(1-|\psi(w)|^2)^{N/q+(\gamma+1)/p}}. \tag{10}$$

Thus by the boundeness of $W_{\varphi,u}-W_{\psi,v}:H_{p,q,\gamma}\to H_\phi^\infty$, we have

$$\begin{aligned}\infty>C&\geq\|(W_{\varphi,u}-W_{\psi,v})g_w\|_{H_\phi^\infty}\geq\phi(w)|g_w(\varphi(w))u(w)-g_w(\psi(w))v(w)|\\&=|I(w)+J(w)|,\end{aligned}\tag{11}$$

where

$$\begin{aligned}I(w)=&(1-|\psi(w)|^2)^{N/q+(\gamma+1)/p}g_w(\psi(w))\Big[\frac{\phi(w)u(w)}{(1-|\varphi(w)|^2)^{N/q+(\gamma+1)/p}}-\\&\frac{\phi(w)v(w)}{(1-|\psi(w)|^2)^{N/q+(\gamma+1)/p}}\Big]\\=&\frac{\phi(w)u(w)}{(1-|\varphi(w)|^2)^{N/q+(\gamma+1)/p}}-\frac{\phi(w)v(w)}{(1-|\psi(w)|^2)^{N/q+(\gamma+1)/p}},\\J(w)=&\frac{\phi(w)u(w)}{(1-|\varphi(w)|^2)^{N/q+(\gamma+1)/p}}\Big[(1-|\varphi(w)|^2)^{N/q+(\gamma+1)/p}g_w(\varphi(w))-\\&(1-|\psi(w)|^2)^{N/q+(\gamma+1)/p}g_w(\psi(w))\Big].\end{aligned}$$

By (2) and Lemma 3 we conclude that

$$\begin{aligned}|J(w)|\leq&C\frac{\phi(w)|u(w)|\rho(\varphi(w),\psi(w))}{(1-|\varphi(w)|^2)^{N/q+(\gamma+1)/p}}\|g_w\|_{H_{p,q,\gamma}}\\\leq&C\frac{\phi(w)|u(w)|}{(1-|\varphi(w)|^2)^{N/q+(\gamma+1)/p}}\rho(\varphi(w),\psi(w))\leq C<\infty.\end{aligned}$$

Thus we obtain that $|J(w)|\leq C<\infty$ for all $w\in B_N$, which, combined with (11), gives $|I(w)|\leq C<\infty$ for all $w\in B_N$. Thus we obtain (4).

(ii)$\Rightarrow$(iii). Assume that (2) and (4) hold. We need only to show (3) holds. Note that the pseudohyperbolic metric ρ is less than 1. Then we have that

$$\begin{aligned}&\frac{\phi(z)|v(z)|}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\rho(\varphi(z),\psi(z))\leq\frac{\phi(z)|u(z)|}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}}\rho(\varphi(z),\psi(z))+\\&\Big|\frac{\phi(z)u(z)}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}}-\frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big|\rho(\varphi(z),\psi(z)).\end{aligned}\tag{12}$$

Then using (2) and (4) in (12) implies that (3) holds.

(iii)$\Rightarrow$(i). Assume that (3) and (4) hold. By Lemmas 2 and 3, for any $f\in H_{p,q,\gamma}$, we have

$$\begin{aligned}&\phi(z)|(W_{\varphi,u}-W_{\psi,v})f(z)|=\phi(z)|f(\varphi(z))u(z)-f(\psi(z))v(z)|\\&=\Big|(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}f(\varphi(z))\Big[\frac{\phi(z)u(z)}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}}-\frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big]+\\&\quad\frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big[(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}f(\varphi(z))-(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}f(\psi(z))\Big]\Big|\\&\leq\|f\|_{H_{p,q,\gamma}}\Big|\frac{\phi(z)u(z)}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}}-\frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big|+\\&\quad C\|f\|_{H_{p,q,\gamma}}\rho(\varphi(z),\psi(z))\frac{\phi(z)|v(z)|}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\\&\leq C\|f\|_{H_{p,q,\gamma}}<\infty,\end{aligned}$$

from which it follows that $W_{\varphi,u}-W_{\psi,v}:H_{p,q,\gamma}\to H_\phi^\infty$ is bounded. The whole proof is completed. $\square$

