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# Global Weakly Discontinuous Solutions for Inhomogeneous Quasilinear Hyperbolic Systems with Characteristics with Constant Multiplicity

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**Abstract** This paper considers the Cauchy problem with a kind of non-smooth initial data for general inhomogeneous quasilinear hyperbolic systems with characteristics with constant multiplicity. Under the matching condition, based on the refined fomulas on the decomposition of waves, we obtain a necessary and sufficient condition to guarantee the existence and uniqueness of global weakly discontinuous solution to the Cauchy problem.

**Keywords** inhomogeneous quasilinear hyperbolic system; characteristic with constant multiplicity; Cauchy problem; global weakly discontinuous solution; weak linear degeneracy; matching condition.

**MR(2010) Subject Classification** 35L45; 35L60

## 1. Introduction and main results

Consider the following first order inhomogeneous quasilinear hyperbolic system

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=B(u), \tag{1.1}$$

where $u=(u_1,\dots,u_n)^{\mathrm{T}}$ is the unknown vector function of $(t,x)$, $A(u)$ is an $n\times n$ matrix with suitably smooth entries $a_{ij}(u)$ $(i,j=1,\dots,n)$, and $B(u)$ is a vector function with suitably smooth elements $b_i(u)$ $(i=1,\dots,n)$.

By hyperbolicity, for any given $u$ on the domain under consideration, $A(u)$ has $n$ real eigenvalues $\lambda_1(u),\dots,\lambda_n(u)$ and a complete set of left (resp., right) eigenvectors. For $i=1,\dots,n$, let $l_i(u)=(l_{i1}(u),\dots,l_{in}(u))$ (resp., $r_i(u)=(r_{i1}(u),\dots,r_{in}(u))^{\mathrm{T}}$) be a left (resp., right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u)=\lambda_i(u)l_i(u)\quad(\text{resp. } A(u)r_i(u)=\lambda_i(u)r_i(u)). \tag{1.2}$$

We have

$$\det|l_{ij}(u)|\neq 0\quad(\text{equivalently, } \det|r_{ij}(u)|\neq 0). \tag{1.3}$$

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Without loss of generality, we assume that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i,j=1,\dots,n) \tag{1.4}$$

and

$$r_i(u)^T r_i(u) \equiv 1 \quad (i=1,\dots,n), \tag{1.5}$$

where $\delta_{ij}$ denotes the Kronecker's symbol.

If $B(u) \equiv 0$, for the initial data

$$t=0: u=\phi(x) \quad (-\infty < x < +\infty), \tag{1.6}$$

where $\phi(x)$ is a $C^1$ vector function with bounded $C^1$ norm and satisfies certain small and decaying property, it was proved that Cauchy problem (1.1) and (1.6) admits a unique global $C^1$ solution $u=u(t,x)$ with small $C^1$ norm for all $t \in \mathbb{R}$, if and only if system (1.1) is weakly linearly degenerate (for strictly hyperbolic system [4, 5, 11, 12]; for the non-strictly hyperbolic system with characteristics with constant multiplicity [7, 14]. Also see [8]).

Recently, Li and Wang [9] studied the Cauchy problem of homogenous quasilinear strictly hyperbolic system (1.1) (i.e., $B(u) \equiv 0$) with a kind of non-smooth initial data

$$t=0: u=\begin{cases} u_l(x), & x \le 0, \\ u_r(x), & x \ge 0, \end{cases} \tag{1.7}$$

where $u_l(x)$ and $u_r(x)$ are $C^1$ vector functions on $x \le 0$ and $x \ge 0$, respectively, with

$$u_l(0)=u_r(0) \text{ and } u_l'(0) \ne u_r'(0) \tag{1.8}$$

and satisfy the following small and decaying property

$$\theta \triangleq \sup_{x\le 0}\{(1-x)^{1+\mu}(|u_l(x)|+|u_l'(x)|)\} + \sup_{x\ge 0}\{(1+x)^{1+\mu}(|u_r(x)|+|u_r'(x)|)\} \ll 1, \tag{1.9}$$

where $\mu > 0$ is a constant. They proved that Cauchy problem (1.1) and (1.7) admits a unique global weakly discontinuous solution $u=u(t,x)$ for all $t \in \mathbb{R}$ if and only if system (1.1) is weakly linearly degenerate. If $B(u)$ satisfies the matching condition, we have generalized their result to the inhomogeneous case [1]. However, in case of $B(u) \equiv 0$, if system (1.1) possesses characteristics with constant multiplicity, under the assumption that normalized coordinates exist, a necessary and sufficient condition to guarantee the existence and uniqueness of global weakly discontinuous solutions has been obtained in [2].

In this paper, we will investigate the inhomogeneous global weakly discontinuous solution to the quasilinear hyperbolic system (1.1) with characteristics with constant multiplicity.

For hyperbolic system (1.1) with characteristics with constant multiplicity, all $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)$ $(i,j=1,\dots,n)$ have the same regularity as $a_{ij}(u)$ $(i,j=1,\dots,n)$.

Without loss of generality, we suppose that, in a neighbourhood of $u=0$,

$$\lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u) \ (p \ge 1), \tag{1.10}$$

where $1 \le p \le n$. As $p=1$, system (1.1) is strictly hyperbolic; as $p > 1$, system (1.1) is a non-strictly hyperbolic systems with characteristics with constant multiplicity. Here we will deal with the latter.

The main difficulty we face is how to deal with the propagation of hyperbolic waves in the inhomogeneous term $B(u)$. For this purpose, we introduce the concept of matching condition (Def. 1.1) and present a more refined formula on the decomposition of waves.

To state our result precisely, we first give the following three definition: the matching condition, normalized coordinates and weak linear degeneracy.

**Definition 1.1** *$B(u)$ satisfies the matching condition if there exists normalized transformation and in normalized coordinates*

$$B\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0,\ \forall |u_h| \text{ small } \ (h=1,\dots,p) \tag{1.11}$$

*and*

$$B(u_j e_j) \equiv 0,\ \forall\ |u_j| \text{ small } \ (j=p+1,\dots,n). \tag{1.12}$$

**Definition 1.2** ([7]) *If there exists an invertible smooth transformation $u = u(\tilde{u})$ $(u(0) = 0)$ such that in $\tilde{u}$-space*

$$\tilde{r}_i\Big(\sum_{h=1}^{p} \tilde{u}_h e_h\Big) \equiv e_i,\ \forall\ |\tilde{u}_h| \ \text{ small } \ (i,h=1,\dots,p) \tag{1.13}$$

*and*

$$\tilde{r}_j(\tilde{u}_j e_j) \equiv e_j, \forall\ |\tilde{u}_j| \text{ small } \ (j=p+1,\dots,n), \tag{1.14}$$

*in which for $k=1,\dots,n$,*

$$e_k = (0,\dots,0,\overset{(k)}{1},0,\dots,0)^{\mathrm{T}}, \tag{1.15}$$

*then the transformation is called a normalized transformation, and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1,\dots,\tilde{u}_n)^{\mathrm{T}}$ are called normalized variables or normalized coordinates.*

