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A Brief Introduction to Fibrewise Topological Spaces Theory

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Abstract Fibrewise topological spaces theory, presented in the recent 20 years, is a new branch of mathematics developed on the basis of General Topology, Algebra topology and Fibrewise spaces theory. It is associated with differential geometry, Lie groups and dynamical systems theory. From the perspective of Category theory, it is in the higher category of general topological space, so the discussion of new properties and characteristics of the variety of fibre topological space has more important significance. This paper introduces the process of the origin and development of Fibrewise topological spaces theory. Then, we study the main contents and important results in this branch. Finally, we review the research status of Fibrewise topological spaces theory and some important topics.

Keywords Fibrewise topology; Fibrewise topological spaces; Fibrewise topological mapping spaces; Absolute(nbd) fibrewise retract; Fibrewise category.

MR(2010) Subject Classification 55R70; 54C10; 54E15; 54E35; 54E50

The concept of the Fiberwise space was developed on the basis of the homotopy and homeomorphism theory, accompanied with the research of the Algebraic topology theory in the 20th century. Firstly, the notion of the "Fibrewise Bundles" originated from the research of the geometry and topology in the 1930s. The first general definitions were given by Whitney [33] in 1935, and Hurcwicz [17, 19] made a contribution for the study of the Fibrewise Bundles. The work of Whitney [34], Hopf [20] and Stiefel [32] demonstrated the importance of the subject for the applications of topology to differential geometry. Since then, many papers dealing with bundles appeared, which presented some of the finest applications of topology to other fields and its cheerful prospect, attracting many researchers' general interest. However, the literature is in a state of partial confusion, mainly due to the experimentation with a variety of definitions of "Fibrewise Bundles". The derivations of analogous conclusions from different hypotheses have produced much overlapping.

In 1951, Steenord [31] wrote the academic monographs "Topology of Fibre Bundles" according to the lectures of fibrewise bundles which he gave at the University of Michigan in 1947, and at Princeton University in 1948. This is the first systematic account of this subject. The book is divided into three parts. The first part of this paper introduces the general theory of the

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bundles, which first introduces the concept of the coordinate bundles and fibrewise bundles, then describes the product bundles, differentiable manifold and tensor bundles, homotopy of bundle maps, cross-section and covering spaces, etc. The second part is devoted to the homotopy theory of bundles, which first introduces homotopy groups and the homotopy sequence of bundles, then describes the fibreing of spheres, the homotopy groups of spheres and classification of sphere bundles over spheres, etc. The third part treats the cohomology theory of bundles, which first introduces the obstruction cohomology class defined in the stepwise extension of a cross section, then introduces the cohomology groups basis on cross section, extention and deformation theory, extention of a function and homotopy classes of maps. Finally, it introduces the Stiefel-Whitney characteristic classes of a sphere bundle and differentiable manifold.

This book marks that the fibrewise bundles theory formed a new discipline, as an indispensable part of differentiable geometry, which plays an important role in contemporary physics.

Then, Serre in the literature [30] defined the covering homotopy property (CHP), then defined the fibrewise space and also discussed some properties of the fibre space. In 1955, Hurewise [18] suitably amended the fibre space given by Serre, gave the generalization definition of the fibre space, and studied some properties that fibrewise space has only in this case. In 1956, Milinor [23, 24] gave the universal fibrewise bundles structure for any topological groups. Since then, Whithery also did a lot of work about the theory of differentiable manifold and sphere bundles theory.

Remarkably, Whyburn [35] used the view of fibrewise to discuss the compactness of maps, and Cain [11] also got some achievement about the compactness of fibrewise maps.