Corollary 1 *Assume that* $0 < p, q < \infty$, $-1 < \gamma < \infty, \varphi \in S(B_N)$ *and* $u \in H(B_N)$. *Then* $W_{\varphi,u} : H_{p,q,\gamma} \to H_\phi^\infty$ *is bounded if and only if*

$$\sup_{z\in B_N} \frac{\phi(z)|u(z)|}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} < \infty. \tag{13}$$

Proof The result follows from the simple case when $v = 0$ in Theorem 1. □

4. The essential norm of $W_{\varphi,u} - W_{\psi,v}$

In this section we characterize the essential norm of $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H_\phi^\infty$. Here we consider the following conditions:

$$\limsup_{|\varphi(z)|\to 1} \frac{\phi(z)|u(z)|}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}}\rho(\varphi(z),\psi(z)), \tag{14}$$

$$\limsup_{|\psi(z)|\to 1} \frac{\phi(z)|v(z)|}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\rho(\varphi(z),\psi(z)), \tag{15}$$

$$\limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1} \Big|\frac{\phi(z)u(z)}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} - \frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big|. \tag{16}$$

Theorem 2 *Assume that* $0 < p, q < \infty$, $-1 < \gamma < \infty$ $u, v \in H(B_N)$ *and* $\varphi, \psi \in S(B_N)$. *If* $W_{\varphi,u}, W_{\psi,v} : H_{p,q,\gamma} \to H_\phi^\infty$ *are bounded operators, then the essential norm* $\|W_{\varphi,u} - W_{\psi,v}\|_{e,H_{p,q,\gamma}\to H_\phi^\infty}$ *is equivalent to the maximum of (14)–(16).*

Proof First we show that the maximum of (14)–(16) is the upper estimate of the essential norm of $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H_\phi^\infty$. Consider the operators on $H(B_N)$ defined by

$$P_k(f)(z) = f(\frac{k}{k+1}z), \quad k \in \mathbb{N}. \tag{17}$$

It is easy to see that they are continuous on the *co* topology and that $P_k(f) \to f$ on compacts of B_N as $k \to \infty$. On the other hand, since the integral means $M_q(r, f) = (\int_{\partial B_N} |f(r\zeta)|^q \mathrm{d}\sigma(\zeta))^{1/q}$ is nondecreasing in r and by the definition of $H_{p,q,\gamma}$ in (1) it follows that

$$\begin{aligned}\|P_k(f)\|^p_{H_{p,q,\gamma}} &= \int_0^1 M_q^p(r, P_k f)(1-r)^\gamma \mathrm{d}r \\ &= \int_0^1 \Big(\int_{\partial B_N} |f(r\frac{k}{k+1}\zeta)|^q \mathrm{d}\sigma(\zeta)\Big)^{p/q}(1-r)^\gamma \mathrm{d}r \\ &\le \int_0^1 \Big(\int_{\partial B_N} |f(r\zeta)|^q \mathrm{d}\sigma(\zeta)\Big)^{p/q}(1-r)^\gamma \mathrm{d}r = \|f\|^p_{H_{p,q,\gamma}}. \end{aligned}\tag{18}$$

From (18) we obtain that $\|P_k(f)\|_{H_{p,q,\gamma}} \le \|f\|_{H_{p,q,\gamma}}$, $k \in \mathbb{N}$, thus we can easily obtain that $\sup\limits_{k\in\mathbb{N}} \|P_k\|_{H_{p,q,\gamma}\to H_{p,q,\gamma}} \le 1$. Moreover, by Lemma 3 it follows that the operators sequence $(P_k)_{k\in\mathbb{N}}$ are also compact on $H_{p,q,\gamma}(B_N)$.

Let $r \in (0, 1)$ be fixed and $f \in H_{p,q,\gamma}(B_N)$ such that $\|f\|_{H_{p,q,\gamma}} \le 1$. Set

$$G_k := (I - P_k)f, \quad k \in \mathbb{N}.$$

Then we can easily obtain $G_k \in H_{p,q,\gamma}(B_N)$, $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}} \|G_k\|_{H_{p,q,\gamma}} \le 2$. We have