**Definition 1.3** ([7]) *The $i$-th characteristic $\lambda_i(u)$ is weakly linearly degenerate, if there exists a normalized transformation and in normalized coordinates*

$$\lambda_i\Big(\sum_{h=1}^{p} \tilde{u}_h e_h\Big) \equiv \lambda(0),\ \forall\ |\tilde{u}_h| \text{ small } (h=1,\dots,p),\ \text{ when } i \in \{1,\dots,p\}; \tag{1.16}$$

$$\lambda_i(\tilde{u}_i e_i) \equiv \lambda_i(0),\ \ \forall\ |\tilde{u}_i| \text{ small },\ \text{ when } i \in \{p+1,\dots,n\}. \tag{1.17}$$

*When all characteristics $\lambda_i(u)$ $(i=1,\dots,n)$ are weakly linearly degenerate, system (1.1) is weakly linearly degenerate.*

Our main result is as follows

**Theorem 1.1** *Suppose that in a neighbourhood of $u=0$, $A(u)$, $B(u) \in C^2$ and the matching condition is satisfied. Furthermore, assume that there exist normalized coordinates. Then there exists $\theta_0 > 0$ so small that for any given initial data satisfying (1.8)–(1.9) with $\theta \in (0,\theta_0]$, Cauchy problem (1.1) and (1.7) admits a unique global weakly discontinuous solution $u = u(t,x)$ containing $n-p+1$ weak discontinuities $x = x_k(t)$ $(k=p,\dots,n)$, where $x = x_k(t)$ $(x_k(0)=0)$ denotes a $k$-th weak discontinuity passing through the origin $(0,0)$, if and only if system (1.1)* *is weakly linearly degenerate. Precisely speaking, the solution $u = u(t,x)$ has the following structure:*

$$u = u(t,x) = \begin{cases} u^{(p-1)}(t,x), (t,x) \in R_{p-1}, \\ u^{(l)}(t,x), (t,x) \in R_l \quad (l = p, \dots, n-1), \\ u^{(n)}(t,x), (t,x) \in R_n, \end{cases} \tag{1.18}$$

*in which $u^{(l)}(t,x) \in C^1$ satisfies system (1.1) in the classical sense on $R_l$ $(l = p-1, \dots, n)$ with*

$$R_l = \begin{cases} \{(t,x) \mid t \geq 0,\ x \leq x_p(t)\} \quad (l = p-1), \\ \{(t,x) \mid t \geq 0,\ x_l(t) \leq x \leq x_{l+1}(t)\} \quad (l = p, \dots, n-1), \\ \{(t,x) \mid t \geq 0,\ x \geq x_n(t)\} \quad (l = n). \end{cases} \tag{1.19}$$

*Moreover, for $k = p, \dots, n$,*

$$u^{(k-1)}(t, x_k(t)) = u^{(k)}(t, x_k(t)), \tag{1.20}$$

$$\frac{\mathrm{d}x_k(t)}{\mathrm{d}t} = \lambda_k(u^{(k-1)}(t, x_k(t))) = \lambda_k(u^{(k)}(t, x_k(t))). \tag{1.21}$$

**Remark 1.1** In Theorem 1.1 some weak discontinuities may degenerate.

## 2. Decomposition of waves

In this section, we will derive a more refined formula on decomposition of waves. To our knowledge, the decomposition of waves is due to Liu [13] to study the formation of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations.

For $i = 1, \dots, n$, let

$$w_i = l_i(u)u_x \tag{2.1}$$

and

$$\beta_i(u) = l_i(u)B(u). \tag{2.2}$$

By (1.4), it is easy to get

$$u_x = \sum_{k=1}^{n} w_k r_k(u) \tag{2.3}$$

and

$$B(u) = \sum_{k=1}^{n} \beta_k(u) r_k(u). \tag{2.4}$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \ (i = 1, \dots, n) \tag{2.5}$$

denote the directional derivative with respect to $t$ along the $i$-th characteristic $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i(u)$. We have

$$\frac{\mathrm{d}u}{\mathrm{d}_i t} = \sum_{k \neq i} (\lambda_i(u) - \lambda_k(u)) w_k r_k(u) + B(u). \tag{2.6}$$

Then, in normalized coordinates (if any!), it is easy to get

$$\frac{\mathrm{d}u_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} \rho_{ijk}(u) u_j w_k + \sum_{j=1}^{n} \Big( \sum_{k=1}^{n} \bar{\rho}_{ijk}(u) \beta_k(u) \Big) u_j + r_{ii}(u)\beta_i(u), \tag{2.7}$$

where

$$\rho_{ijk}(u)=\begin{cases} ll\rho^{(1)}_{ijk}(u), & \text{when } i=1,\dots,p,\\ \rho^{(1)}_{ijk}(u)+\rho^{(2)}_{ijk}(u), & \text{when } i=p+1,\dots,n \end{cases} \tag{2.8}$$

with

$$\rho^{(1)}_{ijk}(u)=\begin{cases} (\lambda_i(u)-\lambda_k(u))\displaystyle\int_0^1 \frac{\partial r_{ki}}{\partial u_j}(\tau u_1,\dots,\tau u_{k-1},u_k,\tau u_{k+1},\dots,\tau u_n)\mathrm{d}\tau,\\ \qquad\qquad j=1,\dots,n,\ k=p+1,\dots,n \text{ and } j\neq k,\\ 0, \qquad\qquad \text{otherwise} \end{cases} \tag{2.9}$$

and

$$\rho^{(2)}_{ijk}(u)=\begin{cases} (\lambda_i(u)-\lambda_k(u))\displaystyle\int_0^1 \frac{\partial r_{ki}}{\partial u_j}(u_1,\dots,u_p,\tau u_{p+1},\dots,\dots,\tau u_n)\mathrm{d}\tau,\\ \qquad\qquad j=p+1,\dots,n \text{ and } k=1,\dots,p,\\ 0, \qquad\qquad \text{otherwise,} \end{cases} \tag{2.10}$$

$$\bar\rho_{ijk}(u)=\begin{cases} \rho^{(3)}_{ijk}(u), & \text{when } i=1,\dots,p,\\ \rho^{(4)}_{ijk}(u), & \text{when } i=p+1,\dots,n \end{cases} \tag{2.11}$$

with

$$\rho^{(3)}_{ijk}(u)=\begin{cases} \displaystyle\int_0^1 \frac{\partial r_{ki}}{\partial u_j}(u_1,\dots,u_p,\tau u_{p+1},\dots,\tau u_n)\mathrm{d}\tau,\\ \qquad\qquad j=p+1,\dots,n, k=1,\dots,p \text{ and } k\neq i,\\ \displaystyle\int_0^1 \frac{\partial r_{ki}}{\partial u_j}(\tau u_1,\dots,\tau u_{k-1},u_k,\tau u_{k+1},\dots,\tau u_n)\mathrm{d}\tau,\\ \qquad\qquad j=1,\dots,n,\ k=p+1,\dots,n \text{ and } j\neq k,\\ 0, \qquad\qquad \text{otherwise} \end{cases} \tag{2.12}$$

and

$$\rho^{(4)}_{ijk}(u)=\begin{cases} \displaystyle\int_0^1 \frac{\partial r_{ki}}{\partial u_j}(u_1,\dots,u_p,\tau u_{p+1},\dots,\tau u_n)\mathrm{d}\tau,\\ \qquad j=p+1,\dots,n, k=1,\dots,p,\\ \displaystyle\int_0^1 \frac{\partial r_{ki}}{\partial u_j}(\tau u_1,\dots,\tau u_{k-1},u_k,\tau u_{k+1},\dots,\tau u_n)\mathrm{d}\tau,\\ \qquad j=1,\dots,n,\ k=p+1,\dots,n, k\neq i \text{ and } j\neq k,\\ 0, \qquad\qquad \text{otherwise} \end{cases} \tag{2.13}$$

