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Some Results on the Problem of Updating the Hyperbolic Matrix Factorizations

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Abstract This paper considers the updating problem of the hyperbolic matrix factorizations. The sufficient conditions for the existence of the updated hyperbolic matrix factorizations are first provided. Then, some differential inequalities and first order perturbation expansions for the updated hyperbolic factors are derived. These results generalize the corresponding ones for the updating problem of the classical QR factorization obtained by Jiguang SUN.

Keywords hyperbolic matrix factorization; hyperbolic QR factorization; hyperbolic polar factorization; updating problem; perturbation analysis.

MR(2010) Subject Classification 15A23; 15A45

1. Introduction

To simplify the presentation, we first introduce some symbols used in this paper. Let $\mathbb{R}^{m\times n}$ be the set of $m\times n$ real matrices and $\mathbb{R}_r^{m\times n}$ be the subset of $\mathbb{R}^{m\times n}$ consisting of matrices with rank r. Let I_r be the identity matrix of order r. Given $A\in\mathbb{R}_r^{m\times n}$, the symbols A^T, $A^\dagger$, $\mathrm{tr}(A)$, $\det(A)$, $\lambda_{\max}(A)$, $\lambda_{\min}(A)$, $N(A)$, $\|A\|_2$, and $\|A\|_F$ stand for its transpose, Moore-Penrose inverse, trace, determinant, largest eigenvalue, smallest eigenvalue, null space, spectral norm, and Frobenius norm, respectively. For a matrix $A=(a_{ij})$, define its condition number by $\kappa(A)=\|A\|_2\|A^\dagger\|_2$ and differential by $dA=(da_{ij})$.

A matrix $Q\in\mathbb{R}_m^{m\times m}$ is said to be J_1-orthogonal if $Q^TJ_1Q=J_1$, where $J_1=\mathrm{diag}(\pm1)\in\mathbb{R}_m^{m\times m}$ is a signature matrix. The definition can be extended to the rectangular matrices. We say that a matrix $Q\in\mathbb{R}_n^{m\times n}$ is (J_1,J_2)-orthogonal if $Q^TJ_1Q=J_2$, where $J_2=\mathrm{diag}(\pm1)\in\mathbb{R}_n^{n\times n}$ is another signature matrix. More on the J-orthogonal matrices can be found in [1]. Considering the generalized orthogonal matrices, several matrix factorizations involved with orthogonal factors can be generalized to the corresponding hyperbolic ones. The hyperbolic QR factorization and the hyperbolic polar factorization are the two typical ones.

We say that a matrix $A\in\mathbb{R}_n^{m\times n}$ admits a hyperbolic QR factorization with respect to the signature matrices J_1 and J_2 if $A=QH$, where $Q\in\mathbb{R}_n^{m\times n}$ is (J_1,J_2)-orthogonal and $H\in\mathbb{R}_n^{n\times n}$ is upper triangular with positive diagonal elements. When the leading principal minors of A^TJ_1A

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have the same signs as the corresponding minors of J_2, the hyperbolic QR factorization of A always exists and is unique [2, 3]. The above condition is also necessary. In addition, there are some other forms of the hyperbolic QR factorization [4, 5]. A matrix $A \in \mathbb{R}_n^{m\times n}$ admits a hyperbolic polar factorization $A = QH$, where $Q \in \mathbb{R}_n^{m\times n}$ is (J_1, J_2)-orthogonal, if and only if there exists a J_2-symmetric matrix $H \in \mathbb{R}_n^{n\times n}$, i.e., the matrix H satisfies $J_2H = H^TJ_2$, such that $J_2A^TJ_1A = H^2$ and $N(A) = N(H)$ (see [1, 6–8]).