$$\begin{aligned}
&\|W_{\varphi,u} - W_{\psi,v}\|_{e,H_{p,q,\gamma}\to H_\phi^\infty} \le \sup_{\|f\|_{H_{p,q,\gamma}}\le 1} \|(W_{\varphi,u} - W_{\psi,v})G_k\|_{H_\phi^\infty}\\
&\quad \le \sup_{\|f\|_{H_{p,q,\gamma}}\le 1} \sup_{z\in B_N} \phi(z)|G_k(\varphi(z))u(z) - G_k(\psi(z))v(z)|\\
&\quad \le \sup_{\|f\|_{H_{p,q,\gamma}}\le 1} \sup_{|\varphi(z)|>r} \phi(z)|G_k(\varphi(z))u(z) - G_k(\psi(z))v(z)| +\\
&\qquad \sup_{\|f\|_{H_{p,q,\gamma}}\le 1} \sup_{|\psi(z)|>r} \phi(z)|G_k(\varphi(z))u(z) - G_k(\psi(z))v(z)| +\\
&\qquad \sup_{\|f\|_{H_{p,q,\gamma}}\le 1} \sup_{\max\{|\varphi(z)|,|\psi(z)|\}\le r} \phi(z)|G_k(\varphi(z))u(z) - G_k(\psi(z))v(z)|\\
&\quad = I_{k,1}(r) + I_{k,2}(r) + I_{k,3}(r).
\end{aligned} \tag{19}$$

First we estimate $I_{k,1}(r)$. By Lemma 2 and using the fact that $\|G_k\|_{H_{p,q,\gamma}} \le 2$, we obtain

$$\begin{aligned}
&\phi(z)|G_k(\varphi(z))u(z) - G_k(\psi(z))v(z)|\\
&\quad \le \frac{\phi(z)|u(z)|}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} \Big|(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p} G_k(\varphi(z)) - (1-|\psi(z)|^2)^{N/q+(\gamma+1)/p} G_k(\psi(z))\Big| +\\
&\qquad (1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}|G_k(\psi(z))| \Big|\frac{\phi(z)u(z)}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} - \frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big| \qquad (20)\\
&\quad \le 2C \frac{\phi(z)|u(z)|}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} \rho(\varphi(z),\psi(z)) +\\
&\qquad 2\Big|\frac{\phi(z)u(z)}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} - \frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big|. \qquad (21)
\end{aligned}$$

A similar estimate is obtained for $I_{k,2}(r)$.

It is clear that for every $f \in H(B_N)$, $\lim_{k\to\infty}(I-P_k)f(z) = 0$ and that the space $H(B_N)$ endowed with compact open topology *co* is a Fréchet space. Hence, by Banach-Steinhaus theorem, $(I-P_k)f$ converges to zero uniformly on compacts of $(H(B_N), co)$ as $k \to \infty$. Since the unit ball of $H_{p,q,\gamma}$ is a compact subset of $(H(B_N), co)$, it follows that

$$\lim_{k\to\infty} \sup_{\|f\|_{H_{p,q,\gamma}}\le 1} \sup_{|\zeta|\le r} |(I-P_k)(f)(\zeta)| = 0. \tag{22}$$

From the boundedness of the operators $W_{\varphi,u}, W_{\psi,v} : H_{p,q,\gamma} \to H_\phi^\infty$ we can easily obtain the boundedness of $W_{\varphi,u} - W_{\psi,v} : H_{p,q,\gamma} \to H_\phi^\infty$, then by Theorem 1 we get (4) holds.

On the other hand, from (22) we get the following equality about the sequence $G_k := (I-P_k)f$, $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \sup_{\|f\|_{H_{p,q,\gamma}}\le 1} \sup_{|\zeta|\le r} |G_k(\zeta)| = 0. \tag{23}$$

Hence for each $r \in (0,1)$ and $|\psi(z)| \le r$, putting (23) and (4) into (20) yields

$$\limsup_{k\to\infty} I_{k,1}(r) \le 2C \sup_{|\varphi(z)|>r} \frac{\phi(z)|u(z)|}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} \rho(\varphi(z),\psi(z)).$$

If $|\psi(z)| > r$, we have that

$$\limsup_{k\to\infty} I_{k,1}(r) \le 2C \sup_{|\varphi(z)|>r} \frac{\phi(z)|u(z)|}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}} \rho(\varphi(z),\psi(z)) +$$

$$2\sup_{\min\{|\varphi(z)|,|\psi(z)|\}>r}\Big|\frac{\phi(z)u(z)}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}}-\frac{\phi(z)v(z)}{(1-|\psi(z)|^2)^{N/q+(\gamma+1)/p}}\Big|.$$

Letting $r\to 1$ in the above inequality, we get an estimate for $\limsup_{r\to1}\limsup_{k\to\infty} I_{k,1}(r)$ in terms of (14) and (16). Similar estimate is obtained for $\limsup_{r\to1}\limsup_{k\to\infty} I_{k,2}(r)$.

