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Congruences on Zappa-Szép Products of Semilattices with An Identity and Groups

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Abstract Let $P = E \bowtie G$ be a Zappa-Szép product of a semilattice E with an identity and a group G. In this paper, we first introduce the concept of congruence pairs for P, and then prove that every congruence on P can be described by such a congruence pair. In fact the congruence lattice on P is lattice-isomorphic to the set of all congruence pairs for P. Finally, we characterize group congruences on P.

Keywords Zappa-Szép product; congruence; congruence pairs.

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The semidirect product of two groups generalizes the direct product of two groups in that only one of the factors is assumed to be normal. The Zappa-Szép product of two groups is a natural generalization of the semidirect product of two groups in that neither factor is required to be normal. The Zappa-Szép product of two semigroups can also be considered as a natural generalization of the Zappa-Szép product of two groups.

Zappa-Szép product arises when an algebraic structure has the property that every element has a unique decomposition as a product of elements from two given substructures. The Zappa-Szép product was developed in [8] and used to discover properties of groups by Rédei, Szép and Tibiletti. Zappa-Szép product of semigroups also appeared in the work of Coleman and Easdown [7] on the structure of a ring R under the binary operation $a \circ b = a + b - ab$. They may also be constructed from actions of two structures on one another, satisfying axioms first formulated by Zappa [8], and have a natural interpretation within automata theory [3, Section 2].

Let E be a semilattice and G a group. Then the semidirect product $E \rtimes G$ is an E-unitary inverse semigroup which is isomorphic to a P-semigroup $P(Y, G; X)$. Using the kernel normal system, Jones in [4] obtained a way of constructing the congruences on a P-semigroup in terms of subsemilattices of Y and subgroups of G. Petrich [6] gave a construction of congruences on a P-semigroup by a congruence on Y and subgroups of G, in a different notation, due to Jones [4]. Notice that any semidirect product $E \rtimes G$ is a Zappa-Szép product $E \bowtie G$ which need not to be an inverse semigroup. It is therefore of interest to look at the description of congruences on a Zappa-Szép product $E \bowtie G$.

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In this paper our aim is to describe congruences and congruence lattice on $P = E \bowtie G$ a Zappa-Szép product of a semilattice E with an identity and a group G. We recall the definition and basic properties of Zappa-Szép products in Section 1. After introducing the concept of congruence pairs for P in Section 2, we prove that every congruence on P can induce a congruence pair. Last section shows that every congruence on P can be constructed by a congruence pair, and the congruence lattice on P is lattice-isomorphic to the set of all congruence pairs for P. Finally, we characterize group congruences on P.

Readers can be referred to [2] for the undefined notion and notations about semigroups in this paper.

1. Definition and basic properties

In this section, in order to give an expression of semigroup theoretic aspects of the Zappa-Szép products of semilattices and groups, we record the definition and basic properties from [3].

Definition 1.1 ([3, Section 2]) *Let A and S be semigroups, and suppose that we are given functions $S \times A \to A, (s,a) \mapsto s \cdot a$ and $S \times A \to S, (s,a) \mapsto s^a$ satisfying the following axioms for all $s,t \in S$ and $a,b \in A$:*

$ZS1$: $s \cdot (t \cdot a) = st \cdot a$;

$ZS2$: $s \cdot (ab) = (s \cdot a)(s^a \cdot b)$;

$ZS3$: $s^{ab} = (s^a)^b$;

$ZS4$: $(st)^a = s^{t \cdot a} t^a$.

Then it is easy to check that the set $A \times S$ endowed with the product $(a,s)(b,t) = (a(s \cdot b), s^b t)$ is a semigroup, the Zappa-Szép product of A and S, which we denote by $A \bowtie S$.

Example 1.1 ([3, Section 2]) If the action of A on S is trivial, we obtain the familiar semidirect product $A \rtimes S$.

Example 1.2 ([3, Section 2]) Let S be an inverse semigroup with semilattice of idempotents $E(S)$. Then we can form the Zappa-Szép product $E(S) \bowtie S$ using the actions

$$s \cdot \alpha = s\alpha s^{-1} \text{ and } s^\alpha = s\alpha \text{ for all } \alpha \in E(S), \quad s \in S.$$

The product in $E(S) \bowtie S$ is then given by

$$(\alpha, s)(\beta, t) = (\alpha s \beta s^{-1}, s\beta t).$$

Let $P = E \bowtie G$ be a Zappa-Szép product of a band E and a group G. We assume the additional two axioms:

$ZS5$: for the identity element $1 \in G$ we have $1 \cdot \alpha = \alpha$;

$ZS6$: if E has an identity $1 \in E$, then $g \cdot 1 = 1$ and $g^1 = g$ for all $g \in G$.

Proposition 1.1 ([3, Lemma 3.1]) *Let E be a band and let G be a group, action on each other so that $[ZS1], \ldots, [ZS5]$ hold. Then for all $g \in G$ and $\alpha, \beta \in E$ with $\alpha \leq \beta$ we have:*

(1) $1^\alpha = 1$;

(2) $(g^{\alpha})^{-1} = (g^{-1})^{g\cdot\alpha}$;

(3) $g \cdot \alpha = (g \cdot \beta)(g^{\beta} \cdot \alpha)$;

(4) $g^{\beta} \cdot \alpha = (g^{\beta} \cdot \alpha)(g \cdot \alpha)$;

(5) *if* $g^{\beta} = 1$, *then* $\alpha = \alpha(g \cdot \alpha)$.