In 1966, Husemoller [21] published the book titled "Fibre Bundles", which summarizes the research achievements about fibrewise bundles generally. The first part of this book introduces the general theory of fibrewise bundles. Firstly, it introduces the basic definition of bundles, then mainly describes the vector bundles, when it gave the definition and examples of the vector bundles, he described the homotopy properties of vector bundles, the construction and the homotopies of Gauss maps, also gave an account of Riemannian-Hermitian metrics of the vector bundles. Secondly, it introduces some properties of fibrewise bundles, especially the construction of fibre bundles by Milnor. The second part describes the basic content of K-theory, which firstly introduces the stability properties of vector bundles, then recounts the relative K-theory, the Adams operations and Adams representations, finally, recounts the application of K-theory in vector field on the sphere. The third part describes the characteristic classes, which are about some result of differential manifold, and the general theory of characteristic classes. This book was reprinted many times, and it also played an active role in promoting the study of fibrewise bundles.

In 1978, in the literature [4, 5], Booth and Brown gave the perfect definition of fibreweise mapping space with fibrewise compact-open topology, and discussed some properties of fibrewise mapping spaces in that case. In 1985, Lewis further promoted the two authors' results, and expanded some conclusions of fibrewise mapping spaces to the fibrewise category.

In 1982, Niefield [27] discussed the fibrewise uniform space in some limiting condition, he restricted that the base space and fibrewise space are uniform, and the projection is uniformly continuous. In 1985, James [16] improved the result of Niefield, he weakened the condition that the base space is uniform to the usual topological space, and the fibrewise space is uniform space, and then had some new results.

The fibrewise space and fibrewise bundles have become the powerful tools of the homotopy group in calculating various spaces which is closely related with homotopy theory, so an active branch of mathematics was formed. Up to now, it is not only connected with algebraic topology, but also has been significantly applied to the theory of differential geometry, lie group, and dynamical system theory, etc.

In 1984, Russia scholar Pasynkov presented some thought about fibrewise topological space in literature [29], and the study of fibrewise topological space began since then. From then on, James as the representative of some researchers began to study the fibrewise topological space in the generalization process of fibrewise bundles.

In 1989, James [15] wrote the book of "Fibrewise Topology" by summarizing the achievements of the past. The book is divided into five chapters, which introduces the detailed contents of the fibrewise topological theory systematically. The first chapter introduces the basic fibrewise topology. Firstly, it gives the notion of fibrewise topology space and the concept of fibrewise mapping between fibrewise topological spacs, then it describes the fibrewise separation conditions, fibrewise compact space, and fibrewise pointed topological space. The second chapter introduces the further fibrewise topology, such as fibrewise compactification, fibrewise mapping-space, and the definition and properties of fibrewise compactly-generated spaces. The third chapter introduces the fibrewise uniform spaces. It firstly narrates fibrewise uniform structures and fibrewise uniform topology, then it describes the Cauchy condition and fibrewise completion. Finally, it gives the concept of fibrewise compactness and precompactness and some corresponding results. The fourth chapter discusses the fibrewise homotopy theory. It firstly narrates fibrewise (pointed) homotopy and fibrewise (pointed) cofibrations, then it describes fibrewise non-degenerate spaces, and fibrewise fibrations. Finally, it gives the relation with equivariant homotopy theory. The fifth chapter provides some miscellaneous topics, such as fibrewise bundles, countable coverings, and fibrewise connectedness.

This book played a very important role in the development of the theory about fibrewise topology spaces, so the following research of this theory was all on the basis of the view about this book. In 1998, James and Crabb [13] wrote the "Fibrewise Homotopy Theory", which further enriched and developed the fibrewise topology space theory.

However, the research of fibrewise topology theory in the view of general topology was sponsored by Russia Pasynkov, Buhagiar and Miwa etc. in 1997 (see [6–9]). They passed many profound results of general topology theory to fibrewise topological space theory by supplementing and perfecting the method of James.

