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Closure Operators and Closure Systems on Quantaloid-Enriched Categories

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Abstract In this paper, we introduce the fundamental notions of closure operator and closure system in the framework of quantaloid-enriched category. We mainly discuss the relationship between closure operators and adjunctions and establish the one-to-one correspondence between closure operators and closure systems on quantaloid-enriched categories.

Keywords quantaloid; enriched category; closure operator; closure system; adjunction.

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1. Introduction

A quantaloid $\mathcal{Q}$ is a category enriched in the closed symmetric monoidal category of suplattices. By a series of papers of Stubbe [16–18], the theory of categories enriched in a quantaloid has been developed. Actually, the study of categories enriched in a quantaloid goes back to a series of work of Rosenthal [12, 14] with the purpose to study automata theory.

Since a quantaloid is a particular bicategory, $\mathcal{Q}$-category theory is a particular case of the theory of "$\mathcal{W}$-category theory" as pioneered by [4, 9, 15, 22]. Nevertheless, as pointed out by Stubbe, this particular case is also of particular interest: many examples of bicategory-enriched categories are really quantaloid-enriched. Also, without becoming trivial, quantaloid enrichment often behaves remarkably better than general bicategory-enrichment: essentially because all diagrams of 2-cells in a quantaloid commute.

On the other hand, since quantaloids are a categorical generalization of quantales [13, 14], $\mathcal{Q}$-categories can be viewed as a generalization of Ω-categories studied by Wagner [20, 21], Lai and Zhang [10]. The main differences between Ω-categories and $\mathcal{Q}$-categories lies in the base structures for enrichment: the former are commutative quantales, categorically speaking, they are symmetric monoidal categories, while the later are quantaloids, categorically speaking, they are bicategories. This difference results in two main differences between Ω-categories and $\mathcal{Q}$-categories, one is that the universe of a $\mathcal{Q}$-category must be a $\mathcal{Q}$-typed set, and the other is that in general it does not make sense for the composition $g \circ f$ of two arrows f, g to be "symmetric"

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in a $\mathcal{Q}$-category. Notably, as a particular case of Ω-category, L-enriched categories (here L is a complete residuated lattice) were studied extensively under the name L-ordered sets [7, 23, 25], with the purpose to provide a framework for quantitative domain theory.

$\mathcal{Q}$-categories have at least two main important applications. One is in theoretical computer science, where they are applied to study fuzzy automata theory [12, 14] and study semantics of programming languages [5, 8, 19–21, 24, 28]. The other is in fuzzy set theory: Zhang [26] used Ω-category to provide a new approach to many valued topology, and more recently he argued that the theory of enriched categories is a usful tool for fuzzy set theorists in [27].

So it is necessary to study the intrinsic qualities of $\mathcal{Q}$-categories. Although many structures related to $\mathcal{Q}$-categories are established in the pioneer work of Stubbe, there are other aspects of $\mathcal{Q}$-categories which need to be explored. As a continuation of Stubbe's work, this paper is devoted to explain the notions of closure operator and closure system in the framework of $\mathcal{Q}$-category. It is well known that closure operators play a significant role in both pure and applied mathematics, which have close relation with another important concept Galois connection [6]. Since such notions as Galois connection are naturally generalized to $\mathcal{Q}$-category, it is natural to ask whether or not we can introduce the concept of closure operator (and some other related concepts) in $\mathcal{Q}$-category. Recently, Bělohlávek studied fuzzy closure and fuzzy interior operators in [1–3], Yao studied kernel systems, Guo studied fuzzy closure systems on L-ordered sets in [7, 25], which really give us inspiration. The difficulty lies in that we must cope with the differences between Ω-categories and $\mathcal{Q}$-categories.

In this paper we will illustrate how to define such notions as closure operator and closure system naturally in $\mathcal{Q}$-categories, so as to smoothly extend the classical results to the new framework. We mainly discuss the relation between closure operators and adjunctions and the relation between closure operators and closure systems.

2. Preliminaries

This paper is based on the theory of categorical structures enriched in a base quantaloid developed by Stubbe. To make this paper reasonably self-contained, we recall some definitions and notations. For details we refer to [14, 16, 17], and we keep all the notations introduced there. For the classical notions of closure operator and closure system on poset please refer to [6], where one can find more applications of them, and for fuzzy versions of such notions and their applications please refer to [1–3, 7, 23, 25].

A quantaloid $\mathcal{Q}$ is a category enriched in the symmetric monoidal closed category **Sup** of complete lattices and morphisms that preserve arbitrary suprema. In elementary terms, a quantaloid $\mathcal{Q}$ is a category whose hom-sets are actually suplattices, in which composition distributes on both sides over arbitrary suprema of morphisms. A homomorphism $F : \mathcal{Q} \to \mathcal{Q}'$ is a functor of (the underlying) categories that preserves arbitrary suprema of morphisms.

In this paper $\mathcal{Q}$ always denotes a small quantaloid, and $\mathcal{Q}_0$ for its set of objects.

Any quantaloid is a closed bicategory: we denote

$$- \circ f \dashv \{f, -\} \text{ and } f \circ - \dashv [f, -]$$

for the adjoints to composition with some morphisms $f : A \to B$ in a quantaloid.

A $\mathcal{Q}$-typed set X is a (small) set X together with a mapping $t : X \to \mathcal{Q}_0 : x \mapsto tx$ sending each element $x \in X$ to its type $tx \in \mathcal{Q}_0$. The notation with a "t" for the types of elements in a $\mathcal{Q}$-typed set is generic.

Definition 2.1 *A $\mathcal{Q}$-enriched category (or $\mathcal{Q}$-category for short) $\mathbb{A}$ consists of*

objects: a $\mathcal{Q}$-typed set $\mathbb{A}_0$,
hom-arrows: for all a, $a' \in \mathbb{A}_0$, an arrow $\mathbb{A}(a', a) : ta \to ta'$ in $\mathcal{Q}$, satisfying
composition-inequalities: for all a, a', $a'' \in \mathbb{A}_0$, $\mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a)$ in $\mathcal{Q}$,
identity-inequalities: for all $a \in \mathbb{A}_0$, $1_{ta} \leq \mathbb{A}(a, a)$ in $\mathcal{Q}$.