If E is a semilattice, then in addition, we have the following:

(6) *the action of G on E is order-preserving;*

(7) $g^{\beta} \cdot \alpha = g \cdot \alpha$;

(8) *if* $g^{\beta} = 1$, *then* $g \cdot \alpha = \alpha$;

(9) *if* $g^{\beta} = 1$, *then* $\beta(g \cdot \gamma) = \beta\gamma$ *for all* $\gamma \in E$.

Proposition 1.2 ([3, Proposition 3.2]) *A Zappa-Szép product $P = E \bowtie G$ of a band E and a group G is a regular semigroup. Moreover:*

(a) *the subset of idempotents is* $E(P) = \{(\alpha, g) : g^{\alpha} = 1\}$;

(b) *if E has an identity $1 \in E$, then P is unit-regular;*

(c) *if E is a semilattice, then the set of inverses of (α, g) is*

$$V(\alpha, g) = \{(g^{-1} \cdot \alpha, x) \in P : x^{\alpha} = (g^{-1})^{\alpha}\},$$

and furthermore P is orthodox and $\mathcal{L}$-unipotent.

2. Congruence pairs

Hereafter, let $P = E \bowtie G$ be a Zappa-Szép product of a semilattice E with an identity and a group G. The object in this section is to introduce the concept of congruence pairs for P. We begin our study of congruences on P by the consideration of a congruence on E and a family of subgroups of G and their mutual relationship. First we need the following lemmas.

Lemma 2.1 *Let ρ be a congruence on P. If $(\alpha, g)\rho(\beta, h)$, then $(\alpha, 1)\rho(\beta, 1)$. Moreover, $(\alpha, g)\rho(\alpha, h)$ and $(\beta, g)\rho(\beta, h)$.*

Proof Assume that $(\alpha, g)\rho(\beta, h)$. Multiplying on the right by $(1, g^{-1})$, we obtain

$$(\alpha, g)(1, g^{-1})\rho(\beta, h)(1, g^{-1}).$$

Since $g \cdot 1 = 1$, $h \cdot 1 = 1$, $g^1 = g$ and $h^1 = h$, we have

$$\begin{aligned}(\alpha, 1) &= (\alpha(g \cdot 1), g^1 g^{-1}) = (\alpha, g)(1, g^{-1})\rho(\beta, h)(1, g^{-1}) \\ &= (\beta(h \cdot 1), h^1 g^{-1}) = (\beta, hg^{-1})\end{aligned}$$

which yields $(\alpha, 1)\rho(\beta, hg^{-1})$. Symmetrically, $(\beta, 1)\rho(\alpha, gh^{-1})$. Then,

$$(\alpha\beta, 1) = (\beta\alpha, 1) = (\beta, 1)(\alpha, 1)\rho(\beta, 1)(\beta, hg^{-1}) = (\beta, hg^{-1})\rho(\alpha, 1)$$

and

$$(\alpha\beta, 1) = (\alpha, 1)(\beta, 1)\rho(\alpha, 1)(\alpha, gh^{-1}) = (\alpha, gh^{-1})\rho(\beta, 1).$$

This implies $(\alpha, 1)\rho(\beta, 1)$, as required.

Furthermore,

$$
\begin{aligned}
(\alpha, g)\rho(\beta, h) &= (\beta, 1)(\beta, h)\rho(\alpha, 1)(\beta, h) = (\alpha\beta, h) = (\beta\alpha, h)\\
&= (\beta, 1)(\alpha, h)\rho(\alpha, 1)(\alpha, h) = (\alpha, h)
\end{aligned}
$$

yielding $(\alpha, g)\rho(\alpha, h)$. Symmetrically, $(\beta, g)\rho(\beta, h)$. $\square$

Let ρ be a congruence on P and $E_P = \{(\alpha, 1) : \alpha \in E\}$. By Proposition 1.1 (1) and Proposition 1.2 (a), we have $E_P \subseteq E(P)$. For any $(\alpha, 1), (\beta, 1) \in E_P$, we have $(\alpha, 1)(\beta, 1) = (\alpha\beta, 1) \in E_P$. In fact, it is easy to see that E_P is a semilattice with an identity. We now define a relation τ^ρ on E by the rule

$$\alpha\tau^\rho\beta \Leftrightarrow (\alpha, 1)\rho(\beta, 1) \text{ for } \alpha, \beta \in E.$$

Since the mapping $(\alpha, 1) \to \alpha$ is an isomorphism of E_P onto E, the next lemma is immediate.