Obviously,

$$\rho_{iji}(u)\equiv 0, \quad \forall\ i,j, \tag{2.14}$$

$$\rho_{ijk}(u)\equiv 0,\ \forall\ i,\ \forall\ j,k\in\{1,\dots,p\} \tag{2.15}$$

and

$$\bar\rho_{ijj}(u)\equiv 0, \quad \forall\ i,j. \tag{2.16}$$

Noting (2.3) and (2.7), we have

$$d[u_i(\mathrm{d}x-\lambda_i(u)\mathrm{d}t)]=\Big[\sum_{j,k=1}^n F_{ijk}(u)u_jw_k+\sum_{j=1}^n\Big(\sum_{k=1}^n\bar\rho_{ijk}(u)\beta_k(u)\Big)u_j+r_{ii}(u)\beta_i(u)\Big]\mathrm{d}t\wedge\mathrm{d}x, \tag{2.17}$$

where

$$F_{ijk}(u) = \rho_{ijk}(u) + \nabla\lambda_j(u) r_k(u)\delta_{ij}. \tag{2.18}$$

By (2.14), we have

$$F_{ijj}(u) \equiv 0, \quad \forall\ j \neq i \tag{2.19}$$

and

$$F_{iii}(u) = \nabla\lambda_i(u) r_i(u), \quad \forall\ i. \tag{2.20}$$

By (2.15), we get

$$F_{ijk}(u) = \nabla\lambda_i(u) r_k(u)\delta_{ij}, \quad \forall\ i, \forall\ j,k \in \{1,\dots,p\}. \tag{2.21}$$

Hence, when $\lambda_i(u)$ is weakly linearly degenerate, in normalized coordinates, from (1.13), (1.14), (1.16) and (1.17) it follows that

$$\begin{gathered} F_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) = \nabla\lambda\Big(\sum_{h=1}^{p} u_h e_h\Big) r_k\Big(\sum_{h=1}^{p} u_h e_h\Big)\delta_{ij} \equiv 0, \\ \forall\ i,j,k \in \{1,\dots,p\},\ \forall\ |u_h| \text{ small } (h = 1,\dots,p) \end{gathered} \tag{2.22}$$

and

$$F_{iii}(u_i e_i) = \nabla\lambda_i(u_i e_i) r_i(u_i e_i) \equiv 0, \quad \forall\ |u_i| \text{ small } (i = p+1,\dots,n). \tag{2.23}$$

On the other hand, we have [1, 15]

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} B_{ijk}(u)\beta_k(u) + \nu_{ij}(u)\Big) w_j, \tag{2.24}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla\lambda_k(u) r_j(u)\delta_{ik} + (j|k)\}, \tag{2.25}$$

$$B_{ijk}(u) = -l_i(u)\nabla r_j(u) r_k(u) \tag{2.26}$$

and

$$\nu_{ij}(u) = l_i(u)\nabla B(u) r_j(u), \tag{2.27}$$

in which $(j|k)$ stands for all terms obtained by changing $j$ and $k$ in the previous terms. Hence

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\ j \neq i \tag{2.28}$$

and

$$\gamma_{iii}(u) = -\nabla\lambda_i(u) r_i(u), \quad \forall\ i. \tag{2.29}$$

Moreover, we have

$$\gamma_{ijk}(u) \equiv 0, \quad \forall\ i \in \{p+1,\dots,n\},\ \forall\ j,k \in \{1,\dots,p\}. \tag{2.30}$$

Furthermore, when $\lambda_i(u)$ is weakly linearly degenerate, in normalized coordinates it follows from (1.13), (1.14), (1.16) and (1.17) that

$$\gamma_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\ i,j,k \in \{1,\dots,p\},\ \forall\ |u_h| \text{ small } (h = 1,\dots,p) \tag{2.31}$$

and

$$\gamma_{iii}(u_i e_i) \equiv 0, \quad \forall\ |u_i| \text{ small } (i = p+1, \ldots, n). \tag{2.32}$$

Noting (2.3), by (2.24) we have

$$d[w_i(\mathrm{d}x - \lambda_i(u)\mathrm{d}t)] = \Big[\sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n}\Big(\sum_{k=1}^{n} B_{ijk}(u)\beta_k(u) + \nu_{ij}(u)\Big) w_j\Big] \mathrm{d}t \wedge \mathrm{d}x, \tag{2.33}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u)) l_i(u)[\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)]. \tag{2.34}$$

Obviously,

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall\ i, j \tag{2.35}$$

and

$$\Gamma_{ijk}(u) \equiv 0, \quad \forall\ i,\ \forall\ j, k \in \{1, \ldots, p\}. \tag{2.36}$$

To simplify (2.7), (2.17), (2.24) and (2.33), similarly to the proof of Lemma 2.1 in [1], we can prove the following lemma, which plays an important role in the proof of Lemma 3.2.

**Lemma 2.1** *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, $B(u) \in C^2$ satisfies the matching condition. Then, in normalized coordinates, $\forall\ |u|$ small, $\forall\ i$, we have*

$$\sum_{j=1}^{n}\Big(\sum_{k=1}^{n} \bar{\rho}_{ijk}(u)\beta_k(u)\Big) u_j + r_{ii}(u)\beta_i(u) = \sum_{j,k=1}^{n} P_{ijk}(u) u_j u_k \tag{2.37}$$

*and*

$$\sum_{j=1}^{n}\Big(\sum_{k=1}^{n} B_{ijk}(u)\beta_k(u) + \nu_{ij}(u)\Big) w_j = \sum_{j,k=1}^{n} Q_{ijk}(u) u_k w_j, \tag{2.38}$$

*where $P_{ijk}(u)$ and $Q_{ijk}(u)$ are continuous functions of $u$ in a neighbourhood of $u = 0$. Moreover, for $i = 1, \ldots, p$, we have*

$$P_{ijk}(u) \equiv 0, \quad \forall\ |u| \text{ small}, \quad \forall\ j, k \in \{1, \ldots, p\}, \tag{2.39}$$

*and*

$$Q_{ijk}(u) \equiv 0, \quad \forall\ |u| \text{ small}, \quad \forall\ j, k \in \{1, \ldots, p\}; \tag{2.40}$$

*while for $i = 1, \ldots, n$, there hold*

$$P_{ijj}(u) \equiv 0, \quad \forall\ |u| \text{ small}, \quad \forall\ j \in \{1, \ldots, n\} \tag{2.41}$$

*and*

$$Q_{ijj}(u) \equiv 0, \quad \forall\ |u| \text{ small}, \quad \forall\ j \in \{1, \ldots, n\}. \tag{2.42}$$

## 3. Proof of Theorem 1.1

The main result in this paper can be proved in a way similar to the proof of Theorem 1.1 in [1]. Here we point out only the essentially different part.