Definition 2.2 *A distributor $\Phi : \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{B}$ between two $\mathcal{Q}$-categories is given by*

distributor-arrows: for all $a \in \mathbb{A}_0, b \in \mathbb{B}_0$, an arrow $\Phi(b, a) : ta \to tb$ in $\mathcal{Q}$ satisfying
action-inequalities: for all $a, a' \in \mathbb{A}_0, b, b' \in \mathbb{B}_0$, $\mathbb{B}(b', b) \circ \Phi(b, a) \leq \Phi(b', a)$ and

$$\Phi(b, a) \circ \mathbb{A}(a, a') \leq \Phi(b, a') \text{ in } \mathcal{Q}.$$

Definition 2.3 *A functor $F : \mathbb{A} \to \mathbb{B}$ between $\mathcal{Q}$-categories is*

object-mapping: a map $F : \mathbb{A}_0 \to \mathbb{B}_0 : a \mapsto Fa$ satisfying
type-equalities: for all $a \in \mathbb{A}_0, ta = t(Fa)$ in $\mathcal{Q}$,
action-inequalities: for all $a, a' \in \mathbb{A}_0, \mathbb{A}(a', a) \leq \mathbb{B}(Fa', Fa)$ in $\mathcal{Q}$.

Proposition 2.4 *$\mathcal{Q}$-categories are the objects, and distributors the arrows, of a quantaloid* **Dist**$(\mathcal{Q})$ *in which*

(i) the composition $\Psi \otimes_{\mathbb{B}} \Phi : \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{C}$ of two distributors $\Phi : \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{B}$ and $\Psi : \mathbb{B} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{C}$ has as distributor-arrows, for $a \in \mathbb{A}_0$ and $c \in \mathbb{C}_0$,

$$(\Psi \otimes_{\mathbb{B}} \Phi)(c, a) = \bigvee_{b \in \mathbb{B}_0} \Psi(c, b) \circ \Phi(b, a);$$

(ii) the identity distributor on a $\mathcal{Q}$-category $\mathbb{A}$ has as distributor-arrows precisely the hom-arrows of the category $\mathbb{A}$ itself, so we simply write it as $\mathbb{A} : \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{A}$;

(iii) the supremum $\bigvee_{i \in I} \Phi_i : \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{B}$ of given distributors $(\Phi_i : \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{B})_{i \in I}$ is calculated elementwise, thus its distributor-arrows are, for $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$,

$$(\bigvee_{i \in I} \Phi_i)(b, a) = \bigvee_{i \in I} \Phi_i(b, a).$$

For examples of quantaloids and $\mathcal{Q}$-categories please refer to [9, 13, 14, 16]. We only recall that **2** is the 2-element Boolean algebra; **2**-categories are orders, distributors are ideal relations, and functors are order-preserving maps; quantales and complete residuated lattices are specific examples of quantaloids, hence Ω-categories [10, 20,21] and L-Fuzzy posets [5, 24], etc are specific examples of $\mathcal{Q}$-categories.

Since $\mathbf{Dist}(\mathcal{Q})$ is a quantaloid, it is in particular closed. Let for example $\Theta : \mathbb{A} \looparrowright \mathbb{C}$, $\Psi : \mathbb{B} \looparrowright \mathbb{C}$ and $\Phi : \mathbb{A} \looparrowright \mathbb{B}$ be distributors between $\mathcal{Q}$-categories. Then $[\Psi, \Theta] : \mathbb{A} \looparrowright \mathbb{B}$ and $\{\Phi, \Theta\} : \mathbb{B} \looparrowright \mathbb{C}$ are the distributors with distributor-arrows, for $a \in \mathbb{A}_0$, $b \in \mathbb{B}_0$, and $c \in \mathbb{C}_0$,

$$[\Psi, \Theta](b, a) = \bigwedge_{c \in \mathbb{C}_0} [\Psi(c, b), \Theta(c, a)],$$

$$\{\Phi, \Theta\}(c, b) = \bigwedge_{a \in \mathbb{A}_0} \{\Phi(b, a), \Theta(c, a)\},$$

where the liftings and extensions on the right are calculated in $\mathcal{Q}$.

Proposition 2.5 *$\mathcal{Q}$-categories are the objects, and functors the arrows, of a category $\mathbf{Cat}(\mathcal{Q})$ in which*

(i) the composition $G \circ F : \mathbb{A} \to \mathbb{C}$ of two functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{C}$ is determined by the composition of object maps $G \circ F : \mathbb{A}_0 \to \mathbb{C}_0 : a \mapsto G(F(a))$;

(ii) the identity functor $1_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ on a $\mathcal{Q}$-category $\mathbb{A}$ is determined by the identity object map $1_{\mathbb{A}} : \mathbb{A}_0 \to \mathbb{A}_0 : a \mapsto a$.

The category $\mathbf{Cat}(\mathcal{Q})$ inherits the local structure from the quantaloid $\mathbf{Dist}(\mathcal{Q})$ via the functor $\mathbf{Cat}(\mathcal{Q}) \to \mathbf{Dist}(\mathcal{Q})$: we put, for two functors $F, G : \mathbb{A} \rightrightarrows \mathbb{B}$,

$$F \le G \Longleftrightarrow \mathbb{B}(-, F-) \le \mathbb{B}(-, G-)(\Longleftrightarrow \mathbb{B}(G-, -) \le \mathbb{B}(F-, -)).$$

For an object A of a quantaloid $\mathcal{Q}$, denote by $*_A$ the one-object $\mathcal{Q}$-category whose homarrow is the identity 1_A. Given a $\mathcal{Q}$-category $\mathbb{A}$, the set $\{a \in \mathbb{A}_0 \mid ta = A\}$ is in bijection with $\mathbf{Cat}(\mathcal{Q})(*_A, \mathbb{A})$: any such object a determines a "constant" functor $\triangle a : *_A \to \mathbb{A}$; and any such functor $F : *_A \to \mathbb{A}$ "picks out" an object $a \in \mathbb{A}$. Hence, without confuse we will not distinguish an element in $\mathbb{A}_0$ from the corresponding functor. The underlying order $(\mathbb{A}_0, \le)$ of a $\mathcal{Q}$-category $\mathbb{A}$ is defined as follows: for two objects $a, a' \in \mathbb{A}_0$ we have that $a' \le a$ if and only if $A := ta = ta'$ and for all $x \in \mathbb{A}_0, \mathbb{A}(x, a') \le \mathbb{A}(x, a)$ in $\mathcal{Q}$, or equivalently $\mathbb{A}(a', x) \ge \mathbb{A}(a, x)$, or equivalently $1_{\mathbb{A}} \le \mathbb{A}(a', a)$. Whenever two objects of $\mathbb{A}$ are equivalent in $\mathbb{A}$'s underlying order ($a \le a'$ and $a' \le a$), then we say that they are isomorphic objects (and write $a \cong a'$).

