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Cauchy Integral Formulae in $\mathbb{R}^n$

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Abstract In this note $p(\underline{D}) = \underline{D}^m + b_1\underline{D}^{m-1} + \cdots + b_m$ is a polynomial Dirac operator in $\mathbb{R}^n$, where $\underline{D} = \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$ is a standard Dirac operator in $\mathbb{R}^n$, b_j are the complex constant coefficients. In this note we discuss all decompositions of $p(\underline{D})$ according to its coefficients b_j, and obtain the corresponding explicit Cauchy integral formulae of f which are the solution of $p(\underline{D})f = 0$.

Keywords Dirac operator; Cauchy integral formula.

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1. Introduction

Denote by $\mathbf{R}_n$ the universal Clifford algebra whose elements are $e_0, e_1, \ldots, e_n, \ldots, e_{j_1} \cdots e_{j_r}, \ldots, e_1 \cdots e_n$, where $1 \le j_1 < \cdots < j_r \le n$, $1 \le r \le n$. $e_0 = 1$ is the unit element. $e_1, \ldots, e_n$ are the vectors satisfying $e_ie_j + e_je_i = -2\delta_{ij}$ ([1]).

Denote $\underline{x} \in \mathbb{R}^n$ as $\underline{x} = x_1e_1 + \cdots + x_ne_n$, $x_j \in \mathbb{R}$, then $\underline{x}^2 = -|\underline{x}|^2$. Any nonzero vector $\underline{x} \in \mathbb{R}^n$ has a unique inverse $\underline{x}^{-1} = \overline{\underline{x}}|\underline{x}|^{-2}$, where $\overline{\underline{x}} = -\underline{x}$. $\mathbb{R}^n$ is called a homogeneous Euclidean space [2]. The Dirac operator in $\mathbb{R}^n$ is defined by $\underline{D} = \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$. The solutions of $(\underline{D}f)(\underline{x}) = \sum_{j=1}^n e_j \frac{\partial f}{\partial_{x_j}} = 0$ are called left monogenic while the solutions of $(f\underline{D})(\underline{x}) = \sum_{j=1}^n \frac{\partial f}{\partial_{x_j}} e_j = 0$ are called right monogenic.

There exists an analogous hyper-complex theory of monogenic functions in $\mathbb{R}^{n+1}$, where $\mathbb{R}^{n+1}$ is called inhomogeneous Euclidean space. In this case $x = x_0e_0 + \underline{x} \in \mathbb{R}^{n+1}$, $D = \frac{\partial}{\partial x_0} + \underline{D}$ is the Dirac operator in $\mathbb{R}^{n+1}$ (see [1, 2]).

Cauchy integral formulae related with both D and $\underline{D}$ are active topics in Clifford analysis. There are some connections as well as differences between the null solutions of D and $\underline{D}$. The standard function theory on D can be found in the book [1]. Function theory of the iterated Dirac operator D^k was discussed in [3]. The structure of the null solutions of $p(D) = D^m + b_1D^{m-1} + \cdots + b_m$ was obtained in [4]. In the case of the homogeneous Euclidean space $\mathbb{R}^n$, Cauchy integral formulae of $\underline{D}$ (see [2]), $\underline{D} - \lambda$ (see [5]) and $\underline{D}^k$ (see [6–8]) have been obtained, respectively. Similar results on Helmholtz-type operators were discussed in Quaternion and Clifford analysis [9, 10].

Some studies have been carried out for the polynomial Dirac operator $p(\underline{D}) = \underline{D}^m + b_1\underline{D}^{m-1} + \cdots + b_m$ in $\mathbb{R}^n$ (see [7, 8]). A Cauchy-Green type formula for the operator $\underline{D}^m +$

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$\sum_{k=1}^{m-1}(-1)^{k-m}b_k\underline{D}^k$ has been obtained in [7] by a fundamental solution of $p(\underline{D})$. In this paper we use the technique in [3, 4] to give out an explicit Cauchy integral formula for the null solutions of $p(\underline{D})$, the method we used is different from that in [7].