Noting (1.10), there exist positive constants $\delta$ and $\delta_0$ so small that

$$\lambda_{i+1}(u) - \lambda_i(u') \geq 2\delta_0, \quad \forall\ |u|,\ |u'| \leq \delta \quad (i = p, \ldots, n-1) \tag{3.1}$$

and

$$|\lambda_i(u) - \lambda_i(u')| \leq \frac{\delta_0}{2}, \quad \forall\ |u|,\ |u'| \leq \delta \quad (i = 1, \ldots, n). \tag{3.2}$$

Without loss of generality, we may assume that

$$\lambda_i(0) > \delta_0 \quad (i = 1, \ldots, n). \tag{3.3}$$

For the time being we assume that on any given existence domain $R(T) = \{(t,x) \mid 0 \leq t \leq T, -\infty < x < \infty\}$ of the weakly discontinuous solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7), we have

$$|u(t,x)| \leq \delta, \quad \forall\ (t,x) \in R(T). \tag{3.4}$$

In the proof of Theorem 1.1, we will explain that this hypothesis is reasonable.

Let

$$R_l(T) = \begin{cases} \{(t,x) \mid 0 \leq t \leq T,\ x \leq x_p(t)\} \quad (l = p-1), \\ \{(t,x) \mid 0 \leq t \leq T,\ x_l(t) \leq x \leq x_{l+1}(t)\} \quad (l = p, \ldots, n-1), \\ \{(t,x) \mid 0 \leq t \leq T,\ x \geq x_n(t)\} \quad (l = n) \end{cases}$$

and

$$D_i^T = \begin{cases} \{(t,x) \mid 0 \leq t \leq T,\ x \leq (\lambda(0) + \delta_0)t\} \quad (i = 1, \ldots, p), \\ \{(t,x) \mid 0 \leq t \leq T,\ (\lambda_i(0) - \delta_0)t \leq x \leq (\lambda_i(0) + \delta_0)t\} \quad (i = p+1, \ldots, n-1), \\ \{(t,x) \mid 0 \leq t \leq T,\ x \geq (\lambda_n(0) - \delta_0)t\} \quad (i = n). \end{cases}$$

Obviously,

$$D_1^{\mathrm{T}} = \cdots = D_p^{\mathrm{T}}$$

and

$$\bigcup_{i=1}^n D_i^{\mathrm{T}} \subset R(T).$$

Let

$$w^{(l)} = (w_1^{(l)}, \ldots, w_n^{(l)}) \quad (l = p-1, \ldots, n)$$

with

$$w_i^{(l)} = l_i(u^{(l)}) u_x^{(l)} \quad (i = 1, \ldots, n),$$

$$W_\infty^c(T) = \max\Big\{ \max_{i=1,\ldots,p} \max_{l=p,\ldots,n} \sup_{(t,x)\in R_l(T)\setminus D_i^{\mathrm{T}}} \{(1 + |x - \lambda_i(0)t|)^{1+\mu} |w_i^{(l)}(t,x)|\},$$
$$\max_{i=p+1,\ldots,n} \max_{l=p-1,\ldots,n} \sup_{(t,x)\in R_l(T)\setminus D_i^{\mathrm{T}}} \{(1 + |x - \lambda_i(0)t|)^{1+\mu} |w_i^{(l)}(t,x)|\}\Big\},$$

$$\widetilde{W}_1(T) = \max\Big\{ \max_{i=1,\ldots,p} \max_{j=p+1,\ldots,n} \{\sup_{c_j} \int_{c_j\cap R_{p-1}(T)} |w_i^{(p-1)}(t,x)| \mathrm{d}t + \sup_{c_j} \int_{c_j\cap R_p(T)} |w_i^{(p)}(t,x)| \mathrm{d}t\},$$
$$\max_{i=p+1,\ldots,n} \max_{j\neq i} \{\sup_{c_j} \int_{c_j\cap R_{i-1}(T)} |w_i^{(i-1)}(t,x)| \mathrm{d}t + \sup_{c_j} \int_{c_j\cap R_i(T)} |w_i^{(i)}(t,x) \mathrm{d}t\}\Big\},$$

where $c_j$ denotes any given $j$-th characteristic on $D_i^{\mathrm{T}}$,

$$W_1(T)=\max\Big\{\max_{i=1,\dots,p}\sup_{0\le t\le T}\Big\{\int_{a(t)}^{x_p(t)}|w_i^{(p-1)}(t,x)|\mathrm{d}x+\int_{x_p(t)}^{b(t)}|w_i^{(p)}(t,x)|\mathrm{d}x\Big\},$$
$$\max_{i=p+1,\dots,n}\sup_{0\le t\le T}\Big\{\int_{a(t)}^{x_i(t)}|w_i^{(i-1)}(t,x)|\mathrm{d}x+\int_{x_i(t)}^{b(t)}|w_i^{(i)}(t,x)|\mathrm{d}x\Big\}\Big\},$$

where

$$a(t)=\begin{cases}-\infty, \ \text{if } i=1,\dots,p,\\ (\lambda_i(0)-\delta_0)t, \ \text{if } i=p+1,\dots,n,\end{cases}$$

$$b(t)=\begin{cases}(\lambda_i(0)+\delta_0)t, \ \text{if } i=1,\dots,n-1,\\ +\infty, \ \text{if } i=n\end{cases}$$

and

$$U_\infty(T)=\|u(t,x)\|_{L^\infty(R(T))},$$

$$W_\infty(T)=\sum_{l=p-1}^{n}\|w^{(l)}(t,x)\|_{L^\infty(R_l(T))}.$$

Similarly, we can define $U_\infty^c(T),\widetilde{U}_1(T)$ and $U_1(T)$.

**Lemma 3.1** ([9]) *On the $p$-th weak discontinuity $x=x_p(t)$, we have*

$$w_i^{(p-1)}=w_i^{(p)}\ (i=p+1,\dots,n); \tag{3.5}$$

*while on the $k$-th weak discontinuity $x=x_k(t)$ $(k=p+1,\dots,n)$, we have*

$$w_i^{(k-1)}=w_i^{(k)}\ (i=1,\dots,k-1,k+1,\dots,n). \tag{3.6}$$

**Lemma 3.2** *Suppose that in a neighbourhood of $u=0$, $A(u)\in C^2$, system (1.1) is weakly linearly degenerate, and $B(u)\in C^2$ satisfies the matching condition. Suppose furthermore that the initial data satisfy (1.9). Suppose finally that there exist normalized coordinates. Then, in normalized coordinates there exists $\theta_0>0$ so small that for any given $\theta\in(0,\theta_0]$, we have the following uniform a priori estimates on $R(T)$:*

$$W_\infty^c(T)\le\kappa_1\theta, \tag{3.7}$$

$$\widetilde{W}_1(T),\ W_1(T)\le\kappa_2\theta, \tag{3.8}$$

$$U_\infty^c(T)\le\kappa_3\theta \tag{3.9}$$

*and*

$$\widetilde{U}_1(T),\ U_1(T)\le\kappa_4\theta, \tag{3.10}$$

*where $\kappa_i$ $(i=1,2,3,4)$ are positive constants independent of $\theta$ and $T$.*

**Proof** For simplicity and without loss of generality, in the sequel we assume $u=(u_1,\dots,u_n)^{\mathrm{T}}$ are normalized variables.