An arrow $F : \mathbb{A} \to \mathbb{B}$ is left adjoint to an arrow $G : \mathbb{B} \to \mathbb{A}$ in $\mathbf{Cat}(\mathcal{Q})$ (and G is then right adjoint to F), written $F \dashv G$, if $1_{\mathbb{A}} \le G \circ F$ and $F \circ G \le 1_{\mathbb{B}}$. Further, $F : \mathbb{A} \to \mathbb{B}$ is an *equivalence* in $\mathbf{Cat}(\mathcal{Q})$ if there exists a $G : \mathbb{B} \to \mathbb{A}$ such that $G \circ F \cong 1_{\mathbb{A}}$ and $F \circ G \cong 1_{\mathbb{B}}$. We say $\mathbb{A}$ and $\mathbb{B}$ are equivalent provided that there is an equivalence $F : \mathbb{A} \to \mathbb{B}$.

Proposition 2.6 *$F : \mathbb{A} \to \mathbb{B}$ is left adjoint to $G : \mathbb{B} \to \mathbb{A}$ in $\mathbf{Cat}(\mathcal{Q})$ if and only if $\mathbb{B}(F-, -) = \mathbb{A}(-, G-) : \mathbb{B} \looparrowright \mathbb{A}$ in $\mathbf{Dist}(\mathcal{Q})$.*

We consider a functor $F : \mathbb{A} \to \mathbb{B}$ and a distributor $\Theta : \mathbb{C} \looparrowright \mathbb{A}$ between $\mathcal{Q}$-categories. A functor $G : \mathbb{C} \to \mathbb{B}$ is the Θ-*weighted colimit* of F if: $\mathbb{B}(G-, -) = [\Theta, \mathbb{B}(F-, -)]$. If the Θ-weighted colimit of F exists, then it is necessarily essentially unique. It therefore makes sense to speak of "the" colimit and to denote it by $\mathrm{colim}(\Theta, F)$; its universal property is thus that

$$\mathbb{B}(\mathrm{colim}(\Theta, F)-, -) = [\Theta, \mathbb{B}(F-, -)] \text{ in } \mathbf{Dist}(\mathcal{Q}).$$

For a distributor $\Phi : \mathbb{B} \looparrowright \mathbb{C}$ in $\mathbf{Dist}(\mathcal{Q})$ and a functor $F : \mathbb{B} \to \mathbb{A}$ in $\mathbf{Cat}(\mathcal{Q})$, the Φ-weighted limit of F is–whenever it exists–a functor $\lim(\Phi, F) : \mathbb{C} \to \mathbb{A}$ in $\mathbf{Cat}(\mathcal{Q})$ such that

$$\mathbb{A}(-, \lim(\Phi, F)-) = \{\Phi, \mathbb{A}(-, F-)\} \text{ in } \mathbf{Dist}(\mathcal{Q}).$$

The following lemma will be used frequently.

Lemma 2.7 *Let $\mathbb{A}$ be a $\mathcal{Q}$-category, $A \in \mathcal{Q}_0$, $\Phi : \mathbb{A} \looparrowright *_A$ be a distributor. Then $x \cong \lim(\Phi, 1_{\mathbb{A}})$ if and only if (1) $\Phi(y) \leq \mathbb{A}(x, y)$ for all $y \in \mathbb{A}_0$, and (2) $\{\Phi, \mathbb{A}(-,-)\}(y) \leq \mathbb{A}(y, x)$.*

Proof Suppose $x \cong \lim(\Phi, 1_{\mathbb{A}})$ i.e., $\mathbb{A}(-, x) = \{\Phi, \mathbb{A}\}$. Then, $\{\Phi, \mathbb{A}\}(y) \leq \mathbb{A}(y, x)$ holds clearly, and $\mathbb{A}(x, x) = \{\Phi, \mathbb{A}\}(x) = \bigwedge\limits_{z \in \mathbb{A}_0} \{\Phi(z), \mathbb{A}(x, z)\} \leq \{\Phi(y), \mathbb{A}(x, y)\}$. Hence, $\Phi(y) \leq \mathbb{A}(x, y)$.

Conversely, by (2) we know $\mathbb{A}(-, x) \geq \{\Phi, \mathbb{A}\}$. Hence, in order to prove $x \cong \lim(\Phi, 1_{\mathbb{A}})$ we only need to show that $\{\Phi, \mathbb{A}\} \geq \mathbb{A}(-, x)$. Since for each $z \in \mathbb{A}_0$ and $y \in \mathbb{A}_0$, $\mathbb{A}(y, x) \circ \Phi(z) \leq \mathbb{A}(y, x) \circ \mathbb{A}(x, z) \leq \mathbb{A}(y, z)$, we have $\mathbb{A}(y, x) \leq \bigwedge_{z \in \mathbb{A}_0} \{\Phi(z), \mathbb{A}(y, z)\} = \{\Phi, \mathbb{A}\}(y)$. □

3. Closure operators on $\mathcal{Q}$-categories

In this section, we will introduce closure operator on $\mathcal{Q}$-categories and discuss their relation to adjunctions on $\mathcal{Q}$-categories.

Definition 3.1 *Let $F : \mathbb{A} \to \mathbb{A}$ be a functor on a $\mathcal{Q}$-category $\mathbb{A}$. Then*

(1) F is called a projection operator if $F \cong F \circ F$,

(2) F is called a closure operator if F is a projection operator with $1_{\mathbb{A}} \leq F$,

(3) F is called a kernel operator if F is a projection operator with $1_{\mathbb{A}} \geq F$.

Proposition 3.2 *Let $F : \mathbb{A} \to \mathbb{A}$ be a functor on a $\mathcal{Q}$-category $\mathbb{A}$. Then*

(1) F is a closure operator if and only if $\mathbb{A}(x, Fy) = \mathbb{A}(Fx, Fy)$ for all $x, y \in \mathbb{A}_0$,

(2) F is a kernel operator if and only if $\mathbb{A}(Fx, y) = \mathbb{A}(Fx, Fy)$ for all $x, y \in \mathbb{A}_0$.

Example 3.3 Take $\mathcal{Q} = \mathbf{2}$. Then closure and kernel operators on skeletal $\mathbf{2}$-categories are just closure and kernel operators on posets.

Example 3.4 Take $\mathcal{Q}$ to be a complete residuated lattice. Then closure and kernel operators on $\mathcal{Q}$-categories were studied under the name fuzzy closure operator by Guo in [7] and L-kernel operator by Yao in [25], respectively.

Hence, all the results in the present paper can be applied to those particular cases, and we will not mention this each time.

We know closure operators and kernel operators have close relation with Golois connections [6]. Now we are going to discuss the relation between closure operators and adjunctions on $\mathcal{Q}$-categories.