Lemma 2.2 *τ^ρ is a congruence on E.*

By the approach from the Theorem VII.2.1 in [6], for every $(\alpha, 1) \in E_P$, we define $H_{\alpha\tau^\rho}$ to be the projection of $(\alpha, 1)\rho$ in G, explicitly

$$H_{\alpha\tau^\rho} = \{g \in G : (\beta, g)\rho(\alpha, 1) \text{ for some } \beta \in E\}.$$

Lemma 2.3 *$H_{\alpha\tau^\rho}$ is a subgroup of G.*

Proof Since $(\alpha, 1)\rho(\alpha, 1)$, by the definition of $H_{\alpha\tau^\rho}$, we have $1 \in H_{\alpha\tau^\rho}$ and so $H_{\alpha\tau^\rho} \neq \emptyset$. Let $g, h \in H_{\alpha\tau^\rho}$. Then

$$
\begin{aligned}
g, h \in H_{\alpha\tau^\rho} &\Rightarrow (\beta, g)\rho(\alpha, 1) \text{ and } (\gamma, h)\rho(\alpha, 1)\\
&\Rightarrow (\alpha, g)\rho(\alpha, 1) \text{ and } (\alpha, h)\rho(\alpha, 1) \quad \text{(by Lemma 2.1)}\\
&\Rightarrow (\alpha, g)\rho(\alpha, h) \Rightarrow (\alpha, g)(1, h^{-1})\rho(\alpha, h)(1, h^{-1})\\
&\Rightarrow (\alpha, gh^{-1})\rho(\alpha, 1) \Rightarrow gh^{-1} \in H_{\alpha\tau^\rho}.
\end{aligned}
$$

This implies that $H_{\alpha\tau^\rho}$ is a subgroup of G, as required. $\square$

As a consequence from the proof of Lemma 2.3, we have

Corollary 2.1 *$gh^{-1} \in H_{\alpha\tau^\rho}$ if and only if $(\alpha, g)\rho(\alpha, h)$.*

Proof From the proof of Lemma 2.3, we have $(\alpha, g)\rho(\alpha, h)$ implies $gh^{-1} \in H_{\alpha\tau^\rho}$. Conversely, we have

$$
\begin{aligned}
gh^{-1} \in H_{\alpha\tau^\rho} &\Rightarrow (\exists\beta \in E)(\beta, gh^{-1})\rho(\alpha, 1)\\
&\Rightarrow (\alpha, gh^{-1})\rho(\alpha, 1) \quad \text{(by Lemma 2.1)}\\
&\Rightarrow (\alpha, gh^{-1})(1, h)\rho(\alpha, 1)(1, h) \Rightarrow (\alpha, g)\rho(\alpha, h). \quad \square
\end{aligned}
$$

For the rest of this section, we now study the mutual relationship between τ^ρ and $\{H_{\alpha\tau^\rho}\}_{\alpha\in E}$. Let P, ρ, τ^ρ and $\{H_{\alpha\tau^\rho}\}_{\alpha\in E}$ be stated as above.

Lemma 2.4 *If $\alpha\tau^\rho\beta$, then $(t \cdot \alpha)\tau^\rho(t \cdot \beta)$ and $t^\alpha(t^\beta)^{-1} \in H_{(t\cdot\alpha)\tau^\rho}$ for any $t \in G$.*

Proof First, we have

$$\begin{aligned}\alpha\tau^\rho\beta &\Rightarrow (\alpha,1)\rho(\beta,1) \Rightarrow (t\cdot\alpha,t)(\alpha,1)\rho(t\cdot\alpha,t)(\beta,1)\\ &\Rightarrow (t\cdot\alpha,t^\alpha)\rho((t\cdot\alpha)(t\cdot\beta),t^\beta) \qquad (*)\\ &\Rightarrow (t\cdot\alpha,1)\rho((t\cdot\alpha)(t\cdot\beta),1) \ \ \text{(by Lemma 2.1)}\\ &\Rightarrow (t\cdot\alpha)\tau^\rho(t\cdot\alpha)(t\cdot\beta).\end{aligned}$$

Symmetrically, $(t\cdot\beta)\tau^\rho(t\cdot\alpha)(t\cdot\beta)$. It follows that $(t\cdot\alpha)\tau^\rho(t\cdot\beta)$.

Next,

$$\begin{aligned}\alpha\tau^\rho\beta &\Rightarrow (t\cdot\alpha,t^\alpha)\rho((t\cdot\alpha)(t\cdot\beta),t^\beta) \ \ \text{(by (*))}\\ &\Rightarrow (t\cdot\alpha,t^\alpha)\rho(t\cdot\alpha,t^\beta) \ \ \text{(by Lemma 2.1)}\\ &\Rightarrow t^\alpha(t^\beta)^{-1}\in H_{(t\cdot\alpha)\tau^\rho} \ \ \text{(by Corollary 2.1).} \ \ \square\end{aligned}$$

Lemma 2.5 *If $g\in H_{\alpha\tau^\rho}$, then $\alpha(g\cdot\nu)\tau^\rho\alpha\nu$ and $g^\nu\in H_{(\alpha\upsilon)\tau^\rho}$ for any $\nu\in E$.*