2. Cauchy integral formulae in $\mathbb{R}^n$

The fundamental solution of $\underline{D}=\sum_{j=1}^n e_j\frac{\partial}{\partial x_j}$ is

$$G_1(\underline{x})=\frac{-1}{\omega_n}\frac{\underline{x}}{|\underline{x}|^n}.$$

$G_1(\underline{x})$ is both left and right monogenic in $\mathbb{R}^n\backslash\{\underline{0}\}$. ω_n denotes the measure of the unit sphere in $\mathbb{R}^n$ (see [2]).

Theorem 2.1 ([2]) *Suppose that* $f:U\subset\mathbb{R}^n\to\mathbf{R}_n$ *is left monogenic,* $\Omega\subset U$ *is a bounded domain with piecewise* C^1 *boundary* $\partial\Omega$*. Then*

$$f(\underline{x})=\int_{\partial\Omega}G_1(\underline{y}-\underline{x})n(\underline{y})f(\underline{y})\mathrm{d}\mu(\underline{y}),\quad \underline{x}\in\Omega,$$

where $\mathrm{d}\mu(\underline{y})$ *is the surface measure of* $\partial\Omega$*,* $n(\underline{y})$ *is the outward unit normal at* $\underline{y}\in\partial\Omega$*.*

The following functions $G_k(\underline{x})$ play important roles in the study [7, 8]:

Case 1 In the case where n is odd, denote

$$G_{2l+1}(\underline{x})=A_{2l+1}\frac{\underline{x}}{|\underline{x}|^{n-2l}},\quad l=0,1,\ldots,$$
$$G_{2l}(\underline{x})=A_{2l}\frac{1}{|\underline{x}|^{n-2l}},\quad l=1,\ldots;$$

Case 2 In the case where n is even, denote

$$G_{2l+1}(\underline{x})=A'_{2l+1}\frac{\underline{x}}{|\underline{x}|^{n-2l}},\quad l=0,1,\ldots,\frac{n}{2}-1,$$
$$G_{2l}(\underline{x})=A'_{2l}\frac{1}{|\underline{x}|^{n-2l}},\quad l=1,\ldots,\frac{n-2}{2};$$
$$G_k(\underline{x})=B_k\underline{x}^{k-n}+C_k\underline{x}^{k-n}\ln(|\underline{x}|),\quad k\geq n.$$

These functions are the building stones in the study of $p(\underline{D})$. The real coefficients A_k, A'_k, B_k and C_k are chosen such that $\underline{D}G_k=G_k\underline{D}=G_{k-1}$ (see [5, 7, 8]), then $\underline{D}^kG_k=G_k\underline{D}^k=0$ in the domain $\mathbb{R}^n\backslash\{\underline{0}\}$. These $G_k(\underline{x})$ are reobtained by use of Fourier transform if the coefficients A_k, A'_k, B_k and C_k are omitted [6].

Assume $\lambda\in\mathbb{C}$. Denote

$$E_\lambda(\underline{x})=\sum_{k=1}^\infty\lambda^{k-1}G_k(\underline{x}), \tag{2.1}$$

then for any fixed $\underline{x}\in\mathbb{R}^n\backslash\{0\}$, the series in (2.1) is convergent for $\lambda\in\mathbb{C}$ (see [5, 8]). $E_\lambda(\underline{x})$ is the fundamental solution of $\underline{D}-\lambda$ implying that $(\underline{D}-\lambda)E_\lambda(\underline{x})=E_\lambda(\underline{x})(\underline{D}-\lambda)=0$ in the domain $\mathbb{R}^n\backslash\{\underline{0}\}$. Obviously, $E_0(\underline{x})=G_1(\underline{x})$ is the fundamental solution of $\underline{D}$.