Proposition 3.5 *If $F : \mathbb{A} \to \mathbb{B}$ is left adjoint to $G : \mathbb{B} \to \mathbb{A}$ in $\mathbf{Cat}(\mathcal{Q})$, then $G \circ F : \mathbb{A} \to \mathbb{A}$ is a closure operator and $F \circ G : \mathbb{B} \to \mathbb{B}$ is a kernel operator.*

Proof By definition of adjoint in $\mathbf{Cat}(\mathcal{Q})$ we know $G \circ F \geq 1_{\mathbb{A}}$ and $F \circ G \leq 1_{\mathbb{B}}$. Hence, $G \circ F = G \circ F \circ 1_{\mathbb{A}} \leq G \circ F \circ G \circ F = G \circ (F \circ G) \circ F \leq G \circ 1_{\mathbb{B}} \circ F = G \circ F$, and $F \circ G = F \circ 1_{\mathbb{A}} \circ G \leq F \circ G \circ F \circ G \leq F \circ G \circ 1_{\mathbb{B}} \leq F \circ G$. Thus, $G \circ F$ is a closure operator and $F \circ G$ is a kernel operator. □

For a functor $F : \mathbb{A} \to \mathbb{A}$ on a $\mathcal{Q}$-category $\mathbb{A}$, denote the image of F by $F\mathbb{A}$, that is to say, $F\mathbb{A}$ is the $\mathcal{Q}$-category with objects $(F\mathbb{A})_0 = \{Fa | a \in \mathbb{A}_0\}$ and hom-arrows inherit from $\mathbb{A}$, i.e., $F\mathbb{A}(b, a) = \mathbb{A}(b, a)$ for all $a, b \in (F\mathbb{A})_0$. Denote the co-restriction of F to its image by $F^\circ : \mathbb{A} \to F\mathbb{A}$, and the inclusion functor of its image into $\mathbb{B}$ by $F_\circ : F\mathbb{A} \to \mathbb{A}$.

Proposition 3.6 *Let $F : \mathbb{A} \to \mathbb{A}$ be a functor on a $\mathcal{Q}$-category $\mathbb{A}$. Then we have the following two groups of equivalent statements:*

(a) F is a closure operator,

(b) $F^\circ : \mathbb{A} \to F\mathbb{A}$ is left adjoint to $F_\circ : F\mathbb{A} \to \mathbb{A}$ in $\mathbf{Cat}(\mathcal{Q})$;

(a′) F is a kernel operator,

(b′) $F_\circ : F\mathbb{A} \to \mathbb{A}$ is left adjoint to $F^\circ : \mathbb{A} \to F\mathbb{A}$ in $\mathbf{Cat}(\mathcal{Q})$.

Proof (a) implies (b). For every $y \in (F\mathbb{A})_0$, there exists an $x \in \mathbb{A}$ such that $Fx = y$. Thus $F^\circ \circ F_\circ(y) = F^\circ \circ F_\circ(Fx) = F^\circ(Fx) \cong F^\circ x = y$, and for every $x \in \mathbb{A}_0$, $F_\circ \circ F^\circ(x) = F^\circ x \geq x$. Therefore, $F^\circ \dashv F_\circ$.

(b) implies (a) is direct consequence of proposition 3.5. □

Lemma 3.7 ([16, Proposition 5.12]) *If $F' : \mathbb{B} \to \mathbb{B}'$ is a left adjoint in $\mathbf{Cat}(\mathcal{Q})$, then it is cocontinuous.*

Corollary 3.8 *The co-restriction $F^\circ : \mathbb{A} \to F\mathbb{A}$ of a closure operator is cocontinuous.*

Theorem 3.9 *Let $F : \mathbb{A} \to \mathbb{A}$ be a closure operator, $I : F\mathbb{A} \to \mathbb{A}$ be the inclusion functor, $\Theta : \mathbb{D} \looparrowright F\mathbb{A}$ be a distributor. If $\mathrm{colim}(\mathbb{A}(-, I-) \otimes_{F\mathbb{A}} \Theta, 1_{\mathbb{A}})$ exists, denote it by G, then $F \circ G \cong \mathrm{colim}(\Theta, 1_{F\mathbb{A}})$.*

The following diagram pictures the situation

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "\Theta"', "\circ" marking] \arrow[dr, dotted, "\mathrm{colim}(\Theta, 1_{F\mathbb{A}})"] \arrow[ddr, dotted, "G"] & \\
F\mathbb{A} \arrow[r, "1_{F\mathbb{A}}"] \arrow[d, "\mathbb{A}(-,I-)"', "\circ" marking] & F\mathbb{A} \\
\mathbb{A} \arrow[r, "1_{\mathbb{A}}"'] & \mathbb{A} \arrow[u, "F"']
\end{tikzcd}$$

Diagram 1 Θ-weighted colimit of $1_{F\mathbb{A}}$

Proof If $\mathrm{colim}(\mathbb{A}(-,I-)\otimes_{F\mathbb{A}}\Theta, 1_{\mathbb{A}})$ exists, then

$$\begin{aligned} &F\mathbb{A}(F\circ \mathrm{colim}(\mathbb{A}(-,I-)\otimes_{F\mathbb{A}}\Theta, 1_{\mathbb{A}})-,-)\\ &\quad=\mathbb{A}(\mathrm{colim}(\mathbb{A}(-,I-)\otimes_{F\mathbb{A}}\Theta, 1_{\mathbb{A}})-,I-)\\ &\quad=[\mathbb{A}(-,I-)\otimes_{F\mathbb{A}}\Theta, \mathbb{A}(-,I-)]\\ &\quad=[\Theta, F\mathbb{A}(-,-)].\ \square \end{aligned}$$

Recall that a functor $F:\mathbb{A}\to\mathbb{B}$ is fully faithful if

$$\forall a,a'\in\mathbb{A}_0 : \mathbb{A}(a',a)=\mathbb{B}(Fa',Fa) \text{ in } \mathcal{Q}.$$

Lemma 3.10 ([16, Corollary 5.13]) *Consider a fully faithful right adjoint $G:\mathbb{A}\to\mathbb{B}$ in* $\mathbf{Cat}(\mathcal{Q})$*; if $\mathbb{B}$ is cocomplete, then so is $\mathbb{A}$.*

Corollary 3.11 *If $F:\mathbb{A}\to\mathbb{A}$ is a closure operator on a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, then $F\mathbb{A}$ is cocomplete.*

Theorem 3.12 *If $F\dashv G:\mathbb{A}\to\mathbb{A}$ in* $\mathbf{Cat}(\mathcal{Q})$*, then the following conditions are equivalent:*

(1) F is a kernel operator;
(2) G is a closure operator;
(3) $F\circ G\cong F$;
(4) $G\circ F\cong G$.