In order to study the metric property of fibrewise mappings, Pasynkov, Miwa, Nordo, Cammaroto and Buhagiar [10, 12, 28] defined TM-maps and MT-maps, and also proved that the metrization theorem on fibrewise maps. In literature [9], Miwa and Buhagiar defined and studied subparacompact-maps and (sub)metacompact maps, which are the fibrewise topological analogues of subparacompact-spaces and (sub)metacompact spaces, respectively. Furthermore, several characterizations and properties of these maps are proved. In literature [1–3], Bai and Miwa defined $\check{C}ech$-complete maps, p-maps, and M-maps, studied the relations with other maps, and discussed the spread of complete compact spaces in fibrewise topology spaces.

Konami and Miwa [22] used the notion of cover to guide the concept of fibrewise covering uniform spaces and its general spaces (fibrewise generalized uniform spaces and fibrewise half-uniform spaces), and they also studied the completeness of fibrewise generalized uniform spaces and fibrewise half-uniform spaces.

Theory of retracts is an important part of general topology, which played a pivotal role in the research of dimension theory, function space theory, shape theory, infinitely dimensional topology, CW-complex theory, algebraic topology, and generalized metric spaces, etc. Crabb, James [13] studied some properties of the absolute neighborhood fibrewise retract ($ANFR$) of fibrewise metric spaces. Miwa [25] further discussed the properties of fibrewise retraction and extension. After that, he generalized the above results to the absolute neighborhood fibrewise retract ($ANFR_B(S_B)$) of fibrewise class $S_B = \{(X, f)|f : X \to B, B \in S\}$, for a space $B \in S$. He also studied the absolute neighborhood fibrewise extension ($ANFE_B(S_B)$). In 2006, Guo and Miwa [14] further studied the absolute neighborhood fibrewise retract ($ANFR$) of fibrewise mapping-spaces, and proved that some fibrewise mapping-spaces $map_B(X, Y)$, for $B \in S$ are stratifible spaces. Further, Y is in $ANFR_B(S_B)$, then $map_B(X, Y)$ is also in $ANFR_B(S_B)$.

At present, one of the main contents of the study of fibrewise topology space theory (TOP_B) is how to extend the main concept and result of general topological space (TOP) to fibrewise topology space, with sufficed $TOP_B = TOP$, when base space $B = \{*\}$ (B is the single-point space). The above content mainly focused on the further study of various map's fibrewise covering property, fibrewise compactification, and absolute neighborhood fibrewise retract. Secondly, in the view of category theory, TOP_B is higher category than TOP, so the research of some new properties and characteristics in various fibrewise topology space is worth discussing. Especially the research about fibrewise category of fibrewise mapping spaces is paid close attention by some scholars.

References

[1] Yunfeng BAI, T. MIWA. *Čech-complete maps*. Glas. Mat. Ser. III 43(63), 2008, **1**: 219–229.

[2] Yunfeng BAI, T. MIWA. *On p-maps and M-maps*. Tsukuba J. Math., 2008, **32**(1): 179–196.

[3] Yunfeng BAI, T. MIWA. *Uniformities and completeness on metrizable maps*. Nanjing Daxue Xuebao Shuxue Bannian Kan, 2009, **26**(1): 1–13.

[4] P. I. BOOTH, R. BROWN. *On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps*. General Topology and Appl., 1978, **8**(2): 165–179.

[5] P. I. BOOTH, R. BROWN. *Spaces of partial maps, fibred mapping spaces and the compact-open topology*. General Topology and Appl., 1978, **8**(2): 181–195.

[6] D. BUHAGIAR. *Paracompact maps*. Questions Answers Gen. Topology, 1997, **15**(2): 203–223.

[7] D. BUHAGIAR. *A category of continuous maps*. RIMS Kokyuroku, 1999, **1107**: 70–83.

[8] D. BUHAGIAR. *The category MAP*. Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci., 2001, **34**: 1–19.

[9] D. BUHAGIAR, T. MIWA. *Covering properties on maps.* Questions Answers Gen. Topology, 1998, **16**(1): 53–66.