Theorem 2.2 ([5, 8]) *Suppose that* $f:U\subset\mathbb{R}^n\to\mathbf{R}_n$ *is* C^1 *satisfying* $(\underline{D}-\lambda)f=0$*. Let* $\Omega\subset U$

be a bounded domain with piecewise C^1 boundary $\partial\Omega$. Then

$$f(\underline{x}) = \int_{\partial\Omega} E_{-\lambda}(\underline{y}-\underline{x})n(\underline{y})f(\underline{y})\mathrm{d}\mu(\underline{y}), \quad \underline{x}\in\Omega.$$

Theorem 2.3 ([7]) *Suppose that $f : U \subset \mathbb{R}^n \to \mathbf{R}_n$ is C^k satisfying $\underline{D}^k f = 0$, $\Omega \subset U$ is a bounded domain with piecewise C^k boundary $\partial\Omega$. Then*

$$f(\underline{x}) = \int_{\partial\Omega} \sum_{j=1}^{k} (-1)^{j-1} G_j(\underline{y}-\underline{x})n(\underline{y})\underline{D}^{j-1} f(\underline{y})\mathrm{d}\mu(\underline{y}), \quad \underline{x}\in\Omega.$$

Theorem 2.4 ([8]) *Let $E_\lambda^{(1)}(\underline{x}) = E_\lambda(\underline{x})$ be a fundamental solution of $\underline{D}-\lambda$. Then*

$$E_\lambda^{(k)}(\underline{x}) = \frac{1}{\Gamma(k)}\frac{d^k}{d\lambda^k}E_\lambda^{(1)}(\underline{x})$$

is a fundamental solution for $(\underline{D}-\lambda)^k$. Moreover,

$$(\underline{D}-\lambda)E_\lambda^{(k)}(\underline{x}) = E_\lambda^{(k)}(\underline{x})(\underline{D}-\lambda) = E_\lambda^{(k-1)}(\underline{x}).$$

Theorem 2.5 *Suppose $f : U \subset \mathbb{R}^n \to \mathbf{R}_n$ is C^k satisfying $(\underline{D}-\lambda)^k f = 0$, $\lambda \in \mathbb{C}$, Ω is a bounded domain in U with piecewise C^k boundary $\partial\Omega$. Then*

$$f(\underline{x}) = \int_{\partial\Omega} \sum_{j=1}^{k} (-1)^{j-1} E_{-\lambda}^{(j)}(\underline{y}-\underline{x})n(\underline{y})(\underline{D}-\lambda)^{j-1} f(\underline{y})\mathrm{d}\mu(\underline{y}), \quad \underline{x}\in\Omega. \tag{2.2}$$

Proof The idea for the proof comes from [3].

Assume that $f, g \in C^k(U)$, $\lambda \in \mathbb{C}$, then

$$\begin{aligned}\int_{\partial\Omega} g(\underline{y})n(\underline{y})f(\underline{y})\mathrm{d}\mu(\underline{y}) &= \int_\Omega (g\underline{D}.f + g.\underline{D}f)\mathrm{d}\underline{y} \\ &= \int_\Omega (g(\underline{D}+\lambda).f + g.(\underline{D}-\lambda)f)\mathrm{d}\underline{y}. \end{aligned}\tag{2.3}$$

Substituting $g(\underline{D}+\lambda)^{k-1-j}$ for g and $(\underline{D}-\lambda)^j f$ for f in (2.3), and summing up for $j = 0, 1, \ldots, k-1$ after multiplying the factor $(-1)^j$ on both sides, we have

$$\begin{aligned}&\int_{\partial\Omega} \sum_{j=0}^{k-1} (-1)^j g(\underline{D}+\lambda)^{k-1-j} n(\underline{y})(\underline{D}-\lambda)^j f\mathrm{d}\mu(\underline{y}) \\ &= \int_\Omega g(\underline{D}+\lambda)^k.f + (-1)^{k-1} g.(\underline{D}-\lambda)^k f\mathrm{d}\underline{y}.\end{aligned}\tag{2.4}$$