Proof (1) implies (2). Suppose F is a kernel operator, then $\mathbb{A}(-,G-)=\mathbb{A}(F-,-)\geq A(-,-)$, i.e., $1_{\mathbb{A}}\leq G$, and $\mathbb{A}(-,G\circ G)=\mathbb{A}(F-,G-)=\mathbb{A}(F\circ F,-)=\mathbb{A}(F-,-)=\mathbb{A}(-,G-)$, i.e., $G\cong G\circ G$. Hence, G is a closure operator.

(2) implies (3). Suppose G is a closure operator, then $\mathbb{A}(F\circ G-,-)=\mathbb{A}(G,G)=\mathbb{A}(G-,G\circ G-)\geq\mathbb{A}(-,G-)=\mathbb{A}(F-,-)$. Hence, $F\circ G\leq F$. Conversely, $F\leq F\circ G$ is obvious, since $G\geq 1_{\mathbb{A}}$. Therefore, $F\circ G\cong F$.

(3) $\iff$ (4). If $F\circ G\cong F$, then $G\circ F\cong G\circ F\circ G\cong G$. Conversely, if $G\circ F\cong G$, then $F\circ G\cong F\circ G\circ F\cong F$.

(3) and (4) imply (1). For every $x\in\mathbb{A}_0$, $\mathbb{A}(Fx,x)\geq\mathbb{A}(Fx,FGx)\circ\mathbb{A}(FGx,x)=\mathbb{A}(Fx,Fx)\circ\mathbb{A}(Gx,Gx)\geq 1_{tx}$, i.e., $F\leq 1_{\mathbb{A}}$, and $\mathbb{A}(F-,F\circ F-)=\mathbb{A}(-,G\circ F\circ F-)=\mathbb{A}(-,G\circ F-)=\mathbb{A}(F-,F-)\geq\mathbb{A}(-,-)$, i.e., $F\leq F\circ F$. Hence we can conclude that F is a kernel operator. $\square$

Similarly, we have

Theorem 3.13 *If $F\dashv G:\mathbb{A}\to\mathbb{A}$ in* $\mathbf{Cat}(\mathcal{Q})$*, then the following conditions are equivalent:*

(1) F is a closure operator;
(2) G is a kernel operator;
(3) $G\circ F\cong F$;
(4) $F\circ G\cong G$.

For a projection operator $F:\mathbb{A}\to\mathbb{A}$ on a $\mathcal{Q}$-category, we define $\mathrm{Fix}_F=\{x\in\mathbb{A}_0|Fx\cong x\}$.

Theorem 3.14 *If $F:\mathbb{A}\to\mathbb{A}$ is a closure operator, $G:\mathbb{A}\to\mathbb{A}$ is a kernel operator, then $F\dashv G$*

if and only if $\mathrm{Fix}_F = \mathrm{Fix}_G$.

Proof If $F \dashv G$, then for every $a \in \mathrm{Fix}_F$, $\mathbb{A}(a, Ga) = \mathbb{A}(Fa, a) = \mathbb{A}(a, a) \geq 1_{ta}$, i.e., $a \leq Ga$. Hence $a \cong Ga$, i.e., $a \in \mathrm{Fix}_G$. Similarly, we can prove $\mathrm{Fix}_G \subseteq \mathrm{Fix}_F$. Thus $\mathrm{Fix}_F = \mathrm{Fix}_G$.

Conversely, if $\mathrm{Fix}_F = \mathrm{Fix}_G$, then for every $x \in \mathbb{A}_0$, since $Fx \in \mathrm{Fix}_F$, $GFx \cong Fx$. Hence, $\mathbb{A}(Fx, y) \leq \mathbb{A}(GFx, Gy) = \mathbb{A}(Fx, Gy) \leq \mathbb{A}(x, Fx) \circ \mathbb{A}(Fx, Gx) \leq \mathbb{A}(x, Gy)$. Similarly, we have $\mathbb{A}(x, Gy) \leq \mathbb{A}(Fx, y)$. Hence, $\mathbb{A}(F-, -) = \mathbb{A}(-, G-)$, i.e., $F \dashv G$ in $\mathbf{Cat}(\mathcal{Q})$. $\square$

4. Closure systems

In this section, we will explain the notion of closure system on $\mathcal{Q}$-categories. As expected we will establish the relation between closure systems and closure operators on $\mathcal{Q}$-categories.

Recall that for a $\mathcal{Q}$-category $\mathbb{A}$ the co-Yoneda embedding $Y'_{\mathbb{A}} : \mathbb{A} \to \mathcal{P}^{\dagger}\mathbb{A}$ is defined as $Y'_{\mathbb{A}}(a) = \mathbb{A}(a, -) : \mathbb{A} \,{-\!\!\circ\!\!\rightarrow}\, *_{ta}$, for each $a \in \mathbb{A}_0$.

Definition 4.1 *Let $\mathbb{A}$ be a $\mathcal{Q}$-category. A sub-$\mathcal{Q}$-category $\mathbb{B}$ of $\mathbb{A}$ is called a closure system, provided that for every $a \in \mathbb{A}_0$, $\lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}})$ exists and $a \leq \lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}})$.*

The following lemma is the dual of Lemma 5.2(4) in [16].

Lemma 4.2 *For $\Phi : \mathbb{A} \,{-\!\!\circ\!\!\rightarrow}\, \mathbb{C}$ and $F : \mathbb{A} \to \mathbb{B}$, $\lim(\Phi, F)$ exists if and only if, for all objects $c \in \mathbb{C}_0$, $\lim(\Phi(c, -), F)$ exists; then $\lim(\Phi, F)(c) \cong \lim(\Phi(c, -), F)$.*

For a sub-$\mathcal{Q}$-category $\mathbb{B}$ of $\mathbb{A}$, with $I : \mathbb{B} \to \mathbb{A}$ as the inclusion functor, we have $Y'_{\mathbb{A}}(a)|_{\mathbb{B}} = \mathbb{A}(a, -)|_{\mathbb{B}} = \mathbb{A}(a, I-)$, hence $\lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}})$ exists if and only if $\lim(\mathbb{A}(a, I-), 1_{\mathbb{B}})$ exists. So we have

Proposition 4.3 *Let $\mathbb{A}$ be a $\mathcal{Q}$-category. Then a sub-$\mathcal{Q}$-category $\mathbb{B}$ of $\mathbb{A}$ is a closure system if and only if $\lim(\mathbb{A}(-, I-), 1_{\mathbb{B}})$ exists and for all $a \in \mathbb{A}_0, a \leq \lim(\mathbb{A}(a, I-), 1_{\mathbb{B}})$.*