In this paper, we consider the following problem. Given $A \in \mathbb{R}_n^{m\times n}$, $X \in \mathbb{R}_r^{m\times r}$, and $Y \in \mathbb{R}_r^{n\times r}$ such that $\operatorname{rank}(A + XY^T) = n$, find a (J_1, J_2)-orthogonal matrix $P \in \mathbb{R}_n^{m\times n}$ and a nonsingular matrix $U \in \mathbb{R}_n^{n\times n}$ such that

$$A + XY^T = PU.$$

The problem is called the rank-r updating problem of the hyperbolic matrix factorizations, and the matrices P and U are referred to as the updated hyperbolic factors. When J_1 and J_2 are identity matrices, the problem reduces to the rank-r updating problem of the corresponding classical matrix factorizations, where the problem of the classical QR factorization has many important applications and has been extensively studied in the literature [9–11]. Sun [10] considered its perturbation analysis. In the present paper, we will provide the perturbation analysis of the rank-r updating problem of the hyperbolic matrix factorizations using the technique from [10]. The sufficient conditions for the existence of two specific updated hyperbolic factorizations, i.e., the hyperbolic QR factorization and the hyperbolic polar factorization, are first given. Then, the differential inequalities and first order perturbation expansions for the general updated hyperbolic factors are obtained.

2. The hyperbolic matrix factorization updating problem

A lemma is first introduced as follows, which can be found in [12] and will be useful to derive the sufficient conditions for the existence of the updated hyperbolic matrix factorizations.

Lemma 2.1 *Let $M \in \mathbb{R}^{n\times n}$ be symmetric and $\|M\|_2 < 1$. Then there exists a lower triangular matrix $L \in \mathbb{R}^{n\times n}$ with positive diagonal elements such that*

$$J + M = LJL^T,$$

where $J = \operatorname{diag}(\pm 1) \in \mathbb{R}_n^{n\times n}$.

Theorem 2.1 *Let $A \in R_n^{m\times n}$ have the following hyperbolic matrix factorization*

$$A = QH, \tag{1}$$

where Q is (J_1, J_2)-orthogonal. Let $X \in \mathbb{R}_r^{m\times r}$ and $Y \in \mathbb{R}_r^{n\times r}$ such that $\operatorname{rank}(A + XY^T) = n$. Define $V = H^{-T}Y$,

$$M = Q^TJ_1XV^T + VX^TJ_1Q + VX^TJ_1XV^T,$$

and assume that $\|M\|_2 < 1$.

i) If H in (1) is upper triangular with positive diagonal elements, then $A+XY^T$ has the unique hyperbolic QR factorization

$$A+XY^T=PU, \tag{2}$$

where P is (J_1,J_2)-orthogonal and U is upper triangular with positive diagonal elements.

ii) If H in (1) is J_2-symmetric and $H^TL=L^TH$, where $J_2+M=LJ_2L^T$, then $A+XY^T$ has the hyperbolic polar factorization

$$A+XY^T=PU, \tag{3}$$

where P is (J_1,J_2)-orthogonal and U is J_2-symmetric.

Proof Since A has the hyperbolic matrix factorization $A=QH$, upon computation, we have

$$(A+XY^T)^TJ_1(A+XY^T)=H^T(J_2+M)H.$$

Note that $\|M\|_2<1$. Applying Lemma 2.1, we have

$$J_2+M=LJ_2L^T,$$

where L is lower triangular with positive diagonal elements. Thus,

$$(A+XY^T)^TJ_1(A+XY^T)=H^TLJ_2L^TH. \tag{4}$$

i) When H is upper triangular with positive diagonal elements, partition the matrices H, L, and J_2 in the same form as follows

$$H=\begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},\ L=\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix},\ J_2=\begin{bmatrix} J_{211} & 0 \\ 0 & J_{222} \end{bmatrix},$$

where $H_{11},L_{11},J_{211}\in\mathbb{R}^{k\times k}$. After some calculation, we have that the k-th leading principal minor of $(A+XY^T)^TJ_1(A+XY^T)$ is

$$\det(H_{11}^TL_{11}J_{211}L_{11}^TH_{11})=\big(\det(H_{11}^TL_{11})\big)^2\det(J_{211}),$$

which has the same sign as the k-th leading principal minor of J_2. Therefore, $A+XY^T$ has the unique hyperbolic QR factorization as in (2).

ii) When H is J_2-symmetric, from (4), it is seen that

$$J_2(A+XY^T)^TJ_1(A+XY^T)=J_2H^TLJ_2L^TH,$$

which combined with $H^TL=L^TH$ gives

$$J_2(A+XY^T)^TJ_1(A+XY^T)=(J_2H^TL)^2.$$

It is easy to check that J_2H^TL is J_2-symmetric. Moreover, since $\operatorname{rank}(A+XY^T)=n$, we have

$$N(A+XY^T)=N(J_2H^TL)=\text{Null}.$$

Therefore, $A+XY^T$ has the hyperbolic polar factorization as in (3). □

In the following, we consider the perturbation analysis for the updating problem of the hyperbolic matrix factorizations. Some differential inequalities for the updated hyperbolic factors are first derived in the following, which generalize the corresponding ones in Theorem 3.1 in [10] and can be used to obtain their first order perturbation expansions.