We first estimate $W_\infty^c(T)$.

For any given $i \in \{1, \ldots, n\}$, passing through any fixed point $A(t,x) \in R(T) \setminus D_i^{\mathrm{T}}$, we draw the $i$-th characteristic $c_i: \ \xi = \xi(\tau)$ $(\tau \le t)$ which intersects the $x$-axis at a point $(0, x_{i0})$. When $A \in R_l(T) \setminus D_i^{\mathrm{T}}$ $(i \in \{1, \ldots, p\}, \ l \in \{p, \ldots, n-1\})$, by (2.38), integrating (2.24) along $c_i$ from 0 to $t$ and noting (2.28) and (3.6) gives

$$w_i^{(l)}(t,x) = I_1^c + I_2^c, \tag{3.11}$$

where

$$\begin{aligned} I_1^c =& w_i^{(n)}(0, x_{i0}) + \int_0^{t_{in}} \sum_{j,k=1}^n \gamma_{ijk}(u^{(n)}) w_j^{(n)} w_k^{(n)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \sum_{m=l+2}^n \int_{t_{im}}^{t_{i,m-1}} \sum_{j,k=1}^n \gamma_{ijk}(u^{(m-1)}) w_j^{(m-1)} w_k^{(m-1)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \int_{t_{i,l+1}}^t \sum_{j,k=1}^n \gamma_{ijk}(u^{(l)}) w_j^{(l)} w_k^{(l)}(\tau, \xi_i(\tau)) \mathrm{d}\tau \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} I_2^c =& \int_0^{t_{in}} \sum_{j,k=1}^n Q_{ijk}(u^{(n)}) w_j^{(n)} u_k^{(n)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \sum_{m=l+2}^n \int_{t_{im}}^{t_{i,m-1}} \sum_{j,k=1}^n Q_{ijm}(u^{(m-1)}) w_j^{(m-1)} u_k^{(m-1)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \int_{t_{i,l+1}}^t \sum_{j,k=1}^n Q_{ijk}(u^{(l)}) w_j^{(l)} u_k^{(l)}(\tau, \xi_i(\tau)) \mathrm{d}\tau; \end{aligned} \tag{3.13}$$

while when $A \in R_n(T) \setminus D_i^{\mathrm{T}} (i \in \{1, \ldots, p\})$, we have

$$\begin{aligned} w_i^{(n)}(t,x) =& w_i^{(n)}(0, x_{i0}) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u^{(n)}) w_j^{(n)} w_k^{(n)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \int_0^t \sum_{j,k=1}^n Q_{ijk}(u^{(n)}) w_j^{(n)} u_k^{(n)}(\tau, \xi_i(\tau)) \mathrm{d}\tau, \end{aligned} \tag{3.14}$$

here and hereafter, $(t_{im}, x_m(t_{im}))$ denotes the intersection point of $c_i$ with the $m$-th weak discontinuity $x = x_m(t)$ $(m = p, \ldots, n)$. Then noting (2.40), (2.42), (3.4) and $|\xi_i(\tau) - \lambda_j(0)\tau| \ge \delta_0 \tau$ when $(\tau, \xi(\tau)) \bar{\in} D_j^T$, by using Lemma 3.2 in [9] and the estimate of $I_1^c$ in [2] we find

$$\begin{aligned} (1 + |x - \lambda_i(0)t|)^{1+\mu} |w_i^{(l)}(t,x)| \le& C\{\theta + W_\infty^c(T)\widetilde{W}_1(T) + (W_\infty^c(T))^2 + \\ & \widetilde{U}_1(T) W_\infty^c(T) + U_\infty^c(T)\widetilde{W}_1(T) + U_\infty^c(T) W_\infty^c(T)\}, \end{aligned} \tag{3.15}$$

here and henceforth, $C$ denotes a different positive constant independent of $\theta$ and $T$.

On the other hand, when $A \in R_l(T) \setminus D_i^T$ $(i \in \{p+1, \ldots, n\}, p \le l < i)$, noting (3.5), (3.6) and integrating (2.24) from 0 to $t$ yields

$$w_i^{(l)}(t,x) = \tilde{I}_1^c + \tilde{I}_2^c, \tag{3.16}$$

where

$$\begin{aligned}\tilde{I}_1^c =& w_i^{(p-1)}(0,x_{i0})+\int_0^{t_{ip}}\sum_{j,k=1}^n \gamma_{ijk}(u^{(p-1)})w_j^{(p-1)}w_k^{(p-1)}(\tau,\xi_i(\tau))\mathrm{d}\tau+\\ &\sum_{m=p}^{l-1}\int_{t_{im}}^{t_{i,m+1}}\sum_{j,k=1}^n \gamma_{ijk}(u^{(m)})w_j^{(m)}w_k^{(m)}(\tau,\xi_i(\tau))\mathrm{d}\tau+\\ &\int_{t_{il}}^{t}\sum_{j,k=1}^n \gamma_{ijk}(u^{(l)})w_j^{(l)}w_k^{(l)}(\tau,\xi_i(\tau))\mathrm{d}\tau \end{aligned}\tag{3.17}$$

and

$$\begin{aligned}\tilde{I}_2^c =& \int_0^{t_{ip}}\sum_{j,k=1}^n Q_{ijk}(u^{(p-1)})w_j^{(p-1)}u_m^{(p-1)}(\tau,\xi_i(\tau))\mathrm{d}\tau+\\ &\sum_{m=p}^{l-1}\int_{t_{im}}^{t_{i,m+1}}\sum_{j,k=1}^n Q_{ijm}(u^{(m)})w_j^{(m)}u_k^{(m)}(\tau,\xi_i(\tau))\mathrm{d}\tau+\\ &\int_{t_{il}}^{t}\sum_{j,k=1}^n Q_{ijk}(u^{(l)})w_j^{(l)}u_k^{(l)}(\tau,\xi_i(\tau))\mathrm{d}\tau; \end{aligned}\tag{3.18}$$

when $A\in R_{p-1}(T)\setminus D_i^T$ $(i\in\{p+1,\dots,n\})$, we obtain

$$\begin{aligned}w_i^{(p-1)}(t,x) =& w_i^{(p-1)}(0,x_{i0})+\int_0^{t}\sum_{j,k=1}^n \gamma_{ijk}(u^{(p-1)})w_j^{(p-1)}w_k^{(p-1)}(\tau,\xi_i(\tau))\mathrm{d}\tau+\\ &\int_0^{t}\sum_{j,k=1}^n Q_{ijk}(u^{(p-1)})w_j^{(p-1)}u_k^{(p-1)}(\tau,\xi_i(\tau))\mathrm{d}\tau. \end{aligned}\tag{3.19}$$

In these two cases, noting (2.28), (2.30) and (2.42), we can get (3.15) similarly.