Lemma 4.4 *Let $\mathbb{B}$ be a closure system in a $\mathcal{Q}$-category $\mathbb{A}$, $A \in \mathcal{Q}_0$ and $\Phi : \mathbb{B} \,{-\!\!\circ\!\!\rightarrow}\, *_A$ be a distributor. If $\lim(\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-, -), 1_{\mathbb{A}})$ exists (denote it by a), then $a \cong \lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}}) \cong \lim(\Phi, 1_{\mathbb{B}})$, where $I : \mathbb{B} \to \mathbb{A}$ is the inclusion functor.*

Proof Denote $a_0 = \lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}})$. Then $\mathbb{A}(a, a_0) \geq 1_{ta}$. And,

$$\begin{aligned}
\mathbb{A}(a_0, a) &= \mathbb{A}(a_0, \lim(\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-, -), 1_{\mathbb{A}})) \\
&= \bigwedge_{y \in \mathbb{A}_0} \{\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-, -)(y), \mathbb{A}(a_0, y)\} \\
&= \bigwedge_{y \in \mathbb{A}_0} \{\bigvee_{b \in \mathbb{B}_0} \Phi(b) \circ \mathbb{A}(b, y), \mathbb{A}(a_0, y)\} \\
&= \bigwedge_{b \in \mathbb{B}_0} \{\Phi(b), \bigwedge_{y \in \mathbb{A}_0} \{\mathbb{A}(b, y), \mathbb{A}(a_0, y)\}\} \\
&= \bigwedge_{b \in \mathbb{B}_0} \{\Phi(b), \mathbb{A}(a_0, b)\}
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{b\in\mathbb{B}_0}\{\Phi(b), \mathbb{B}(\lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}}), b)\}\\
&\geq \bigwedge_{b\in\mathbb{B}_0}\{\Phi(b), Y'_{\mathbb{A}}(a)|_{\mathbb{B}}(b)\}\\
&= \bigwedge_{b\in\mathbb{B}_0}\{\Phi(b), \mathbb{A}(\lim(\Phi\otimes_{\mathbb{B}}\mathbb{A}(I-,-), 1_{\mathbb{A}}), b)\}\\
&\geq \bigwedge_{b\in\mathbb{B}_0}\{\Phi(b), \Phi\otimes_{\mathbb{B}}\mathbb{A}(I-,-)(b)\}\\
&\geq \bigwedge_{b\in\mathbb{B}_0}\{\Phi(b), \Phi(b)\}\\
&\geq 1_A.
\end{aligned}
$$

Hence, $a\cong a_0$. For every $b\in\mathbb{B}_0$, we have

$$
\begin{aligned}
\mathbb{B}(b,a) &= \mathbb{A}(b, \lim(\Phi\otimes_{\mathbb{B}}\mathbb{A}(I-,-), 1_{\mathbb{A}}))\\
&= \{\Phi\otimes_{\mathbb{B}}\mathbb{A}(I-,-), \mathbb{A}\}(b)\\
&= \bigwedge_{x\in\mathbb{A}_0}\{\Phi\otimes_{\mathbb{B}}\mathbb{A}(I-,-)(x), \mathbb{A}(b,x)\}\\
&= \bigwedge_{x\in\mathbb{A}_0}\{\bigvee_{y\in\mathbb{B}_0}\Phi(y)\circ\mathbb{A}(y,x), \mathbb{A}(b,x)\}\\
&= \bigwedge_{y\in\mathbb{B}_0}\{\Phi(y), \bigwedge_{x\in\mathbb{A}_0}\{\mathbb{A}(y,x), \mathbb{A}(b,x)\}\}\\
&= \bigwedge_{y\in\mathbb{B}_0}\{\Phi(y), \mathbb{B}(b,y)\}\\
&= \{\Phi, \mathbb{B}(-, 1_{\mathbb{B}}-)\}(b).
\end{aligned}
$$

Thus $a\cong\lim(\Phi, 1_{\mathbb{B}})$. □

Theorem 4.5 *Let $\mathbb{B}$ be a closure system in $\mathbb{A}$, $I:\mathbb{B}\to\mathbb{A}$ be the inclusion fuctor, $\Phi: \mathbb{B} \circ\!\!\to \mathbb{C}$ be a distributor. If $\lim(\Phi\otimes_{\mathbb{B}}\mathbb{A}(I-,-), 1_{\mathbb{A}})$ exists (denote it by F), then F is equivalent to $I\circ\lim(\Phi, 1_{\mathbb{B}})$.*

The following diagram pictures the situation

$$
\begin{tikzcd}
\mathbb{C} \arrow[dr, dotted, "\lim(\Phi, 1_{\mathbb{B}})"] \arrow[ddr, dotted, "F"] & \\
\mathbb{B} \arrow[u, "\Phi", "\circ" marking] \arrow[r, "1_{\mathbb{B}}"] & \mathbb{B} \arrow[d, "I"] \\
\mathbb{A} \arrow[u, "\mathbb{A}(I-,-)", "\circ" marking] \arrow[r, "1_{\mathbb{A}}"'] & \mathbb{A}
\end{tikzcd}
$$

Diagram 2 Φ-weighted limit of $1_{\mathbb{B}}$

Proof If $\lim(\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-,-), 1_{\mathbb{A}})$ exists, then for every $c \in \mathbb{C}_0$, $\lim(\Phi(c,-) \otimes_{\mathbb{B}} \mathbb{A}(I-,-), 1_{\mathbb{A}})$ exists and is equivalent to $\lim(\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-,-), 1_{\mathbb{A}})(c)$. By Lemma 4.4, we know $\lim(\Phi(c,-) \otimes_{\mathbb{B}} \mathbb{A}(I-,-), 1_{\mathbb{A}})$ is equivalent to $\lim(\Phi(c,-), 1_{\mathbb{B}})$. Hence, $\lim(\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-,-), 1_{\mathbb{A}})(c)$ is equivalent to $\lim(\Phi(c,-), 1_{\mathbb{B}})$. Thus, $\lim(\Phi, 1_{\mathbb{B}})$ exists and for every $c \in \mathbb{C}_0$, $\lim(\Phi, 1_{\mathbb{B}})(c) \cong \lim(\Phi(c,-), 1_{\mathbb{B}}) \cong \lim(\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-,-), 1_{\mathbb{A}})(c)$. Therefore, $\lim(\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-,-), 1_{\mathbb{A}}) \cong I \circ \lim(\Phi, 1_{\mathbb{B}})$. $\square$