Proof First, we have

$$\begin{aligned}g\in H_{\alpha\tau^\rho} &\Rightarrow (\alpha,g)\rho(\alpha,1) \ \ \text{(by Corollary 2.1)}\\ &\Rightarrow (\alpha,g)(\nu,1)\rho(\alpha,1)(\nu,1)\Rightarrow(\alpha(g\cdot\nu),g^\nu)\rho(\alpha\nu,1) \qquad (**)\\ &\Rightarrow (\alpha(g\cdot\nu),1)\rho(\alpha\nu,1) \ \ \text{(by Lemma 2.1)}\\ &\Rightarrow \alpha(g\cdot\nu)\tau^\rho\alpha\nu.\end{aligned}$$

Next,

$$\begin{aligned}g\in H_{\alpha\tau^\rho} &\Rightarrow (\alpha(g\cdot\nu),g^\nu)\rho(\alpha\nu,1) \ \ \text{(by (**))}\\ &\Rightarrow (\alpha\nu,g^\nu)\rho(\alpha\nu,1) \ \ \text{(by Lemma 2.1)}\\ &\Rightarrow g^\nu\in H_{(\alpha\nu)\tau^\rho} \ \ \text{(by Corollary 2.1).} \ \ \square\end{aligned}$$

Lemma 2.6 *If $\alpha\le\beta$, then $t^\alpha H_{\beta\tau^\rho}(t^\alpha)^{-1}\subseteq H_{(t\cdot\alpha)\tau^\rho}$ for any $t\in G$.*

Proof To obtain that $t^\alpha H_{\beta\tau^\rho}(t^\alpha)^{-1}\subseteq H_{(t\cdot\alpha)\tau^\rho}$, we need only to prove that $t^\alpha g(t^\alpha)^{-1}\in H_{(t\cdot\alpha)\tau^\rho}$ for any $t\in G$, $g\in H_{\beta\tau^\rho}$.

Let $g\in H_{\beta\tau^\rho}$. By Corollary 2.1, we have $(\beta,g)\rho(\beta,1)$. Since $\alpha\le\beta$, we have $\alpha\beta=\alpha$. Then

$$(\alpha,g)=(\alpha\beta,g)=(\alpha,1)(\beta,g)\rho(\alpha,1)(\beta,1)=(\alpha\beta,1)=(\alpha,1).$$

It follows that

$$\begin{aligned}(t\cdot\alpha,t)(\alpha,g)\rho(t\cdot\alpha,t)(\alpha,1) &\Rightarrow (t\cdot\alpha,t^\alpha g)\rho(t\cdot\alpha,t^\alpha)\\ &\Rightarrow t^\alpha g(t^\alpha)^{-1}\in H_{(t\cdot\alpha)\tau^\rho} \ \ \text{(by Corollary 2.1).} \ \ \square\end{aligned}$$

Now we introduce the concept of congruence pairs for P as follows.

Definition 2.1 *A pair $(\tau,\{H_{\alpha\tau}\}_{\alpha\in E})$ is a congruence pair for P if*

(i) τ is a congruence on E;

(ii) $\{H_{\alpha\tau}\}_{\alpha\in E}$ *is a family of subgroups on* G*;*

(iii) τ *and* $\{H_{\alpha\tau}\}_{\alpha\in E}$ *satisfy:*

(a) *If* $\alpha\tau\beta$*, then* $(t\cdot\alpha)\tau(t\cdot\beta)$ *and* $t^{\alpha}(t^{\beta})^{-1}\in H_{(t\cdot\alpha)\tau}$ *for any* $t\in G$*;*

(b) *If* $g\in H_{\alpha\tau}$*, then* $\alpha(g\cdot\nu)\tau\alpha\nu$ *and* $g^{\nu}\in H_{(\alpha\nu)\tau}$ *for any* $\nu\in E$*;*

(c) *If* $\alpha\leq\beta$*, then* $t^{\alpha}H_{\beta\tau}(t^{\alpha})^{-1}\subseteq H_{(t\cdot\alpha)\tau}$ *for any* $t\in G$*.*

The following result is immediate from the (c) of Definition 2.1 (iii).

Lemma 2.7 *Let* $(\tau,\{H_{\alpha\tau}\}_{\alpha\in E})$ *be a congruence pair for* P*. If* $\alpha\leq\beta$*, then* $H_{\beta\tau}\subseteq H_{\alpha\tau}$*.*

Proposition 2.1 *Let* $P=E\bowtie G$*, the Zappa-Szép product of a semilattice* E *with an identity and a group* G*. If* ρ *is a congruence on* P*, then* $(\tau^{\rho},\{H_{\upsilon\tau^{\rho}}\}_{\upsilon\in E})$ *is a congruence pair.*

Proof It follows from Lemmas 2.1-2.6. □

3. Congruences and congruence lattice

To describe congruences on P, the set of all congruences on P is denoted by $\mathrm{Con}(P)$. In this section, we are finally ready to establish the relation between congruence lattice $\mathrm{Con}(P)$ and the set of all congruence pairs for P.