While when $A\in R_l(T)\setminus D_i^{\mathrm{T}}$ $(i\in\{p+1,\dots,n\}, l\ge i)$, noting (2.28) and (3.6), (3.11) still holds. Note when $i=n$, by the definition of $D_n^{\mathrm{T}}$, (3.11) disappears. In this case, we deduce (3.15) similarly.

Thus, we have

$$\begin{aligned}W_\infty^c(T) \le& C\{\theta+W_\infty^c(T)\widetilde{W}_1(T)+(W_\infty^c(T))^2+\\ &\widetilde{U}_1(T)W_\infty^c(T)+U_\infty^c(T)\widetilde{W}_1(T)+U_\infty^c(T)W_\infty^c(T)\}. \end{aligned}\tag{3.20}$$

We now estimate $\widetilde{W}_1(T)$.

For $i=1,\dots,p$, passing through any given point $A(t,x)\in D_i^T\bigcap R_p(T)$, we draw the $j$-th characteristic $c_j:\xi=\xi_j(\tau)$ $(j>p,\ \tau\le t)$, which intersects the $p$-th weak discontinuity $x=x_p(t)$ at a point $B(t_B,x_B)$. In the meantime, the $i$-th characteristic $c_i:\xi=\xi_i(\tau)$ $(\tau\le t)$ passing through $A$ intersects the boundary $x=(\lambda(0)+\delta_0)t$ of $D_i^{\mathrm{T}}$ at a point $C(t_C,x_C)$. By (2.33) and (2.38), using Stokes' formula on the domain $ABOC$, we get

$$\begin{aligned}&\int_{t_B}^{t}|w_i^{(p)}\big(\lambda_j(u^{(p)})-\lambda(u^{(p)})\big)(\tau,\xi_j(\tau))|\mathrm{d}\tau\\ \le& \int_{OC}|w_i^{(p)}\big(\lambda(0)+\delta_0-\lambda(u^{(p)})\big)(\tau,(\lambda(0)+\delta_0)\tau)|\mathrm{d}\tau+\end{aligned}$$

$$\iint_{ABOC}\Big|\sum_{j,k=1}^{n}\Gamma_{ijk}(u^{(p)})w_j^{(p)}w_k^{(p)}(t,x)\Big|\mathrm{d}t\mathrm{d}x+$$
$$\iint_{ABOC}\Big|\sum_{j,k=1}^{n}Q_{ijk}(u^{(p)})w_j^{(p)}u_k^{(p)}(t,x)\Big|\mathrm{d}t\mathrm{d}x. \tag{3.21}$$

In view of (2.40), (2.42) and (3.4), the third term on the right hand of the above inequality can be rewritten as

$$\iint_{ABOC}\Big|\sum_{j,k=1}^{n}Q_{ijk}(u^{(p)})w_j^{(p)}w_k^{(p)}(t,x)\Big|\mathrm{d}t\mathrm{d}x \tag{3.22}$$
$$=\iint_{ABOC}\Big|\Big(\sum_{\substack{j\in\{1,\dots,p\}\\ k\in\{p+1,\dots,n\}}}+\sum_{\substack{j\in\{p+1,\dots,n\}\\ k\in\{1,\dots,p\}}}+\sum_{\substack{j,k\in\{p+1,\dots,n\}\\ j\neq k}}\Big)Q_{ijk}(u^{(p)})w_j^{(p)}w_k^{(p)}(t,x)\Big|\mathrm{d}t\mathrm{d}x.$$

Then noting (3.1), (3.4) and the estimate of the first and second terms on the right hand side of (3.21), from (3.21) and (3.22) it follows that

$$\int_{c_j}|w_i^{(p)}|\mathrm{d}\tau=\int_{t_B}^{t}|w_i^{(p)}(\tau,\xi_j(\tau))|\mathrm{d}\tau\le C\{W_\infty^c(T)+W_\infty^c(T)W_1(T)+(W_\infty^c(T))^2+$$
$$U_\infty^c(T)W_1(T)+U_1(T)W_\infty^c(T)+U_\infty^c(T)W_\infty^c(T)\} \tag{3.23}$$

by Lemma 3.2 in [9].

For $i=p+1,\dots,n-1$, passing through any given point $A(t,x)\in D_i^{\mathrm{T}}\bigcap R_i(T)$, we draw the $j$-th characteristic $c_j:\xi=\xi_j(\tau)$ $(\tau\le t)$. When $j>i$, $c_j$ intersects the $i$-th weak discontinuity $x=x_i(t)$ at a point $B(t_B,x_B)$; while when $j<i$, $c_j$ intersects the boundary $x=(x_i(0)+\delta_0)t$ of the domain $D_i^{\mathrm{T}}$ at a point $\widetilde{B}(t_{\widetilde{B}},x_{\widetilde{B}})$. In the meantime, the $i$-th characteristic $c_i:\xi=\xi_i(\tau)$ $(\tau\le t)$ passing through $A$ intersects the boundary $x=(\lambda_i(0)+\delta_0)t$ of $D_i^{\mathrm{T}}$ at a point $C(t_C,x_C)$. Thanks to (2.35), (2.42), (3.1) and (3.4), using Stokes' formula on the domain $ABOC$ or $AC\tilde{B}$, by Lemma 3.2 in [9] we obtain

$$\int_{c_j}|w_i^{(i)}|\mathrm{d}\tau=\int_{t_B(\text{or } t_{\tilde{B}})}^{t}|w_i^{(i)}(\tau,\xi_j(\tau))|\mathrm{d}\tau\le C\{W_\infty^c(T)+W_\infty^c(T)W_1(T)+(W_\infty^c(T))^2+$$
$$U_\infty^c(T)W_1(T)+U_1(T)W_\infty^c(T)+U_\infty^c(T)W_\infty^c(T)\}. \tag{3.24}$$

For $i=n$, passing through any given point $A(t,x)\in D_n^T\bigcap R_n(T)$, both the $j$-th characteristic $c_j:\xi=\xi_j(\tau)$ $(\tau\le t,\ j<n)$ and the $n$-th characteristic $c_n:\xi=\xi_n(\tau)$ $(\tau\le t)$ intersects the $x$-axis at points $B(0,x_B)$ and $C(0,x_C)$ respectively. By involving Stokes' formula on the domain $ACB$, similarly we have

$$\int_{c_j}|w_n^{(n)}|\mathrm{d}\tau\le C\{W_\infty^c(T)+W_\infty^c(T)W_1(T)+(W_\infty^c(T))^2+$$
$$U_\infty^c(T)W_1(T)+U_1(T)W_\infty^c(T)+U_\infty^c(T)W_\infty^c(T)\}. \tag{3.25}$$

On the other hand, we can similarly estimate

$$\int_{c_j\cap R_{p-1}(T)}|w_i^{(p-1)}(t,x)|\mathrm{d}t\quad(i=1,\dots,p,\ j=p+1,\dots,n) \tag{3.26}$$

and

$$\int_{c_j \cap R_{i-1}(T)} |w_i^{(i-1)}(t,x)| \mathrm{d}t \quad (i = p+1, \ldots, n,\ j \neq i). \tag{3.27}$$

Therefore, we infer that

$$\begin{aligned}\widetilde{W}_1(T) \leq & C\{W_\infty^c(T) + W_\infty^c(T)W_1(T) + (W_\infty^c(T))^2 + \\ & U_\infty^c(T)W_1(T) + U_1(T)W_\infty^c(T) + U_\infty^c(T)W_\infty^c(T)\}.\end{aligned} \tag{3.28}$$

Similarly, we get

$$\begin{aligned}W_1(T) \leq & C\{W_\infty^c(T) + W_\infty^c(T)W_1(T) + (W_\infty^c(T))^2 + \\ & U_\infty^c(T)W_1(T) + U_1(T)W_\infty^c(T) + U_\infty^c(T)W_\infty^c(T)\}.\end{aligned} \tag{3.29}$$

We now estimate $U_\infty^c(T)$.

