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# Hyper Order of Solutions of Higher Order L. D. E. with Coefficients Being Lacunary Series

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**Abstract** In this paper, we investigate the hyper order of the solutions of high order linear differential equations with some dominated coefficient being lacunary series, and obtain some results which improve and extend previous results.

**Keywords** linear differential equations; hyper order; lacunary series.

**MR(2010) Subject Classification** 30D35

## 1. Introduction and results

In this paper, we shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions [8, 13]. In addition, we use $\sigma(f), \lambda(f)$ and $\overline{\lambda}(f)$ to denote the order, exponent of convergence of zeros and exponent of convergence of distinct zeros of meromorphic function $f$, respectively. We use $\sigma_2(f)$ to denote the hyper order of $f(z)$, which is defined to be [14]

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_2 T(r,f)}{\log r}.$$

The hyper exponent of convergence of zeros and distinct zeros of $f(z)$ are respectively defined to be [5]

$$\lambda_2(f) = \overline{\lim_{r\to\infty}} \frac{\log_2 N(r,f)}{\log r}, \quad \overline{\lambda}_2(f) = \overline{\lim_{r\to\infty}} \frac{\log_2 \overline{N}(r,f)}{\log r}.$$

We denote the linear measure of a set $E \subset [1,\infty)$ by $mE = \int_E \mathrm{d}t$ and the logarithmic measure of $E$ by $m_lE = \int_E \frac{\mathrm{d}t}{t}$. The upper and lower logarithmic density of $E$ are defined by

$$\overline{\log \mathrm{dens}}(E) = \overline{\lim_{r\to\infty}} \frac{m_l(E\cap[1,r])}{\log r}$$

and

$$\underline{\log \mathrm{dens}}(E) = \underline{\lim_{r\to\infty}} \frac{m_l(E\cap[1,r])}{\log r},$$

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respectively. It is easy to see that $m_l E = \infty$ if $\overline{\log\mathrm{dens}}(E) > 0$ or $\underline{\log\mathrm{dens}}(E) > 0$.

For the higher order linear differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = F, \tag{1.1}$$

where $A_j(z)$ $(j = 0, 1, \ldots, k-1, k \geq 2)$, $F(z)$ are entire functions of finite order, it is well known that every solution of (1.1) is entire when the coefficients $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ and $F(z)$ are entire functions. As to equation (1.1), a classical problem is whether every transcendental solution of (1.1) has infinite order or not if there exists some coefficient $A_d(z)(1 \leq d \leq k-1)$ such that $\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \sigma(A_d)$. Many people investigated this problem. In 1991, Hellerstein, Miles and Rossi proved the following result.

**Theorem 1.1** ([10]) *Let $A_0, \ldots, A_{k-1}$, $F$ be entire functions. Suppose that there exists an $A_d$ $(0 \leq d \leq k-1)$ such that*

$$\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \sigma(A_d) \leq \frac{1}{2}.$$

*Then every solution of (1.1) is either a polynomial or an entire function of infinite order.*

By the definition of hyper order, we can easily obtain $\sigma(f) = \infty$ if $\sigma_2(f) > 0$ and the growth of infinite order solutions of (1.1) can be estimated more precisely. Then the above problem becomes that under what conditions can we get the result $\sigma_2(f)=\sigma(A_d)$ for every transcendental solution of (1.1) if there exists some coefficient $A_d(z)(1 \leq d \leq k-1)$ satisfying $\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \sigma(A_d)$.

In 2000, Chen and Yang gave the more precise estimate of hyper order of the solutions of (1.1) and obtained the following results.

**Theorem 1.2** ([4]) *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$, $F(z)$ be entire functions. Suppose that there exists some $d \in \{0, \ldots, k-1\}$ such that transcendental entire function $A_d(z)$ satisfies*

$$\max\{\sigma(F), \sigma(A_j)(j \neq d)\} < \sigma(A_d) < \frac{1}{2},$$

*then every transcendental solution of (1.1) satisfies $\sigma_2(f) = \sigma(A_d)$. Furthermore, if $F(z) \not\equiv 0$, then $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_d)$.*

From Theorem 1.2, we have $\sigma_2(f) = \sigma(A_d)$ for every transcendental solution of (1.1) if $\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \sigma(A_d) < 1/2$. In order to remove the condition $\sigma(A_d) < 1/2$, we introduce the lacunary series in the following.