Next we estimate $\lim_{k\to\infty} I_{k,3}(r)$. Since the operators $W_{\varphi,u}, W_{\psi,v} : H_{p,q,\gamma} \to H^\infty_\phi$ are bounded, we take the test function $f(z) = 1 \in H_{p,q,\gamma}$. Then we can easily get $u, v \in H^\infty_\phi$. Combining this with (23) gives

$$\begin{aligned}\lim_{k\to\infty} I_{k,3}(r) &:= \lim_{k\to\infty}\sup_{\|f\|_{H_{p,q,\gamma}}\le 1}\sup_{\max\{|\varphi(z)|,|\psi(z)|\}\le r}\phi(z)|G_k(\varphi(z))u(z)-G_k(\psi(z))v(z)|\\ &\le \lim_{k\to\infty}\sup_{\|f\|_{H_{p,q,\gamma}}\le 1}\sup_{\max\{|\varphi(z)|,|\psi(z)|\}\le r}\phi(z)|u(z)||G_k(\varphi(z))|+\\ &\quad \lim_{k\to\infty}\sup_{\|f\|_{H_{p,q,\gamma}}\le 1}\sup_{\max\{|\varphi(z)|,|\psi(z)|\}\le r}\phi(z)|v(z)||G_k(\psi(z))|\\ &=0. \end{aligned}\tag{24}$$

Thus from (24) we obtain that $\lim_{k\to\infty} I_{k,3}(r) = 0$. From all these facts the desired upper estimate follows.

Next we show that the maximum of (14)–(16) is a lower bound for the essential norm. Choose a sequence $(z_k)_{k\in\mathbb{N}}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$ and

$$\lim_{k\to\infty}\frac{(1-|z_k|^2)|u(z_k)|\rho(\varphi(z_k),\psi(z_k))}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}}=\limsup_{|\varphi(z)|\to1}\frac{\phi(z)|u(z)|\rho(\varphi(z),\psi(z))}{(1-|\varphi(z)|^2)^{N/q+(\gamma+1)/p}}.$$

If such a sequence does not exist, then the estimate vacuously holds. Since $|\varphi(z_k)|\to 1$ as $k\to\infty$, by Lemma 4 we can find a sequence $f_k \in H^\infty(B_N)$, $k\in\mathbb{N}$, such that

$$\sum_{k=1}^{\infty}|f_k(z)|\le 1,\tag{25}$$

for all $z\in B_N$, and

$$f_k(\varphi(z_k)) > 1-\frac{1}{2^k},\quad k\in\mathbb{N}.\tag{26}$$

Define the test function

$$H_k(z)=f_k(z)\frac{(1-|\varphi(z_k)|^2)^{b-(\gamma+1)/p}}{(1-\langle z,\varphi(z_k)\rangle)^{N/q+b}}\cdot\frac{\langle\varphi_{\psi(z_k)}(z),\varphi_{\psi(z_k)}(\varphi(z_k))\rangle}{|\varphi_{\psi(z_k)}(\varphi(z_k))|}\tag{27}$$

when $\varphi(z_k)\ne\psi(z_k)$, and $H_k(z)=0$, when $\varphi(z_k)=\psi(z_k)$. Thus we can easily obtain

$$\sup_{k\in\mathbb{N}}\|H_k\|_{H_{p,q,\gamma}}\le C.$$

It is obvious that

$$H_k(\varphi(z_k)) = f_k(\varphi(z_k))\frac{\rho(\varphi(z_k),\psi(z_k))}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}},\ H_k(\psi(z_k))=0.\tag{28}$$

Moreover, it is obvious that $H_k\to 0$ uniformly on the compact subsets of B_N as $k\to\infty$. Then for each compact operator $K : H_{p,q,\gamma}\to H^\infty_\phi$ we have $\lim_{k\to\infty}\|KH_k\|_{H^\infty_\phi}=0$. From this and (28) we get

$$C\|W_{\varphi,u}-W_{\psi,v}-K\|_{H_{p,q,\gamma}\to H^\infty_\phi}\ge\limsup_{k\to\infty}\|(W_{\varphi,u}-W_{\psi,v})H_k\|_{H^\infty_\phi}-\limsup_{k\to\infty}\|KH_k\|_{H^\infty_\phi}$$

$$\begin{aligned}
&\geq \limsup_{k\to\infty} \phi(z_k)|H_k(\varphi(z_k))u(z_k) - H_k(\psi(z_k))v(z_k)| \\
&= \limsup_{k\to\infty} \phi(z_k)\Big| f_k(\varphi(z_k)) \frac{u(z_k)\rho(\varphi(z_k),\psi(z_k))}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}}\Big| \\
&\geq \limsup_{k\to\infty} \frac{\phi(z_k)|u(z_k)|}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}} \rho(\varphi(z_k),\psi(z_k)). \qquad (29)
\end{aligned}$$

From (29) it follows that expression (14) is a lower bound for the essential norm. That the expression in (15) is also a lower bound can be proved similarly, so we omit it.