Theorem 3.1 *Let* $P=E\bowtie G$*, the Zappa-Szép product of a semilattice* E *with an identity and a group* G*. If* ρ *is a congruence on* P*, then* $(\tau^{\rho},\{H_{\upsilon\tau^{\rho}}\}_{\upsilon\in E})$ *is a congruence pair. Conversely, if* $(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})$ *is a congruence pair for* P*, a relation* $\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}$ *defined by*

$$(\alpha,g)\ \rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}\ (\beta,h)\Leftrightarrow\alpha\tau\beta,\quad gh^{-1}\in H_{\alpha\tau},$$

is a congruence on P*. Moreover,*

$$\tau^{\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}}=\tau, H_{\alpha\tau^{\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}}}=H_{\alpha\tau},\quad \rho_{(\tau^{\rho},\{H_{\upsilon\tau^{\rho}}\}_{\upsilon\in E})}=\rho.$$

Proof By Proposition 2.1, we have already established the first statement in this theorem. Thus, we suppose now that $(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})$ is a congruence pair, and let $\rho=\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}$ be as defined. It is immediate that ρ is reflexive and symmetric. In order to verify that ρ is transitive, let $(\alpha,g)\rho(\beta,h)$ and $(\beta,h)\rho(\upsilon,k)$ for $(\alpha,g),(\beta,h),(\upsilon,k)\in P$. By the definition of $\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}$, we have

$$(\alpha,g)\rho(\beta,h)\Rightarrow\alpha\tau\beta \text{ and } gh^{-1}\in H_{\alpha\tau},\quad (\beta,h)\rho(\upsilon,k)\Rightarrow\beta\tau\upsilon \text{ and } hk^{-1}\in H_{\beta\tau}.$$

This implies that $\alpha\tau\beta\tau\upsilon$, and so $\alpha\tau\upsilon$. By $\alpha\tau\beta$, we have $H_{\alpha\tau}=H_{\beta\tau}$ and so $gk^{-1}=(gh^{-1})(hk^{-1})\in H_{\alpha\tau}$. It follows that $(\alpha,g)\rho(\upsilon,k)$. Therefore ρ is an equivalence relation.

To show that ρ is a congruence, we prove that ρ is a right congruence and a left congruence. Let $(\alpha,g)\rho(\beta,h)$ and $(\upsilon,k)\in P$. Then, $\alpha\tau\beta$ and $gh^{-1}\in H_{\alpha\tau}$. Further,

$$\begin{aligned} gh^{-1}\in H_{\alpha\tau} &\Rightarrow \alpha(gh^{-1}\cdot(h\cdot\upsilon))\tau\alpha(h\cdot\upsilon)\ \ \text{(by (2) of Definition 2.1 (iii))}\\ &\Rightarrow \alpha(g\cdot\upsilon)\tau\alpha(h\cdot\upsilon)\Rightarrow\alpha(g\cdot\upsilon)\tau\beta(h\cdot\upsilon)\ \ \text{(by } \alpha\tau\beta) \qquad (***)\end{aligned}$$

and

$$\begin{aligned} gh^{-1} \in H_{\alpha\tau} &\Rightarrow (gh^{-1})^{h\cdot\upsilon} \in H_{\alpha(h\cdot\upsilon)\tau} \quad \text{(by (2) of Definition 2.1 (iii))} \\ &\Rightarrow g^{h^{-1}\cdot(h\cdot\upsilon)}(h^{-1})^{h\cdot\upsilon} \in H_{\alpha(h\cdot\upsilon)\tau} \\ &\Rightarrow g^{\upsilon}(h^{\upsilon})^{-1} \in H_{\alpha(h\cdot\upsilon)\tau} \quad \text{(since } h^{-1}\cdot(h\cdot\upsilon)=\upsilon \text{ and } (h^{-1})^{h\cdot\upsilon}=(h^{\upsilon})^{-1}) \\ &\Rightarrow g^{\upsilon}(h^{\upsilon})^{-1} \in H_{\beta(h\cdot\upsilon)\tau} \quad \text{(since } \alpha\tau\beta \text{ we have } \alpha(h\cdot\upsilon)\tau\beta(h\cdot\upsilon)) \\ &\Rightarrow g^{\upsilon}(h^{\upsilon})^{-1} \in H_{\alpha(g\cdot\upsilon)\tau} \quad \text{(by } (***)). \end{aligned}$$

Furthermore, $(g^{\upsilon}k)(h^{\upsilon}k)^{-1} = (g^{\upsilon}k)(k^{-1}(h^{\upsilon})^{-1}) = g^{\upsilon}(h^{\upsilon})^{-1} \in H_{\alpha(g\cdot\upsilon)\tau}$. Therefore,

$$(\alpha,g)(\upsilon,k) = (\alpha(g\cdot\upsilon), g^{\upsilon}k)\rho(\beta(h\cdot\upsilon), h^{\upsilon}k) = (\beta,h)(\upsilon,k).$$

This implies that ρ is a right congruence.