Let $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be a lacunary series of finite order, where the sequence of exponents $\{\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots\}$ is an increasing sequence of nonnegative integers satisfying the Fabry gap condition

$$\frac{\lambda_n}{n} \to \infty, \quad n \to \infty. \tag{1.2}$$

In 2009, Tu and Liu proved the following result.

**Theorem 1.3** ([12]) *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$, $F(z)$ be entire functions satisfying $\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \sigma(A_d) < \infty$ $(1 \leq d \leq k-1)$. Suppose that $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is an entire function of regular growth such that the sequence of exponents $\{\lambda_n\}$ satisfies (1.2), then* *(i) If $F(z) \equiv 0$, then every transcendental solution $f(z)$ of (1.1) satisfies $\sigma_2(f) = \sigma(A_d)$;*

*(ii) If $F(z) \not\equiv 0$, then every transcendental solution $f(z)$ of (1.1) satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_d)$.*

In this paper, we remove the condition that $A_d(z)$ is of regular growth and obtain the same result as that in Theorem 1.3.

**Theorem 1.4** *Let $A_j(z)$ $(j = 0, 1, \dots, k-1)$, $F(z)$ be entire functions satisfying $\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \sigma(A_d) < \infty (1 \leq d \leq k-1)$. Suppose that $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is a lacunary series such that the sequence of exponents $\{\lambda_n\}$ satisfies (1.2), then*

*(i) Every transcendental solution $f(z)$ of (1.1) satisfies $\sigma_2(f) = \sigma(A_d)$;*

*(ii) If $F(z) \not\equiv 0$, then every transcendental solution $f(z)$ of (1.1) satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_d)$;*

*(iii) If $f(z)$ is a polynomial solution of (1.1), then $f(z)$ must be a polynomial with degree less than $d$.*

*(iv) If $d = 1$, then every non-constant solution $f(z)$ of (1.1) satisfies $\sigma_2(f) = \sigma(A_d)$.*

## 2. Lemmas

**Lemma 2.1** ([7]) *Let $f(z)$ be a transcendental meromorphic function and $\alpha > 1$ be a given constant. For any given $\varepsilon > 0$, there exist a set $E_1 \subset [1, \infty)$ that has finite logarithmic measure and a constant $B > 0$ that depends only on $\alpha$ and $(i, j)(i, j \in \{0, \dots, k\}$ with $i < j)$ such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\Big(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r) \log T(\alpha r, f)\Big)^{j-i}. \tag{2.1}$$

**Lemma 2.2** ([3]) *Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < \infty$. Then for any given $\varepsilon > 0$, there is a set $E_2 \subset [1, \infty)$ that has finite linear measure and finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0, 1] \bigcup E_2$, we have*

$$\exp\{-r^{\sigma+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\sigma+\varepsilon}\}. \tag{2.2}$$

**Lemma 2.3** ([6]) *Let $f(z) = \sum_{n=1}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function of finite order. If the sequence of exponents $\{\lambda_n\}$ satisfies (1.2), then for any given $\varepsilon$ $(0 < \varepsilon < 1)$,*

$$\log L(r, f) > (1-\varepsilon) \log M(r, f) \tag{2.3}$$

*holds outside a set $E_3$ of logarithmic density 0, where $M(r, f) = \sup_{|z|=r} |f(z)|$, $L(r, f) = \inf_{|z|=r} |f(z)|$.*