Corollary 4.6 *If $\mathbb{B}$ is a closure system in a complete $\mathcal{Q}$ category $\mathbb{A}$, then for every distributor $\Phi: \mathbb{B} \to\!\!\!\!\!\circ\!\!\to \mathbb{C}$, $\lim(\Phi, 1_{\mathbb{B}})$ exists and $I \circ \lim(\Phi, 1_{\mathbb{B}}) \cong \lim(\Phi \otimes_{\mathbb{B}} \mathbb{A}(I-,-), 1_{\mathbb{A}})$.*

Proposition 4.7 *If $F: \mathbb{A} \to \mathbb{A}$ is a closure operator, then $F\mathbb{A}$ is a closure system.*

Proof Take $a \in \mathbb{A}_0$. Then for every $b \in (F\mathbb{A})_0$, we have

$$\begin{aligned}
\{Y'_{\mathbb{A}}(a)|_{F\mathbb{A}}, F\mathbb{A}(-,-)\}(b) &= \bigwedge_{x \in (F\mathbb{A})_0} \{Y'_{\mathbb{A}}(a)|_{F\mathbb{A}}(x), F\mathbb{A}(b,x)\} \\
&= \bigwedge_{x \in (F\mathbb{A})_0} \{\mathbb{A}(a,x), F\mathbb{A}(b,x)\} \\
&= \bigwedge_{x \in (F\mathbb{A})_0} \{F\mathbb{A}(Fa,x), F\mathbb{A}(b,x)\} \\
&= F\mathbb{A}(-,Fa)(b).
\end{aligned}$$

Hence, $Fa \cong \lim(Y'_{\mathbb{A}}(a)|_{F\mathbb{A}}, 1_{F\mathbb{A}})$ and $Y'_{\mathbb{A}}(a)(\lim(Y'_{\mathbb{A}}(a)|_{F\mathbb{A}}, 1_{F\mathbb{A}})) = Y'_{\mathbb{A}}(a)(Fa) = \mathbb{A}(a, Fa) \geq 1_{ta}$. Therefore, $F\mathbb{A}$ is a closure system. $\square$

If $\mathbb{B}$ is a closure system on a $\mathcal{Q}$-category $\mathbb{A}$, then by definition for each $a \in \mathbb{A}_0$, $\lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}})$ exists, and such a limit "picks out" an object of type ta in $\mathbb{B}$. Therefore, we will treat it just as that object. Hence, for each $a \in \mathbb{A}_0$, we can assign an element $\lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}})$ to it, that is to say, we can define a map $F_{\mathbb{B}}: \mathbb{A}_0 \to \mathbb{A}_0$ by $F_{\mathbb{B}}(a) = \lim(Y'_{\mathbb{A}}(a)|_{\mathbb{B}}, 1_{\mathbb{B}})$.

Proposition 4.8 *Let $\mathbb{B}$ be a closure system on a $\mathcal{Q}$-category $\mathbb{A}$. Then the map $F_{\mathbb{B}}$ is a closure operator on $\mathbb{A}$ with $F_{\mathbb{B}}\mathbb{A}$ equivalent to $\mathbb{B}$.*

Proof (1) For $x, y \in \mathbb{A}_0$ we have

$$\begin{aligned}
\mathbb{B}(Fx, Fy) &= \mathbb{B}(Fx, \lim(Y'_{\mathbb{A}}(y)|_{\mathbb{B}}, 1_{\mathbb{B}})) \\
&= \bigwedge_{z \in \mathbb{B}_0} \{Y'_{\mathbb{A}}(y)(z), \mathbb{B}(Fx, z)\} \\
&= \bigwedge_{z \in \mathbb{B}_0} \{Y'_{\mathbb{A}}(y)(z), \mathbb{B}(\lim(Y'_{\mathbb{A}}(x)|_{\mathbb{B}}, 1_{\mathbb{B}}), z)\} \\
&= \bigwedge_{z \in \mathbb{B}_0} \{\mathbb{A}(y,z), \bigwedge_{b \in \mathbb{B}_0} [\mathbb{B}(b, \lim(Y'_{\mathbb{A}}(x)|_{\mathbb{B}}, 1_{\mathbb{B}})), \mathbb{B}(b,z)]\} \\
&= \bigwedge_{z \in \mathbb{B}_0} \bigwedge_{b \in \mathbb{B}_0} \{\mathbb{A}(y,z), [\bigwedge_{p \in \mathbb{B}_0} \{\mathbb{A}(x,p), \mathbb{B}(b,p)\}, \mathbb{B}(b,z)]\} \\
&\geq \bigwedge_{z \in \mathbb{B}_0} \bigwedge_{b \in \mathbb{B}_0} \{\mathbb{A}(y,z), [\{\mathbb{A}(x,z), \mathbb{B}(b,z)\}, \mathbb{B}(b,z)]\}
\end{aligned}$$

$$\geq \bigwedge_{z\in\mathbb{B}_0}\{\mathbb{A}(y,z),\mathbb{A}(x,z)\}$$
$$\geq \mathbb{A}(x,y).$$

(2) For every $x\in\mathbb{A}_0$, $\mathbb{A}(x,Fx)=\mathbb{A}(x,\lim(Y'_{\mathbb{A}}(x)|_{\mathbb{B}},1_{\mathbb{B}}))\geq 1_{tx}$.

(3) For every $x\in\mathbb{A}_0$, $\mathbb{A}(FFx,Fx)=\mathbb{A}(\lim(Y'_{\mathbb{A}}(Fx)|_{\mathbb{B}},1_{\mathbb{B}}),Fx)\geq Y'_{\mathbb{A}}(Fx)|_{\mathbb{B}}(Fx)=\mathbb{A}(Fx,Fx)\geq 1_{tx}$.

By (1)–(3), we can conclude that F is a closure operator.

For every $b\in\mathbb{B}_0$, since $Y'_{\mathbb{A}}(b)|_{\mathbb{B}}=Y'_{\mathbb{B}}(b)$, whence $\lim(Y'_{\mathbb{A}}(b)|_{\mathbb{B}},1_{\mathbb{B}})\cong\lim(Y'_{\mathbb{B}}(b),1_{\mathbb{B}})\cong b$. Hence, $F_{\mathbb{B}}(b)\cong b$. Therefore, we can deduce that the inclusion functor $I:F_{\mathbb{B}}\mathbb{A}\to\mathbb{B}$ is an equivalence. □

Convention 4.9 If $J:\mathbb{A}\to\mathbb{A}$ is a closure operator, denote by $\mathbb{T}_J$ the closure system induced by J. Conversely, if $\mathbb{T}$ is a closure system on $\mathbb{A}$, denote by $J_{\mathbb{T}}$ the closure operator induced by $\mathbb{T}$ as defined in Proposition 4.8, then we have:

Theorem 4.10 *Let $\mathbb{A}$ be a $\mathcal{Q}$-category, $J:\mathbb{A}\to\mathbb{A}$ be a closure operator, $\mathbb{T}$ be a closure system. Then:*

(1) $J_{\mathbb{T}_J}\cong J$,

(2) $\mathbb{T}_{J_{\mathbb{T}}}\cong\mathbb{T}$.