Next we show (16) is also the lower bounded for the essential norm. Let the sequence $(z_k)_{k\in\mathbb{N}}$ satisfy $\min\{|\varphi(z_k)|,|\psi(z_k)|\}\to 1$ as $k\to\infty$, and

$$\begin{aligned}
&\limsup_{k\to\infty} \Big| \frac{(1-|z_k|^2)u(z_k)}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}} - \frac{(1-|z_k|^2)v(z_k)}{(1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}}\Big| \\
&\quad = \limsup_{\min\{|\varphi(z_k)|,|\psi(z_k)|\}\to 1} \Big| \frac{(1-|z_k|^2)u(z_k)}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}} - \frac{(1-|z_k|^2)v(z_k)}{(1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}}\Big|.
\end{aligned}$$

We may assume that there is the following

$$l := \lim_{k\to\infty} \rho(\varphi(z_k),\psi(z_k)) \geq 0.$$

If $l>0$ when $\min\{|\varphi(z_k)|,|\psi(z_k)|\}\to 1$ as $k\to\infty$, then (16) follows from (14) and (15). Thus we can assume that $l=0$ when $\min\{|\varphi(z_k)|,|\psi(z_k)|\}\to 1$ as $k\to\infty$.

Let $(f_k)_{k\in\mathbb{N}}$ be the sequence satisfying (25) and (26). Then we choose the function

$$F_k(z) = f_k(z)\frac{(1-|\varphi(z_k)|^2)^{b-(\gamma+1)/p}}{(1-\langle z,\varphi(z_k)\rangle)^{N/q+b}}, \quad k\in\mathbb{N}.$$

Thus we can easily obtain $\sup_{k\in\mathbb{N}}\|F_k\|_{H_{p,q,\gamma}} \leq C$. Moreover, it is obvious that $F_k\to 0$ uniformly on the compact subsets of B_N as $k\to\infty$. Then for each compact operator $K: H_{p,q,\gamma}\to H_\phi^\infty$ we have $\lim_{k\to\infty}\|KF_k\|_{H_\phi^\infty}=0$. From this, Lemma 2 and the property of the sequence $(f_k)_{k\in\mathbb{N}}$ we obtain that

$$\begin{aligned}
&C\|W_{\varphi,u}-W_{\psi,v}-K\|_{H_{p,q,\gamma}\to H_\phi^\infty} \geq \limsup_{k\to\infty}\|(W_{\varphi,u}-W_{\psi,v})F_k\|_{H_\phi^\infty} \\
&\quad\geq \limsup_{k\to\infty}\phi(z_k)|F_k(\varphi(z_k))u(z_k)-F_k(\psi(z_k))v(z_k)| \\
&\quad\geq \limsup_{k\to\infty}\phi(z_k)\Big|\frac{u(z_k)f_k(\varphi(z_k))}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}} - \frac{v(z_k)f_k(\varphi(z_k))}{(1-|\psi(k)|^2)^{N/q+(\gamma+1)/p}}\Big| - \\
&\qquad \limsup_{k\to\infty}\phi(z_k)\Big|\frac{v(z_k)f_k(\varphi(z_k))}{(1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}} - F_k(\psi(z_k))v(z_k)\Big| \\
&\quad\geq \limsup_{k\to\infty}\Big|\frac{\phi(z_k)u(z_k)}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}} - \frac{\phi(z_k)v(z_k)}{(1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}}\Big|(1-\frac{1}{2^k}) - \\
&\qquad \limsup_{k\to\infty}\frac{\phi(z_k)|v(z_k)|}{(1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}}\Big|(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}F(\varphi(z_k)) - \\
&\qquad (1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}F(\psi(z_k))\Big| \\
&\quad\geq \limsup_{k\to\infty}\Big|\frac{\phi(z_k)u(z_k)}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}} - \frac{\phi(z_k)v(z_k)}{(1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}}\Big|(1-\frac{1}{2^k}) -
\end{aligned}$$

$$C \limsup_{k\to\infty} \frac{\phi(z_k)|v(z_k)|}{(1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}} \rho(\varphi(z_k),\psi(z_k))$$
$$= \limsup_{k\to\infty} \Big| \frac{\phi(z_k)u(z_k)}{(1-|\varphi(z_k)|^2)^{N/q+(\gamma+1)/p}} - \frac{\phi(z_k)v(z_k)}{(1-|\psi(z_k)|^2)^{N/q+(\gamma+1)/p}} \Big|. \tag{30}$$