Note that $E_{-\lambda}^{(k)}(\underline{y}-\underline{x})(\underline{D}+\lambda)^k = \delta(\underline{y}-\underline{x})$, $(\underline{D}-\lambda)^k f = 0$. Substituting in (2.4) $E_{-\lambda}^{(k)}(\underline{y}-\underline{x})$ for g and taking f to be a solution of $(\underline{D}-\lambda)^k f = 0$, the right side of (2.4) becomes

$$\int_\Omega E_{-\lambda}^{(k)}(\underline{y}-\underline{x})(\underline{D}+\lambda)^k.f(\underline{y}) + (-1)^{k-1}E_{-\lambda}^{(k)}(\underline{y}-\underline{x}).(\underline{D}-\lambda)^k f\mathrm{d}\underline{y} = f(\underline{x})$$

and we finally have

$$f(\underline{x}) = \int_{\partial\Omega} \sum_{j=0}^{k-1} (-1)^j E_{-\lambda}^{(k)}(\underline{y}-\underline{x})(\underline{D}+\lambda)^{k-1-j} n(\underline{y})(\underline{D}-\lambda)^j f\mathrm{d}\mu(\underline{y})$$

$$= \int_{\partial\Omega} \sum_{j=0}^{k-1} (-1)^j E_{-\lambda}^{(j+1)}(\underline{y}-\underline{x}) n(\underline{y}) (D-\lambda)^j f(\underline{y}) \mathrm{d}\mu(\underline{y}),$$

where $E_{-\lambda}^{(k)}(\underline{y}-\underline{x})(\underline{D}+\lambda) = E_{-\lambda}^{(k-1)}(\underline{y}-\underline{x})$ are used in the above equation. □

Remark 2.6 If $\lambda = 0$, then $E_{\lambda}^{(k)}(\underline{x})|_{\lambda=0} = G_j(\underline{x})$. In this case the Cauchy integral formula (2.2) is just the result stated in Theorem 2.3. If $k = 1$, then (2.2) reduces to the Cauchy integral formula in Theorem 2.2.

Denote $p(\underline{D}) = \underline{D}^m + b_1 \underline{D}^{m-1} + \cdots + b_m$, where $b_j \in \mathbb{C}$. Next we use the method in [4] to derive the Cauchy integral formula for the null solutions of $p(\underline{D})$.

Denote $p(\lambda) = \lambda^m + b_1\lambda^{m-1} + \cdots + b_m$. By Gauss's Theorem, $p(\lambda)$ has different complex roots $\lambda_1, \ldots, \lambda_l$ with orders $m_1, \ldots, m_l$, respectively, where all m_k are positive integers, $m_1 + \cdots + m_l = m$. So $p(\lambda)$ is rewritten as

$$\pi(\lambda) = (\lambda-\lambda_1)^{m_1} \cdots (\lambda-\lambda_l)^{m_l}.$$

Correspondingly $p(\underline{D})$ may be rewritten as

$$\pi(\underline{D}) = (\underline{D}-\lambda_1)^{m_1} \cdots (\underline{D}-\lambda_l)^{m_l}. \tag{2.5}$$

Obviously, $\underline{D} - \lambda_k$ commutes with the other $\underline{D} - \lambda_i$. Denote

$$\pi_{k,s}(\underline{D}) = (\underline{D}-\lambda_1)^{m_1} \cdots (\underline{D}-\lambda_{k-1})^{m_{k-1}} (\underline{D}-\lambda_k)^{m_k - s} (\underline{D}-\lambda_{k+1})^{m_{k+1}} \cdots (\underline{D}-\lambda_l)^{m_l},$$

where $1 \le s \le m_k$. For any function f which is smooth enough, f has the decomposition

$$f = \sum_{k=1}^{l} \sum_{s=1}^{m_k} \frac{1}{(m_k - s)!} \Big[\frac{d^{m_k - s}}{d\lambda^{m_k - s}} \frac{(\lambda-\lambda_k)^s}{\pi(\lambda)} \Big]_{\lambda=\lambda_k} \pi_{k,s}(\underline{D}) f. \tag{2.6}$$