Theorem 2.2 *Assume that* $A \in \mathbb{R}_n^{m\times n}$ *and* $A + XY^T \in R_n^{m\times n}$ *have the hyperbolic matrix factorizations as in (1) and (2), respectively. Define*

$$W = Q^T X,\ V = H^{-T} Y,\ S(Q,W,V,X) = Q^T Q + WV^T + VW^T + VX^T XV^T,$$

$$\delta(Q,W,V,X) = \lambda_{\min}(S(Q,W,V,X)),\ \rho(Q,W,V,X) = \lambda_{\max}(S(Q,W,V,X)),$$

and let the eigenvalues of $J_2 dUU^{-1}$ *be real. Then*

$$\|dU\|_F \leq \sqrt{2}\|P\|_2^3\Big(\kappa(A+XY^T)(\|dA\|_F + \|X\|_2\|dY\|_F) + \frac{\|A+XY^T\|_2\|V\|_2}{(\delta(Q,W,V,X))^{1/2}}\|dX\|_F\Big), \quad (5)$$

$$\|dU\|_F \leq \sqrt{2}(\frac{\rho(Q,W,V,X)}{\delta(Q,W,V,X)})^{1/2}\|P\|_2^3(\kappa(A)\|Q\|_2^2(\|dA\|_F + \|X\|_2\|dY\|_F) + \|Q\|_2\|A\|_2\|V\|_2\|dX\|_F), \quad (6)$$

$$\|dP\|_F \leq \sqrt{2}\kappa(\widetilde{P})\|P\|_2\Big(\|(A+XY^T)^\dagger\|_2(\|dA\|_F + \|X\|_2\|dY\|_F) + \frac{\|V\|_2}{(\delta(Q,W,V,X))^{1/2}}\|dX\|_F\Big), \quad (7)$$

$$\|dP\|_F \leq \frac{\sqrt{2}\kappa(\widetilde{P})\|P\|_2}{(\delta(Q,W,V,X))^{1/2}}(\|A^\dagger\|_2\|Q\|_2(\|dA\|_F + \|X\|_2\|dY\|_F) + \|V\|_2\|dX\|_F), \quad (8)$$

where $\widetilde{P} = [P, P_{[\perp]}]$ *satisfies*

$$\widetilde{P}^T J_1 \widetilde{P} = \begin{bmatrix} J_2 & 0 \\ 0 & J_3 \end{bmatrix} \in \mathbb{R}_m^{m\times m},\ J_3 = \mathrm{diag}(\pm 1) \in \mathbb{R}_{(m-n)}^{(m-n)\times(m-n)}. \quad (9)$$

Proof From (2), we know that P and U are the differentiable functions of the elements of A, X, and Y. Thus, differentiating the equation (2), we have

$$dA + dXY^T + XdY^T = dPU + PdU. \quad (10)$$

Premultiplying (10) by $P^T J_1$ and postmultiplying it by U^{-1}, and noting $P^T J_1 P = J_2$ gives

$$P^T J_1 dAU^{-1} + P^T J_1 dXY^T U^{-1} + P^T J_1 XdY^T U^{-1} = P^T J_1 dP + J_2 dUU^{-1}. \quad (11)$$