**Lemma 2.4** *Let $f(z)$ be an entire function of order $0 < \sigma(f) = \sigma < \infty$. Then for any $\beta < \sigma$, there exists a set $E_4$ with positive upper logarithmic density such that for all $|z| = r \in E_4$, we have*

$$\log M(r, f) > r^{\beta}, \tag{2.4}$$

*where $M(r, f) = \sup_{|z|=r} |f(z)|$.*

**Proof** By the definition of the order, there exists a sequence $\{r_n\}$ tending to $\infty$ such that for any given $\varepsilon > 0$, we have

$$\log M(r_n, f) > r_n^{\sigma-\varepsilon}.$$

Since $\beta < \sigma$, we can choose $\varepsilon$ (sufficiently small) and $\alpha$ to satisfy $1 < \alpha < \frac{\sigma-\varepsilon}{\beta}$. Then for all $r \in [r_n, r_n^{\alpha}]$ $(n \geq 1)$, we have

$$\log M(r, f) \geq \log M(r_n, f) > r_n^{\sigma-\varepsilon} \geq r^{\frac{\sigma-\varepsilon}{\alpha}} > r^{\beta}.$$

Setting $E_4 = \bigcup_{n=1}^{\infty}[r_n, r_n^{\alpha}]$, we have

$$\overline{\log\mathrm{dens}}E_4 \geq \overline{\lim_{n\to\infty}}\frac{m_l(E_4 \cap [1, r])}{\log r} \geq \overline{\lim_{n\to\infty}}\frac{m_l(E_4 \cap [1, r_n^{\alpha}])}{\log r_n^{\alpha}} \geq \lim_{n\to\infty}\frac{m_l([r_n, r_n^{\alpha}])}{\log r_n^{\alpha}} = \frac{\alpha-1}{\alpha} > 0.$$

Thus, Lemma 2.4 is proved. □

**Lemma 2.5** *Let $f(z) = \sum_{n=1}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function of order $0 < \sigma(f) = \sigma < \infty$. If the sequence of exponents $\{\lambda_n\}$ satisfies (1.2), then for any $\beta < \sigma(f)$, there exists a set $E_5$ with positive upper logarithmic density such that for all $|z| = r \in E_5$, we have*

$$\log L(r, f) > r^{\beta}, \tag{2.5}$$

*where $L(r, f) = \inf_{|z|=r} |f(z)|$.*

**Proof** By Lemma 2.3, for any given $\varepsilon(> 0)$, there exists a set $E_3$ with $\underline{\underline{\log\mathrm{dens}}}E_3 = 1$ such that for all $r \in E_3$, we have (2.3). By Lemma 2.4, there exists a set $E_4$ with $\overline{\log\mathrm{dens}}E_4 > 0$ such that for all $r \in E_4$, we have

$$\log M(r, f) > r^{\sigma-\varepsilon}. \tag{2.6}$$

Then for any $\beta < \sigma$, we can choose $\varepsilon$ to satisfy $0 < \varepsilon < \min\{\sigma(f) - \beta, 1\}$. By (2.3) and (2.6), we have that for all $r \in E_3 \cap E_4$,

$$\log L(r, f) > (1-\varepsilon)\log M(r, f) > (1-\varepsilon)r^{\sigma-\varepsilon} > r^{\beta}.$$

Note that the set $E_5 = E_3 \cap E_4$ has a positive upper logarithmic density. In fact, we have

$$\overline{\log\mathrm{dens}}(E_3 \cap E_4) + \overline{\log\mathrm{dens}}(E_3 \cup E_4) \geq \underline{\log\mathrm{dens}}E_3 + \overline{\log\mathrm{dens}}E_4.$$

Consequently, we have

$$\overline{\log\mathrm{dens}}(E_5) \geq \underline{\log\mathrm{dens}}E_3 + \overline{\log\mathrm{dens}}E_4 - 1 > 0.$$

Thus, Lemma 2.5 is proved. □

**Lemma 2.6** ([2]) *Let $f(z)$ be a transcendental entire function. Then there is a set $E_6 \subset [1, +\infty)$ having finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin E_6$ and $|f(z)| = M(r, f)$, we have*

$$\Big|\frac{f(z)}{f^{(s)}(z)}\Big| \leq 2r^{s}, \quad s \in N. \tag{2.7}$$

**Lemma 2.7** ([9,11]) *Let $f(z)$ be a transcendental entire function, and let $z$ be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z| = r$ outside a set $E_7$ of $r$ of finite logarithmic* *measure, we have*

$$\frac{f^{(i)}(z)}{f(z)} = \Big(\frac{\nu_f(r)}{z}\Big)^i(1+o(1)), \quad i \in N,\ r \notin E_7, \tag{2.8}$$