Hence from (30) we know that the expression (16) is also a lower bound for the essential norm, as claimed. The whole proof is completed. □

Acknowledgement The authors would like to express their thanks to Prof. Zehua ZHOU and the referees for their helpful suggestions.

References

[1] K. D. BIERSTEDT, J. BONET, J. TASKINEN. *Associated weights and spaces of holomorphic functions.* Studia Math., 1998, **127**(2): 137–168.
[2] J. BONET, P. DOMAŃSKI, M. LINDSTRÖM, et al. *Composition operators between weighted Banach spaces of analytic functions.* J. Austral. Math. Soc. Ser. A, 1998, **64**(1): 101–118.
[3] J. BONET, P. DOMAŃSKI, M. LINDSTRÖM. *Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions.* Canad. Math. Bull., 1999, **42**(2): 139–148.
[4] J. BONET, M. LINDSTRÖM, E. WOLF. *Differences of composition operators between weighted Banach spaces of holomorphic functions.* J. Aust. Math. Soc., 2008, **84**(1): 9–20.
[5] K. D. BIERSTEDT, W. H. SUMMERS. *Biduals of weighted Banach spaces of analytic functions.* J. Austral. Math. Soc. Ser. A, 1993, **54**(1): 70–79.
[6] C. C. COWEN, B. D. MACCLUER. *Composition Operators on Spaces of Analytic Functions.* CRC Press, Boca Raton, FL, 1995.
[7] Ji'neng DAI, Caiheng OUYANG. *Differences of weighted composition operators on $H_\alpha^\infty(B_N)^*$.* J. Inequal. Appl., 2009, 1–19.
[8] Zhijie JIANG, S. STEVIĆ. *Compact differences of weighted composition operators from weighted Bergman spaces to weighted-type spaces.* Appl. Math. Comput., 2010, **217**(7): 3522–3530.
[9] A. MONTES-RODRIGUEZ. *Weighted composition operators on weighted Banach spaces of analytic functions.* J. London Math. Soc. (2), 2000, **61**(3): 872–884.
[10] S. OHNO, K. STROETHOFF, Ruhan ZHAO. *Weighted composition operators between Bloch-type spaces.* Rocky Mountain J. Math., 2003, **33**(1): 191–215.
[11] J. H. SHAPIRO. *Composition Operators and Classical Function Theory.* Springer-Verlag, New York, 1993.
[12] S. STEVIĆ, E. WOLF. *Differences of composition operators between weighted-type spaces of holomorphic functions on the unit ball of $\mathbb{C}^N$.* Appl. Math. Comput., 2009, **215**(5): 1752–1760.
[13] S. STEVIĆ. *Weighted composition operators between mixed-norm spaces and H_α^∞ spaces in the unit ball.* J. Inequal. Appl., 2007, 1–9.
[14] A. L. SHIELDS, D. L. WILLIAMS. *Bonded projections, duality, and multipliers in spaces of analytic functions.* Trans. Amer. Math. Soc., 1971, **162**: 287–302.
[15] Zehua ZHOU, Renyu CHEN. *Weighted composition operators from $F(p,q,s)$ to Bloch type spaces on the unit ball.* Internat. J. Math., 2008, **19**(8): 899–926.
[16] Kehe ZHU. *Operator Theory in Function Spaces.* Marcel Dekker, Inc., New York, 1990.
[17] Kehe ZHU. *Spaces of Holomorphic Functions in the Unit Ball.* Springer-Verlag, New York, 2005.
[18] Zehua ZHOU, Jihuai SHI. *Compactness of composition operators on the Bloch space in classical bounded symmetric domains.* Michigan Math. J., 2002, **50**(2): 381–405.
[19] Honggang ZENG, Zehua ZHOU. *An estimate of the essential norm of a composition operator from $F(p,q,s)$ to β^α in the unit ball.* J. Inequal. Appl., 2010, 1–22.