When $A(t,x) \in R_l(T) \setminus D_i^{\mathrm{T}}$ $(i \in \{1, \ldots, p\},\ l \in \{p, \ldots, n-1\})$, integrating (2.7) along $c_i$ from 0 to $t$ and noting (2.37) gives

$$u_i^{(l)}(t,x) = J_1^c + J_2^c, \tag{3.30}$$

where $c_i$ is the $i$-th characteristic passing through the point $A$, and

$$\begin{aligned}J_1^c = & u_i^{(n)}(0, x_{i0}) + \int_0^{t_{in}} \sum_{j,k=1}^n \rho_{ijk}(u^{(n)}) u_j^{(n)} w_k^{(n)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \sum_{m=l+2}^n \int_{t_{im}}^{t_{i,m-1}} \sum_{j,k=1}^n \rho_{ijk}(u^{(m-1)}) u_j^{(m-1)} w_k^{(m-1)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \int_{t_{i,l+1}}^t \sum_{j,k=1}^n \rho_{ijk}(u^{(l)}) u_j^{(l)} w_k^{(l)}(\tau, \xi_i(\tau)) \mathrm{d}\tau\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}J_2^c = & \int_0^{t_{in}} \sum_{j,k=1}^n P_{ijk}(u^{(n)}) u_j^{(n)} u_k^{(n)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \sum_{m=l+2}^n \int_{t_{im}}^{t_{i,m-1}} \sum_{j,k=1}^n P_{ijm}(u^{(m-1)}) u_j^{(m-1)} u_k^{(m-1)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \int_{t_{i,l+1}}^t \sum_{j,k=1}^n P_{ijk}(u^{(l)}) u_j^{(l)} u_k^{(l)}(\tau, \xi_i(\tau)) \mathrm{d}\tau;\end{aligned} \tag{3.32}$$

while when $A \in R_n(T) \setminus D_i^{\mathrm{T}} (i \in \{1, \ldots, p\})$, we have

$$\begin{aligned}u_i^{(n)}(t,x) = & u_i^{(n)}(0, x_{i0}) + \int_0^t \sum_{j,k=1}^n \rho_{ijk}(u^{(n)}) u_j^{(n)} u_k^{(n)}(\tau, \xi_i(\tau)) \mathrm{d}\tau + \\ & \int_0^t \sum_{j,k=1}^n P_{ijk}(u^{(n)}) u_j^{(n)} u_k^{(n)}(\tau, \xi_i(\tau)) \mathrm{d}\tau,\end{aligned} \tag{3.33}$$

Using (2.14), (2.15) and (2.39), by an analogous proof to (3.15) we find

$$(1 + |x - \lambda_i(0)t|)^{1+\mu} |u_i^{(l)}(t,x)| \leq C\{\theta + U_\infty^c(T)\widetilde{W}_1(T) + U_\infty^c(T)W_\infty^c(T) +$$

$$\widetilde{U}_1(T)W^c_\infty(T)+U^c_\infty(T)\widetilde{U}_1(T)+(U^c_\infty(T))^2\}. \tag{3.34}$$

On the other hand, when $A(t,x)\in R_l(T)\setminus D_i^{\mathrm{T}}$ ($i\in\{p+1,\ldots,n\}$, $l\in\{p-1,\ldots,n\}$), noting (2.14), (2.15) and (2.41), we can similarly estimate. Thus, we have

$$\begin{aligned}U^c_\infty(T)\le & C\{\theta+U^c_\infty(T)\widetilde{W}_1(T)+U^c_\infty(T)W^c_\infty(T)+\\ & \widetilde{U}_1(T)W^c_\infty(T)+U^c_\infty(T)\widetilde{U}_1(T)+(U^c_\infty(T))^2\}.\end{aligned} \tag{3.35}$$

We now estimate $\widetilde{U}_1(T)$.

For $i=1,\ldots,p$, similarly to (3.21), by (2.17) and noting (2.37), using Stokes' formula on the domain $ABOC$, we get

$$\begin{aligned}&\int_{t_B}^t |u_i^{(p)}\big(\lambda_j(u^{(p)})-\lambda(u^{(p)})\big)(\tau,\xi_j(\tau))|\mathrm{d}\tau\\ \le&\int_{OC}|u_i^{(p)}\big(\lambda(0)+\delta_0-\lambda(u^{(p)})\big)(\tau,(\lambda(0)+\delta_0)\tau)|\mathrm{d}\tau+\\ &\iint_{ABOC}\Big|\sum_{j,k=1}^n F_{ijk}(u^{(p)})u_j^{(p)}w_k^{(p)}(t,x)\Big|\mathrm{d}t\mathrm{d}x+\\ &\iint_{ABOC}\Big|\sum_{j,k=1}^n P_{ijk}(u^{(p)})u_j^{(p)}u_k^{(p)}(t,x)\Big|\mathrm{d}t\mathrm{d}x.\end{aligned} \tag{3.36}$$

Applying (2.39) and (2.41), the third term on the right hand of the above inequality can be rewritten as

$$\begin{aligned}&\iint_{ABOC}\Big|\sum_{j,k=1}^n P_{ijk}(u^{(p)})u_j^{(p)}u_k^{(p)}(t,x)\Big|\mathrm{d}t\mathrm{d}x\\ =&\iint_{ABOC}\Big|\Big(\sum_{\substack{j\in\{1,\ldots,p\}\\ k\in\{p+1,\ldots,n\}}}+\sum_{\substack{j\in\{p+1,\ldots,n\}\\ k\in\{1,\ldots,p\}}}+\sum_{\substack{j,k\in\{p+1,\ldots,n\}\\ j\neq k}}\Big)P_{ijk}(u^{(p)})u_j^{(p)}u_k^{(p)}(t,x)\Big|\mathrm{d}t\mathrm{d}x.\end{aligned} \tag{3.37}$$

Taking into account the estimate of the first and second terms on the right hand side of (3.36), by (3.1), (3.4) and Lemma 3.2 in [9], from (3.36) and (3.37) it follows that

$$\begin{aligned}\int_{c_j}|u_i^{(p)}|\mathrm{d}\tau\le & C\{U^c_\infty(T)+U_1(T)W^c_\infty(T)+U^c_\infty(T)W_1(T)+U^c_\infty(T)W^c_\infty(T)+\\ & U_\infty(T)U^c_\infty(T)W_1(T)+U_1(T)U^c_\infty(T)+(U^c_\infty(T))^2\}.\end{aligned} \tag{3.38}$$