Proof (1) Since for every $x\in\mathbb{A}_0$, $y\in(\mathbb{T}_J)_0$, $Y'_{\mathbb{A}}(x)|_{\mathbb{T}_J}(y)=\mathbb{A}(x,y)\leq\mathbb{A}(Jx,Jy)=\mathbb{A}(Jx,y)$ and $Y'_{\mathbb{A}}(x)|_{\mathbb{T}_J}(Jx)=\mathbb{A}(x,Jx)\geq 1_{tx}$. Hence, by Lemma 2.7, $Jx\cong\lim(Y'_{\mathbb{A}}(x)|_{\mathbb{T}_J},1_{\mathbb{T}_J})=J_{\mathbb{T}_J}(x)$.

(2) For every $x\in\mathbb{T}_0$, $J_{\mathbb{T}}(x)=\lim(Y'_{\mathbb{A}}(x)|_{\mathbb{T}},1_{\mathbb{T}})\cong x$. If $x\in(\mathbb{T}_{J_{\mathbb{T}}})_0$, then there exists $y\in\mathbb{A}_0$ such that $x=J_{\mathbb{T}}(y)\in\mathbb{T}_0$. Hence, $(\mathbb{T}_{J_{\mathbb{T}}})_0\subseteq(\mathbb{T})_0$. Thus it is easy to see that the inclusion functor $I:\mathbb{T}_{J_{\mathbb{T}}}\to\mathbb{T}$ is an equivalence. □

References

[1] R. BĚLOHLÁVEK. *Fuzzy closure operators*. J. Math. Anal. Appl., 2001, **262**(2): 473–489.

[2] R. BĚLOHLÁVEK. *Fuzzy Relational Systems: Foundations and Principles*. Kluwer Academic Publishers, Plenum Publishers, New York, 2002.

[3] R. BĚLOHLÁVEK, T. FUNIOKOVA. *Fuzzy interior operators*. Int. J. Gen. Syst., 2004, **33**(4): 315–330.

[4] J. BÉNABOU. *Introduction to bicategories*. Lecture Notes in Mathematics, 1967, **47**: 1–77.

[5] Lei FAN. *A new approach to quantitative domain theory*. Electronic Notes in Theoretical Computer Science, 2001, **45**: 77–87.

[6] G. GIERZ, K. H. HOFMANN, K. KEIMEL, et al. *Continuous Lattices and Domains*. Cambridge University Press, Cambridge, 2003.

[7] Lankun GUO, Guoqiang ZHANG, Qingguo LI. *Fuzzy closure systems on L-ordered sets*. Math. Log. Quart., 2011, **57**(3): 281–291.

[8] D. HOFMANN, P. L. WASZKIEWICZ. *Approximation in quantale-enriched categories*. Topology Appl., 2011, **158**(8): 963–977.

[9] G. M. KELLY. *Basic Concepts of Enriched Category Theory*. Cambridge University Press, Cambridge-New York, 1982.

[10] Hongliang LAI, Dexue ZHANG. *Complete and directed complete Ω-categories*. Theoret. Comput. Sci., 2007, **388**(1-3): 1–25.

[11] F. W. LAWVERE. *Metric spaces, generalized logic, and closed categories*. Repr. Theory Appl. Categ., 2002, **1**: 1–37.

[12] K. I. ROSENTHAL. *Quantaloids, enriched categories and automata theory*. Appl. Categ. Structures, 1995, **3**(3): 279–301.
[13] K. I. ROSENTHAL. *Quantales and Their Applications*. John Wiley & Sons, Inc., New York, 1990.
[14] K. I. ROSENTHAL. *The Theory of Quantaloids*. Longman, Harlow, 1996.
[15] R. H. STREET. *Enriched categories and cohomology*. Repr. Theory Appl. Categ., 2005, **14**: 1–18.
[16] I. STUBBE. *Categorical structures enriched in a quantaloid: categories, distributors and functors*. Theory Appl. Categ., 2005, **14**(1): 1–45.
[17] I. STUBBE. *Categorical structures enriched in a quantaloid: tensored and cotensored categories*. Theory Appl. Categ., 2006, **16**(14): 283–306.
[18] I. STUBBE. *Categorical structures enriched in a quantaloid: orders and ideals over a base quantaloid*. Appl. Categ. Structures, 2005, **13**(3): 235–255.
[19] I. STUBBE. *Towards "dynamic domains": totally continuous cocomplete $\mathcal{Q}$-categories*. Theoret. Comput. Sci., 2007, **373**(1-2): 142–160.
[20] K. R. WAGNER. *Solving Recursive Domain Equations With Enriched Categories*. School of Computer Science, Carnegie-Mellon University, Pittsburgh, 1994.
[21] K. R. WAGNER. *Liminf convergence in Ω-categories*. Theoret. Comput. Sci., 1997, **184**(1-2): 61–104.
[22] R. F. C. WALTERS. *Sheaves and Cauchy-complete categories*. Cahiers Topologie Géom. Différentielle, 1981, **22**(3): 283–286.
[23] Wei YAO, Lingxia LU. *Fuzzy Galois connections on fuzzy posets*. Math. Log. Quart., 2009, **55**(1): 105–112.
[24] Wei YAO. *Quantitative domains via fuzzy sets: Part I: continuity of fuzzy directed complete posets*. Fuzzy Sets and Systems, 2010, **161**(7): 973–987.
[25] Wei YAO, Bin ZHAO. *Kernel systems on L-ordered sets*. Fuzzy Sets and Systems, 2011, **182**(1): 101–109.
[26] Dexue ZHANG. *An enriched category approach to many valued topology*. Fuzzy Sets and Systems, 2007, **158**(4): 349–366.
[27] Dexue ZHANG. *Implication structures, fuzzy subsets, and enriched categories*. Fuzzy Sets and Systems, 2010, **161**(9): 1205–1223.
[28] Qiye ZHANG, Lei FAN. *Continuity in quantitative domains*. Fuzzy Sets and Systems, 2005, **154**(1): 118–131.