On the other hand,

$$\begin{aligned} \alpha\tau\beta &\Rightarrow (k\cdot\alpha)\tau(k\cdot\beta) \quad \text{(by (1) of Definition 2.1 (iii))} \\ &\Rightarrow \upsilon(k\cdot\alpha)\tau\upsilon(k\cdot\beta) \end{aligned}$$

and

$$\begin{aligned} & gh^{-1} \in H_{\alpha\tau} \\ &\Rightarrow k^{\alpha}gh^{-1}(k^{\alpha})^{-1} \in H_{(k\cdot\alpha)\tau} \quad \text{(since } \alpha \le \alpha \text{ and (3) of Definition 2.1 (iii))} \\ &\Rightarrow k^{\alpha}gh^{-1}(k^{\beta})^{-1} = (k^{\alpha}gh^{-1}(k^{\alpha})^{-1})(k^{\alpha}(k^{\beta})^{-1}) \in H_{(k\cdot\alpha)\tau} \quad \text{(by (1) of Definition 2.1 (iii))} \\ &\Rightarrow k^{\alpha}gh^{-1}(k^{\beta})^{-1} \in H_{\upsilon(k\cdot\alpha)\tau} \quad \text{(since } \upsilon(k\cdot\alpha) \le (k\cdot\alpha) \text{ and Lemma 2.7).} \end{aligned}$$

Therefore,

$$(\upsilon,k)(\alpha,g) = (\upsilon(k\cdot\alpha), k^{\alpha}g)\rho(\upsilon(k\cdot\beta), k^{\beta}h) = (\upsilon,k)(\beta,h).$$

This implies that ρ is a left congruence.

It is clear that $\tau^{\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}} = \tau$.

Next,

$$g \in H_{\alpha\tau} \Leftrightarrow (\alpha,g)\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}(\alpha,1) \Leftrightarrow g \in H_{\alpha\tau^{\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}}}$$

which yields $H_{\alpha\tau^{\rho_{(\tau,\{H_{\upsilon\tau}\}_{\upsilon\in E})}}} = H_{\alpha\tau}$.

Finally,

$$\begin{aligned} (\alpha,g)\rho(\beta,h) &\Leftrightarrow (\alpha,1)\rho(\beta,1) \text{ and } (\alpha,g)\rho(\alpha,h) \Leftrightarrow \alpha\tau^{\rho}\beta \text{ and } gh^{-1} \in H_{\alpha\tau^{\rho}} \\ &\Leftrightarrow (\alpha,g)\rho_{(\tau^{\rho},\{H_{\upsilon\tau^{\rho}}\}_{\upsilon\in E})}(\beta,h) \end{aligned}$$

which yields $\rho = \rho_{(\tau^{\rho},\{H_{\upsilon\tau^{\rho}}\}_{\upsilon\in E})}$. $\square$

Theorem 3.2 *Let* $\mathrm{Con}(P)$ *be the congruence lattice of all congruences on* P. *Let* Ω_P *be the poset of all congruences pairs for* P *with the partial order given by*

$$(\tau,\{H_{\alpha\tau}\}_{\alpha\in E}) \le (\tau',\{H_{\alpha\tau'}\}_{\alpha\in E}) \Leftrightarrow \tau \subseteq \tau', H_{\alpha\tau} \subseteq H_{\alpha\tau'} \text{ for each } \alpha \in E.$$

Then the mapping

$$\theta : \mathrm{Con}(P) \longrightarrow \Omega_P, \quad \rho \longrightarrow (\tau^{\rho}, \{H_{\alpha\tau^{\rho}}\}_{\alpha\in E})$$

is a lattice isomorphism from $\mathrm{Con}(P)$ *onto* Ω_P.

Proof According to Theorem 3.1, we know that θ is bijective. Since

$$\rho \leq \sigma \Leftrightarrow \tau^\rho \subseteq \tau^\sigma, H_{\alpha\tau^\rho} \subseteq H_{\alpha\tau^\sigma} \text{ for each } \alpha \in E \Leftrightarrow (\tau^\rho, \{H_{\alpha\tau^\rho}\}_{\alpha\in E}) \leq (\tau^\sigma, \{H_{\alpha\tau^\sigma}\}_{\alpha\in E}),$$

θ is preserving order mapping. Therefore θ is a lattice isomorphism. $\square$

For the rest of this section, we characterize group congruences on P.

Proposition 3.1 *Let* $P = E \bowtie G$, *a Zappa-Szép product of a semilattice* E *with an identity and a group* G. *Let* H *be a normal subgroup of* G *satisfying:*

(1) $t^\alpha(t^\beta)^{-1} \in H$ *for any* $\alpha, \beta \in E, t \in G$;

(2) If $g \in H$, *then* $g^\upsilon \in H$ *for any* $\upsilon \in E$.

Define a relation ρ_H *on* P *by*

$$(\alpha, g)\rho_H(\beta, h) \Leftrightarrow gh^{-1} \in H.$$

Then ρ_H *is a group congruence on* P. *Conversely, every group congruence on* P *can be constructed in this way.*