For $i=p+1,\ldots,n$, noting (2.41), we can similarly deduce that

$$\begin{aligned}\int_{c_j}|u_i^{(i)}|\mathrm{d}\tau\le & C\{\theta+U^c_\infty(T)+U_1(T)W^c_\infty(T)+U^c_\infty(T)W_1(T)+U^c_\infty(T)W^c_\infty(T)+\\ & U_\infty(T)U^c_\infty(T)W_1(T)+U_1(T)U^c_\infty(T)+(U^c_\infty(T))^2\}.\end{aligned} \tag{3.39}$$

On the other hand, we can similarly estimate

$$\int_{c_j\cap R_{p-1}(T)}|u_i^{(p-1)}(t,x)|\mathrm{d}t\quad (i=1,\ldots,p,\ j=p+1,\ldots,n) \tag{3.40}$$

and

$$\int_{c_j\cap R_{i-1}(T)}|u_i^{(i-1)}(t,x)|\mathrm{d}t\quad (i=p+1,\ldots,n,\ j\neq i). \tag{3.41}$$

Hence, we get

$$\begin{aligned}\widetilde{U}_1(T) \leq & C\{\theta + U^c_\infty(T) + U_1(T)W^c_\infty(T) + U^c_\infty(T)W_1(T) + U^c_\infty(T)W^c_\infty(T)+ \\ & U_\infty(T)U^c_\infty(T)W_1(T) + U_1(T)U^c_\infty(T) + (U^c_\infty(T))^2\}.\end{aligned} \tag{3.42}$$

By an analogous argument, we can prove

$$\begin{aligned}U_1(T) \leq & C\{\theta + U^c_\infty(T) + U_1(T)W^c_\infty(T) + U^c_\infty(T)W_1(T) + U^c_\infty(T)W^c_\infty(T)+ \\ & U_\infty(T)U^c_\infty(T)W_1(T) + U_1(T)U^c_\infty(T) + (U^c_\infty(T))^2\}.\end{aligned} \tag{3.43}$$

The combination of (3.20), (3.28), (3.29), (3.35), (3.42) and (3.43) gives (3.7)–(3.10) (see [12]). This completes the proof of Lemma 3.2. $\square$

**Proof of Theorem 1.1** To prove the sufficiency part of Theorem 1.1, we only need to estimate $U_\infty(T)$ and $W_\infty(T)$. For any given point $(t,x) \in R(T)$, similarly to [2], by Lemma 3.2 we can get

$$|u(t,x)| \leq C\{\theta + W^c_\infty(T) + \widetilde{\widetilde{W}}_1(T) + U^c_\infty(T) + \widetilde{U}_1(T)\} \leq C\theta. \tag{3.44}$$

This gives the validity of hypothesis (3.4), and

$$\begin{aligned}W_\infty(T) \leq & C\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)W_\infty(T) + U^c_\infty(T)(W_\infty(T))^2+ \\ & U^c_\infty(T)W^c_\infty(T) + U_\infty(T)W^c_\infty(T) + U^c_\infty(T)W_\infty(T)\} \\ \leq & C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\},\end{aligned} \tag{3.45}$$

which implies

$$W_\infty(T) \leq C\theta. \tag{3.46}$$

Finally, we prove the necessity part of Theorem 1.1. In normalized coordinates, by (1.13), for $i = 1, \ldots, p$, there holds

$$a_{ik}\Big(\sum_{h=1}^p u_h e_h\Big) \equiv \begin{cases} \lambda\Big(\sum_{h=1}^p u_h e_h\Big), & k = i, \\ 0, & k \neq i \end{cases} \tag{3.47}$$

and by (1.14), for $i = p+1, \ldots, n$, there holds

$$a_{ik}(u_i e_i) \equiv \begin{cases} \lambda_i(u_i e_i), & k = i, \\ 0, & k \neq i. \end{cases} \tag{3.48}$$

Then similarly to the proof of the necessity part of Theorem 1.1 in [2], noting (1.16) and (1.17), we can prove the necessity part. $\square$

## References


[1] Fei GUO. *Global weakly discontinuous solutions to the Cauchy problem with a kind of non-smooth initial data for inhomogeneous quasilinear hyperbolic systems.* Gongcheng Shuxue Xuebao, 2007, **24**(3): 414–430.

[2] Fei GUO. *Global weakly discontinuous solutions to the Cauchy problem for general quasilinear hyperbolic systems with characteristics with constant multiplicity.* submitted to Appl. Math. J. Chinese Univ. B.

[3] Dexing KONG. *Cauchy Problem for Quasilinear Hyperbolic Systems.* Mathematical Society of Japan, Tokyo, 2000.

[4] Tatsien LI. *Une remarque sur les coordonnées normalisées et ses applications aux systèmes hyperboliques quasi linéaires*. C. R. Acad. Sci. Paris Sér. I Math., 2000, **331**(6): 447–452.

[5] Tatsien LI. *A Remark on the Normalized Coordinates and Its Applications to Quasilinear Hyperbolic Systems*. in Optimal Control and Partial Differential Equations, J. L. Menaldi et al. (eds.), IOS Press, 2001, 181-187.

[6] Tatsien LI. *Global Classical Solutions for Quasilinear Hyperbolic Systems*. John Wiley & Sons, Ltd., Chichester, 1994.

[7] Tatsien LI, Dexing KONG, Yi ZHOU. *Global classical solutions for quasilinear nonstrictly hyperbolic systems*. Nonlinear Stud., 1996, **3**(2): 203–229.

[8] Tatsien LI, Libin WANG. *Global existence of classical solutions to the Cauchy problem on a semi-bounded initial axis for quasilinear hyperbolic systems*. Nonlinear Anal., 2004, **56**(7): 961–974.

[9] Tatsien LI, Libin WANG. *Global existence of weakly discontinuous solutions to the Cauchy problem with a kind of non-smooth initial data for quasilinear hyperbolic systems*. Chinese Ann. Math. Ser. B, 2004, **25**(3): 319–334.

[10] Tatsien LI, Wenci YU. *Boundary Value Problems for Quasilinear Hyperbolic Systems*. Duke University, Mathematics Department, Durham, NC, 1985.

[11] Tatsien LI, Yi ZHOU, Dexing KONG. *Weak linear degeneracy and global classical solutions for general quasilinear hyperbolic systems*. Comm. Partial Differential Equations, 1994, **19**(7-8): 1263–1317.

[12] Tatsien LI, Yi ZHOU, Dexing KONG. *Global classical solutions for general quasilinear hyperbolic systems with decay initial data*. Nonlinear Anal., 1997, **28**(8): 1299–1332.

[13] Taiping LIU. *Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations*. J. Differential Equations, 1979, **33**(1): 92–111.

[14] Libin WANG. *A remark on the Cauchy problem for quasilinear hyperbolic systems with characteristics of constant multiplicity*. J. Fudan Univ. Nat. Sci., 2001, **40**(6): 633–636. (in Chinese)

[15] Peixia WU. *Global classical solutions to Cauchy problem for general first order quasilinear hyperbolic systems*. Chinese Ann. Math. Ser. A, 2006, **27**(1): 93–108.