Theorem 2.7 (Cauchy integral formula for the solutions of $\pi(\underline{D})f = 0$) *Assume that* Ω *is a bounded domain in* U *with piecewise* C^m *boundary* $\partial\Omega$, $\pi(\underline{D})$ *is given by (2.5),* f *is a null solutions of* $\pi(\underline{D})f = 0$ *in* $U \subset \mathbb{R}^n$. *Then for any* $\underline{x} \in \Omega$,

$$\begin{aligned} f(\underline{x}) = & \sum_{k=1}^{l} \sum_{s=1}^{m_k} \frac{1}{(m_k - s)!} \Big[\frac{d^{m_k - s}}{d\lambda^{m_k - s}} \frac{(\lambda-\lambda_k)^{m_k}}{\pi(\lambda)} \Big]_{\lambda=\lambda_k} \\ & \int_{\partial\Omega} \sum_{j=1}^{s} (-1)^{j-1} E_{-\lambda_k}^{(j)}(\underline{y}-\underline{x}) n(y) (\underline{D}-\lambda_k)^{j-1} \pi_{k,s}(\underline{D}) f(\underline{y}) \mathrm{d}\mu(\underline{y}). \end{aligned} \tag{2.7}$$

Proof Since $\pi(\underline{D})f = 0$ on the domain U, $\pi_{k,s}(\underline{D})f$ is a null solution of $(\underline{D}-\lambda_k)^s$. So by Theorem 2.5,

$$\pi_{k,s}(\underline{D}) f = \int_{\partial\Omega} \sum_{j=1}^{s} (-1)^{j-1} E_{-\lambda_k}^{(j)}(\underline{y}-\underline{x}) n(y) (\underline{D}-\lambda_k)^{j-1} \pi_{k,s}(\underline{D}) f(\underline{y}) \mathrm{d}\mu(\underline{y}). \tag{2.8}$$

Formula (2.7) is derived after inserting (2.8) into (2.6). □

Corollary 2.8 *Denote* $\pi(\underline{D}) = (\underline{D}-\lambda_1)\cdots(\underline{D}-\lambda_m)$, f *a null solution of* $\pi(\underline{D})f = 0$ *in* $U \subset \mathbb{R}^n$. *Then*

$$f(\underline{x}) = \sum_{k=1}^{m} \int_{\partial\Omega} E_{-\lambda_k}(\underline{y}-\underline{x}) n(\underline{y}) \frac{\Pi_{j=1, j\neq k}^{m} (\underline{D}-\lambda_j)}{\Pi_{j=1, j\neq k}^{m} (\lambda_k - \lambda_j)} f(\underline{y}) \mathrm{d}\mu(\underline{y}), \quad \underline{x} \in \Omega. \tag{2.9}$$

Remark 2.9 By substituting $E^{(j)}_{-\lambda_k}(\underline{y}-\underline{x})$ by $E^{(j)}_{\lambda_k}(\underline{x})$ in (2.7) and $E_{-\lambda_k}(\underline{y}-\underline{x})$ by $E_{\lambda_k}(\underline{x})$ in (2.9), respectively, then summing up for $k=1,\ldots,l$, we get the functions

$$\sum_{k=1}^{l}\sum_{s=1}^{m_k}\frac{1}{(m_k-s)!}\Big[\frac{d^{m_k-s}}{d\lambda^{m_k-s}}\frac{(\lambda-\lambda_k)^{m_k}}{\pi(\lambda)}\Big]_{\lambda=\lambda_k}\sum_{j=1}^{s}(-1)^{j-1}E^{(j)}_{\lambda_k}(\underline{x})$$

and

$$\sum_{k=1}^{m}\frac{E_{\lambda_k}(\underline{x})}{\Pi_{j\neq k}(\lambda_k-\lambda_j)},$$

which are the fundamental solutions in [8] for $(\underline{D}-\lambda_1)^{m_1}\cdots(\underline{D}-\lambda_l)^{m_l}$ and $(\underline{D}-\lambda_1)\cdots(\underline{D}-\lambda_m)$, respectively. In [7], John Ryan obtained another fundamental solution of $p(\underline{D})$ and a Cauchy-Green formula for the null solutions of $\underline{D}^m+\sum_{k=1}^{m-1}(-1)^{k-m}b_k\underline{D}^k$. Obviously, the obtained results in this note show that the fundamental solutions of $p(\underline{D})$ are not the same as the kernel functions used in its Cauchy integral formulae in the case where some $\lambda_k\neq 0$. It is also worth pointing out that the method we used is different from that in [7], it is simpler and more constructive than that in [7].

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