[10] D. BUHAGIAR, T. MIWA, B. A. PASYNKOV. *On metrizable type (MT-) maps and spaces.* Topology Appl., 1999, **96**(1): 31–51.

[11] G. L. CAIN. *Compactification of mappings.* Proc. Amer. Math. Soc., 1969, **23**: 298–303.

[12] F. CAMMAROTO, B. A. PASYNKOV. *Some metrization theorems for continuous mappings.* Questions Answers Gen. Topology, 2002, **20**(1): 13–32.

[13] M. CRABB, I. M. JAMES. *Fibrewise Homotopy Theory.* Springer-Verlag London, Ltd., London, 1998.

[14] Baolin GUO, T. MIWA. *Fibrewise ANR of fibrewise mapping-spaces.* Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci., 2006, **39**: 15–23.

[15] I. M. JAMES. *Fibrewise Topology.* Cambridge Univ. Press., Cambridge, 1989.

[16] I. M. JAMES. *Uniform spaces over a base.* J. London Math. Soc. (2), 1985, **32**(2): 328–336.

[17] W. HUREWICZ. *Beitrage Topologie der Deformationen (I-IV).* Pro. Akad. Amsterdam, 1935, **38**: 112–119, 521–528; 1936, **39**: 117–126, 215–224.

[18] W. HUREWICZ. *On the concept of fiber space.* Proc. Nat. Acad. Sci. U. S. A., 1955, **41**: 956–961.

[19] W. HUREWICZ, N. STEENROD. *Homotopy relations in fibre spaces.* Proc. Nat. Acad. Sci. U. S. A., 1941, **27**: 60–64.

[20] H. HOPF, M. RUEFF, *Über faserungstreue Abbildungen der Sphären.* Comm. Math. Helv., 1938, **11**: 49–61. (in German)

[21] D. HUSEMOLLER. *Fibre Bundles.* Springer-Verlag, New York, 1966.

[22] Y. KONAMI, T. MIWA. *Fibrewise covering uniformities and completions.* Acta Math. Hungar., 2008, **119**(1-2): 127–157.

[23] J. MILNOR. *Construction of universal bundles (I).* Ann. of Math. (2), 1956, **63**: 272–284.

[24] J. MILNOR. *Construction of universal bundles (II).* Ann. of Math. (2), 1956, **63**: 430–436.

[25] T. MIWA. *On fibrewise retraction and extension.* Houston J.Math., 2000, **26**: 811–831.

[26] J. MILNOR. *Fibrewise ANR in stratiable maps.* Houston J.Math., 2003, **29**: 1013–1025.

[27] S. B. NIEFIELD. *Cartesianness: topological spaces, uniform spaces, and affine schemes.* J. Pure Appl. Algebra, 1982, **23**(2): 147–167.

[28] G. NORDO, B. A. PASYNKOV. *On trivially metrizable mappings.* Q & A in Gen. Top., 2000, **15**: 203–223.

[29] B. A. PASYNKOV. *Extension to Mappings of Certain Notions and Assertions Concerning Spaces.* Moskov. Gos. Univ., Moscow, 1984.

[30] J. P. SERRE. *Homologie singulière des espaces fibrés.* Ann. of Math. (2), 1951, **54**: 425–505. (in French)

[31] N. STEENROD. *Topology of Fibre Bundles.* Princeton University Press, Princeton, N. J., 1951.

[32] STIEFEL E. *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten.* Comment. Math. Helv., 1935, **8**(1): 305–353. (in German)

[33] H. WHITNEY. *Sphere spaces.* Proc. Nat. Acad. Sci., 1935, **21**: 462–468.

[34] H. WHITNEY. *Topological properties of differentiable manifolds.* Bull. Amer. Math. Soc., 1937, **43**(12): 785–805.

[35] G. T. WHYBURN. *Compactification of mappings.* Math. Ann., 1966, **166**: 168–174.