Proof Direct part. It is easy to verify that ρ_H is an equivalent relation. To show that ρ_H is a congruence, let $(\alpha, g)\rho_H(\beta, h)$ and $(\upsilon, k) \in P$. Then

$$\begin{aligned}(\alpha, g)\rho_H(\beta, h) &\Rightarrow gh^{-1} \in H \Rightarrow (gh^{-1})^{h\cdot\upsilon} \in H \quad \text{(by (2))}\\ &\Rightarrow g^{h^{-1}\cdot(h\cdot\upsilon)}(h^{-1})^{h\cdot\upsilon} \in H\\ &\Rightarrow g^\upsilon(h^\upsilon)^{-1} \in H \quad \text{(by Proposition 1.1 (2))}\\ &\Rightarrow (g^\upsilon k)(h^\upsilon k)^{-1} \in H\end{aligned}$$

and

$$\begin{aligned}(\alpha, g)\rho_H(\beta, h) &\Rightarrow gh^{-1} \in H \quad \text{(by the definition of } \rho_H)\\ &\Rightarrow k^\alpha gh^{-1}(k^\beta)^{-1} = (k^\alpha gh^{-1}(k^\alpha)^{-1})(k^\alpha(k^\beta)^{-1}) \in H\\ &\qquad \text{(since } H \text{ is a normal subgroup and } k^\alpha(k^\beta)^{-1} \in H).\end{aligned}$$

It follows that

$$(\alpha, g)(\upsilon, k) = (\alpha(g\cdot\upsilon), g^\upsilon k)\rho_H(\beta(h\cdot\upsilon), h^\upsilon k) = (\beta, h)(\upsilon, k)$$

and

$$(\upsilon, k)(\alpha, g) = (\upsilon(k\cdot\alpha), k^\alpha g)\rho_H(\upsilon(k\cdot\beta), k^\beta h) = (\upsilon, k)(\beta, h).$$

Therefore, ρ is a congruence.

By Proposition 1.2, P is regular implies that P/ρ_H is regular. It remains to show that P/ρ_H is cancellative. We prove that P/ρ_H is left cancellative and right cancellative. Now

$$\begin{aligned}&(\upsilon, k)\rho_H(\alpha, g)\rho_H = (\upsilon, k)\rho_H(\beta, h)\rho_H\\ &\quad\Rightarrow (\upsilon, k)(\alpha, g)\rho_H(\upsilon, k)(\beta, h) \Rightarrow (\upsilon(k\cdot\alpha), k^\alpha g)\rho_H(\upsilon(k\cdot\beta), k^\beta h)\end{aligned}$$

$$\begin{aligned}
&\Rightarrow k^{\alpha}gh^{-1}(k^{\beta})^{-1}\in H \quad \text{(by the definition of } \rho_H)\\
&\Rightarrow gh^{-1}=[(k^{\alpha})^{-1}(k^{\alpha}gh^{-1}(k^{\beta})^{-1})k^{\alpha}]((k^{\alpha})^{-1}k^{\beta})\in H\\
&\quad \text{(since } H \text{ is a normal subgroup)}\\
&\quad \text{(by (1) we have } (k^{\alpha})^{-1}k^{\beta}=(k^{-1})^{k\cdot\alpha})((k^{-1})^{k\cdot\beta})^{-1}\in H)\\
&\Rightarrow (\alpha,g)\rho_H(\beta,h) \quad \text{(by the definition of } \rho_H)\Rightarrow(\alpha,g)\rho_H=(\beta,h)\rho_H.
\end{aligned}$$

This implies that P/ρ_H is left cancellative.

$$\begin{aligned}
&(\alpha,g)\rho_H(\upsilon,k)\rho_H=(\beta,h)\rho_H(\upsilon,k)\rho_H\Rightarrow(\alpha,g)(\upsilon,k)\rho_H(\beta,h)(\upsilon,k)\\
&\quad\Rightarrow(\alpha,g)(\upsilon,k)(1,k^{-1})\rho_H(\beta,h)(\upsilon,k)(1,k^{-1})\Rightarrow(\alpha,g)(\upsilon,1)\rho_H(\beta,h)(\upsilon,1)\\
&\quad\Rightarrow(\alpha,g)(1,1)\rho_H(\beta,h)(1,1) \quad (\text{since } (\upsilon,1)\rho_H(1,1))\\
&\quad\Rightarrow(\alpha,g)\rho_H(\beta,h)\Rightarrow(\alpha,g)\rho_H=(\beta,h)\rho_H.
\end{aligned}$$

This implies that P/ρ_H is right cancellative. Therefore, ρ_H is a group congruence on P.

Converse part. Let ρ be a group congruence on P. Now we define a subset H of G as follows

$$H=\{g\in G:(\alpha,g)\rho(\beta,1) \text{ for some } \alpha,\beta\in E\}.$$

It is convenient to present the next phases of the arguments as a lemma.

Firstly, notice that if ρ is a group congruence on P, then $(\alpha,g)\rho(\beta,h)$ for all $(\alpha,g),(\beta,h)\in E(P)$.