*where $\nu_f(r)$ is the central index of $f(z)$.*

**Lemma 2.8** ([5]) *Let $f(z)$ be an entire function of infinite order satisfying $\sigma_2(f) = \sigma$. Then*

$$\overline{\lim_{r\to\infty}}\frac{\log_2 \nu_f(r)}{\log r} = \sigma, \tag{2.9}$$

*where $\nu_f(r)$ is the central index of $f(z)$.*

**Lemma 2.9** ([1, 2]) *Let $A_j(z)$ $(j = 0, \ldots, k-1)$ be entire functions satisfying* $\max\{\sigma(A_j), j = 0, \ldots, k-1\} \leq \sigma < \infty$. *If $f(z)$ is a solution of*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0, \tag{2.10}$$

*then $\sigma_2(f) \leq \sigma$.*

By Lemmas 2.7, 2.8 and the same arguments as in the proof of Lemma 2.9, we can easily obtain the following result.

**Lemma 2.10** *Let $A_j(z)$ $(j = 0, \ldots, k-1)$, $F(z)$ be entire functions satisfying* $\max\{\sigma(A_j), j = 0, \ldots, k-1, \sigma(F)\} \leq \sigma < \infty$. *If $f(z)$ is a solution of (1.1), then $\sigma_2(f) \leq \sigma$.*

## 3. Proof of Theorem 1.4

**Proof of Theorem 1.4** (i) Assume that $f(z)$ is transcendental solution of (1.1). By (1.1), we have

$$|A_d| \leq \Big|\frac{f^{(k)}}{f^{(d)}}\Big| + \cdots + |A_{d+1}|\Big|\frac{f^{(d+1)}}{f^{(d)}}\Big| + \Big|\frac{f}{f^{(d)}}\Big|\Big(|A_{d-1}|\Big|\frac{f^{(d-1)}}{f}\Big| + \cdots + |A_0| + \Big|\frac{F}{f}\Big|\Big). \tag{3.1}$$

By Lemma 2.1, there exists a set $E_1 \subset [1, \infty)$ having finite logarithmic measure and a constant $B > 0$ such that

$$\Big|\frac{f^{(j)}(z)}{f^{(i)}(z)}\Big| \leq B(T(2r, f))^{2k}, \quad 0 \leq i < j \leq k \tag{3.2}$$

holds for all $|z| = r \notin E_1$ and for sufficiently large $r$. Since $\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \sigma(A_d)$, we choose $\alpha_1, \beta_1$ to satisfy $\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \alpha_1 < \beta_1 < \sigma(A_d)$. By Lemma 2.2, there exists a set $E_2 \subset [1, \infty)$ having finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin E_2$ and for sufficiently large $r$, we have

$$|A_j(z)| \leq \exp\{r^{\alpha_1}\}. \tag{3.3}$$

Since $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ and $\{\lambda_n\}$ satisfies (1.2), by Lemma 2.5, there exists a set $E_5 \subset [1, \infty)$ having infinite logarithmic measure such that for all $z$ satisfying $|z| = r \in E_5$, we have

$$|A_d(z)| \geq \inf_{|z|=r} |A_d(z)| \geq \exp\{r^{\beta_1}\}. \tag{3.4}$$

By Lemmas 2.2 and 2.6, there exist two sets $E_6, E_2 \subset [1, \infty)$ having finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin (E_6 \bigcup E_2)$ and $|f(z)| = M(r, f)$, we have

$$\Big|\frac{f(z)}{f^{(d)}(z)}\Big| \le 2r^d, \quad \Big|\frac{F(z)}{f(z)}\Big| \le |F(z)| \le \exp\{r^{\alpha_1}\}. \tag{3.5}$$

Hence from (3.1)–(3.5), for all $z$ satisfying$|z| = r \in E_5 \backslash (E_1 \bigcup E_2 \bigcup E_6)$ and $|f(z)| = M(r, f)$, we have

$$\exp\{r^{\beta_1}\} \le (k+1)r^d \exp\{r^{\alpha_1}\} \cdot (T(2r, f))^{2k}. \tag{3.6}$$

Since $\beta_1$ is arbitrarily close to $\sigma(A_d)$, by (3.6), we have

$$\varlimsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r} \ge \sigma(A_d).$$

On the other hand, by Lemma 2.10, we have $\sigma_2(f) \le \sigma(A_d)$. Therefore, $\sigma_2(f) = \sigma(A_d)$.