Lemma 3.1 *Let ρ be a group congruence on P and $g,h\in G$. Then, $(\alpha,g)\rho(\beta,h)$ if and only if $(\gamma,g^{\upsilon})\rho(\delta,h^{\omega})$ for any $\gamma,\delta,\upsilon,\omega\in E$.*

Proof If $(\alpha,g)\rho(\beta,h)$, then

$$\begin{aligned}
(\alpha,g)(\upsilon,1)\rho(\beta,h)(\omega,1)&\Rightarrow(\alpha(g\cdot\upsilon),g^{\upsilon})\rho(\beta(h\cdot\omega),h^{\omega})\Rightarrow(\gamma,1)(\alpha(g\cdot\upsilon),g^{\upsilon})\rho(\delta,1)(\beta(h\cdot\omega),h^{\omega})\\
&\Rightarrow(\gamma\alpha(g\cdot\upsilon),g^{\upsilon})\rho(\delta\beta(h\cdot\omega),h^{\omega})\Rightarrow(\alpha(g\cdot\upsilon)\gamma,g^{\upsilon})\rho(\beta(h\cdot\omega)\delta,h^{\omega})\\
&\Rightarrow(\alpha(g\cdot\upsilon),1)(\gamma,g^{\upsilon})\rho(\beta(h\cdot\omega),1)(\delta,h^{\omega})\Rightarrow(\gamma,g^{\upsilon})\rho(\delta,h^{\omega})\\
&\quad(\text{since}(\alpha(g\cdot\upsilon),1)\rho(\beta(h\cdot\omega),1)) \text{ and } \rho \text{ is a group congruence}).
\end{aligned}$$

Conversely, it is clear. □

Let us now return to the main proof. Since $(\alpha,1)\rho(\alpha,1)$, by the definition of H, we have $1\in H$ and so $H\neq\emptyset$. If $g\in H$ and $h\in H$, then

$$\begin{aligned}
&(\exists\alpha,\beta,\gamma,\delta\in E)(\alpha,g)\rho(\beta,1) \text{ and } (\gamma,h)\rho(\delta,1) \quad \text{(by the definition of } H)\\
&\quad\Rightarrow(\alpha,g)\rho(\gamma,h) \quad (\text{by } (\beta,1)\rho(\delta,1))\\
&\quad\Rightarrow(\alpha,g)\rho(\alpha,h) \quad \text{(by Lemma 3.1)}\\
&\quad\Rightarrow(\alpha,g)(1,h^{-1})\rho(\alpha,h)(1,h^{-1})\Rightarrow(\alpha,gh^{-1})\rho(\alpha,1)\\
&\quad\Rightarrow gh^{-1}\in H \quad \text{(by the definition of } H).
\end{aligned}$$

This implies that H is a subgroup. To show that H is a normal subgroup of G, we need to prove that $tgt^{-1} \in H$ for any $t \in G$, $g \in H$. Let $g \in H$ and $t \in G$. Then

$$\begin{aligned}&(\exists \alpha, \beta \in E)\ (\alpha, g)\rho(\beta, 1)\ \ \text{(by the definition of } H)\\ &\Rightarrow (1, g)\rho(1, 1)\ \ \text{(by Lemma 3.1)}\\ &\Rightarrow (1, t)(1, g)(1, t^{-1})\rho(1, t)(1, 1)(1, t^{-1})\\ &\Rightarrow (1, tgt^{-1})\rho(1, 1) \Rightarrow tgt^{-1} \in H.\end{aligned}$$

In order to establish condition (1), let $\alpha, \beta \in E$. Then

$$\begin{aligned}(1, t)(\alpha, 1)\rho(1, t)(\beta, 1) &\Rightarrow (t \cdot \alpha, t^{\alpha})\rho(t \cdot \beta, t^{\beta})\\ &\Rightarrow (t \cdot \alpha, t^{\alpha})(1, (t^{\beta})^{-1})\rho(t \cdot \beta, t^{\beta})(1, (t^{\beta})^{-1})\\ &\Rightarrow (t \cdot \alpha, t^{\alpha}(t^{\beta})^{-1})\rho(t \cdot \beta, 1) \Rightarrow t^{\alpha}(t^{\beta})^{-1} \in H.\end{aligned}$$

Further, let $g \in H$. Then

$$\begin{aligned}(\exists \alpha, \beta \in E)(\alpha, g)\rho(\beta, 1) &\Rightarrow (\alpha, g^{\upsilon})\rho(\beta, 1)\ \ \text{(by Lemma 3.1)}\\ &\Rightarrow g^{\upsilon} \in H.\end{aligned}$$

We have proved that H satisfies the condition (2).

Finally, we prove that $\rho_H = \rho$. Now

$$\begin{aligned}(\alpha, g)\rho_H(\beta, h) &\Rightarrow gh^{-1} \in H \Rightarrow (\gamma, gh^{-1})\rho(\delta, 1)\\ &\Rightarrow (\gamma, gh^{-1})(1, h)\rho(\delta, 1)(1, h)\\ &\Rightarrow (\gamma, g)\rho(\delta, h) \Rightarrow (\alpha, g)\rho(\beta, h)\end{aligned}$$

which yields $\rho_H \subseteq \rho$.

$$\begin{aligned}(\alpha, g)\rho(\beta, h) &\Rightarrow (\alpha, g)(1, h^{-1})\rho(\beta, h)(1, h^{-1}) \Rightarrow (\alpha, gh^{-1})\rho(\beta, 1)\\ &\Rightarrow gh^{-1} \in H \Rightarrow (\alpha, g)\rho_H(\beta, h)\end{aligned}$$

which yields $\rho \subseteq \rho_H$. $\square$

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