(ii) Assume that if $f(z)$ is a transcendental solution of (1.1), by $(i)$, we have $\sigma_2(f) = \sigma(A_d)$. Next we show that $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$ if $F \not\equiv 0$. From (1.1), we have

$$\frac{1}{f} = \frac{1}{F}\Big(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\Big). \tag{3.7}$$

By (3.7), it is easy to see that if $f(z)$ has a zero at $z_0$ of order $m$ more than $k$, then $F$ must have a zero at $z_0$ of order $m-k$. Hence we get

$$N(r, \frac{1}{f}) \le k\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{F}). \tag{3.8}$$

Also by (3.7), we have

$$m(r, \frac{1}{f}) \le \sum_{j=1}^{k} m(r, \frac{f^{(j)}}{f}) + \sum_{j=0}^{k-1} m(r, A_j) + m(r, \frac{1}{F}). \tag{3.9}$$

By (3.8), (3.9) and the lemma of logarithmic derivative, we obtain that

$$T(r, f) \le k\overline{N}(r, \frac{1}{f}) + M(\log(rT(r, f))) + T(r, F) + \sum_{j=0}^{k-1} T(r, A_j), \quad r \notin E, \tag{3.10}$$

where $E \subset (0, +\infty)$ is a set having finite linear measure, $M > 0$ is a constant, not necessarily the same at each occurrence. For sufficiently large $r \notin E$ and for any given $\varepsilon > 0$, we have

$$M(\log(rT(r, f))) \le \frac{1}{2}T(r, f), \tag{3.11}$$

$$\sum_{j=0}^{k-1} T(r, A_j) + T(r, F) \le (k+1)r^{\sigma(A_d)+\varepsilon}. \tag{3.12}$$

By (3.11) and (3.12), we have

$$T(r, f) \le 2k\overline{N}(r, \frac{1}{f}) + 2(k+1)r^{\sigma(A_d)+\varepsilon}, \tag{3.13}$$

hence $\sigma_2(f) \le \overline{\lambda}_2(f)$ by (3.13). Therefore, $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$.

(iii) Suppose that $f(z)$ is a polynomial solution of (1.1) with degree not less than $d$. By (1.1), we have

$$|A_d f^{(d)}(z)| \le |f^{(k)}(z)| + \cdots + |A_{d+1} f^{(d+1)}(z)| + |A_{d-1} f^{(d-1)}(z)| + \cdots + |A_0 f(z)| + |F(z)|. \tag{3.14}$$

By the proof of (i), we know that there exists a set $E_5$ with infinite logarithmic measure such that for all $z$ satisfying $|z| = r \in E_5$, we have

$$|A_d f^{(d)}(z)| \geq r^M \cdot \exp\{r^{\beta_1}\}. \tag{3.15}$$

On the other hand, since $\max\{\sigma(A_j), j \neq d, \sigma(F)\} < \alpha_1$, by Lemma 2.2 and (3.14), for sufficiently large $|z| = r \notin E_2$, we have

$$\begin{aligned} |A_d f^{(d)}(z)| &\leq |f^{(k)}(z)| + \cdots + |A_{d+1} f^{d+1}(z)| + |A_{d-1} f^{d-1}(z)| + \cdots + |A_0 f(z)| + |F(z)| \\ &\leq r^M \cdot \exp\{r^{\alpha_1}\}. \end{aligned} \tag{3.16}$$

Since $\alpha_1 < \beta_1$, (3.15) is a contradiction with (3.16), thus, the degree of $f(z)$ must be less than $d$.

(iv) If $d = 1$ and $f(z)$ is a polynomial solution of (1.1), by (iii), we obtain that $f(z)$ must be a constant. In addition, by (i), we obtain that every non-constant solution $f(z)$ of (1.1) satisfies $\sigma_2(f)=\sigma(A_d)$. $\square$

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