Note that differentiating the equation $P^T J_1 P = J_2$ leads to

$$dP^T J_1 P + P^T J_1 dP = 0. \quad (12)$$

Thus, considering (11) and (12), and setting $\Phi = P^T J_1 dAU^{-1} + P^T J_1 dXY^T U^{-1} + P^T J_1 XdY^T U^{-1}$, we have

$$\Phi + \Phi^T = J_2 dUU^{-1} + (J_2 dUU^{-1})^T. \quad (13)$$

From the Schur's triangulariation theorem on real matrix [13, Theorem 2.3.4] and the hypothesis that the eigenvalues of $J_2 dUU^{-1}$ are real, there exists an orthogonal matrix G such that

$$J_2 dUU^{-1} = GRG^T, \quad (14)$$

where R is upper triangular. Then

$$\|J_2dUU^{-1} + (J_2dUU^{-1})^T\|_F = \|GR^TG^T + GRG^T\|_F = \|R^T + R\|_F. \tag{15}$$

Applying the technique of [14], we have

$$\|R + R^T\|_F \geq \sqrt{2}\|R\|_F = \sqrt{2}\|GRG^T\|_F = \sqrt{2}\|J_2dUU^{-1}\|_F. \tag{16}$$

Furthermore, from the definition of the Frobenius norm, it follows that

$$\begin{aligned}\|J_2dUU^{-1}\|_F &= (\mathrm{tr}((J_2dU)(U^{-1}U^{-T})(J_2dU)^T))^{1/2}\\ &\geq (\mathrm{tr}((J_2dU)\lambda_{\min}(U^{-1}U^{-T})(J_2dU)^T))^{1/2} = \|U\|_2^{-1}\|dU\|_F,\end{aligned}$$

which together with (16), (15), and (13) leads to

$$\begin{aligned}\|dU\|_F \leq&\sqrt{2}\|U\|_2\|P^TJ_1dAU^{-1} + P^TJ_1dXY^TU^{-1} + P^TJ_1XdY^TU^{-1}\|_F\\ \leq&\sqrt{2}\|U\|_2\|P\|_2(\|U^{-1}\|_2(\|dA\|_F + \|X\|_2\|dY\|_F) + \|U^{-T}Y\|_2\|dX\|_F).\end{aligned} \tag{17}$$

In the following, we establish the upper bounds of $\|U\|_2, \|U^{-1}\|_2$, and $\|U^{-T}Y\|_2$, respectively.

Premultiplying (2) by $J_2P^TJ_1$ and noting $P^TJ_1P = J_2$ and $J_2J_2 = I_n$ leads to

$$J_2P^TJ_1(A + XY^T) = U, \tag{18}$$

which in turn implies

$$\|U\|_2 \leq \|P\|_2\|A + XY^T\|_2. \tag{19}$$

From the fact that $\mathrm{rank}(A + XY^T) = n$, it follows that

$$(A + XY^T)^\dagger(A + XY^T) = I_n,$$

which together with (2) leads to

$$U^{-1} = (A + XY^T)^\dagger P. \tag{20}$$

Then

$$\|U^{-1}\|_2 \leq \|P\|_2\|(A + XY^T)^\dagger\|_2. \tag{21}$$

In order to present the other forms of the upper bounds of $\|U\|_2$ and $\|U^{-1}\|_2$, and derive the upper bound of $\|U^{-T}Y\|_2$, we now prove the fact that if $\mathrm{rank}(A+XY^T) = n$, then $S(Q, W, V, X)$ is positive definite.

Take $Q_{[\perp]} \in \mathbb{R}^{m\times(m-n)}_{(m-n)}$ such that $\widetilde{Q} = [Q, Q_{[\perp]}]$ satisfies

$$\widetilde{Q}^TJ_1\widetilde{Q} = \begin{bmatrix} J_2 & 0\\ 0 & J_4\end{bmatrix} \in \mathbb{R}^{m\times m}_m,\ J_4 = \mathrm{diag}(\pm1) \in \mathbb{R}^{(m-n)\times(m-n)}_{(m-n)}.$$

Premultiplying the first equation above by $\widetilde{Q}\begin{bmatrix} J_2 & 0\\ 0 & J_4\end{bmatrix}$ and postmultiplying it by $\widetilde{Q}^{-1}J_1$, and noting

$$\begin{bmatrix} J_2 & 0\\ 0 & J_4\end{bmatrix}\begin{bmatrix} J_2 & 0\\ 0 & J_4\end{bmatrix} = I_m \text{ and } J_1J_1 = I_m$$

gives

$$\widetilde{Q}\begin{bmatrix} J_2 & 0 \\ 0 & J_4 \end{bmatrix}\widetilde{Q}^T = J_1,$$

from which we have

$$QJ_2Q^T + Q_{[\perp]}J_4Q_{[\perp]}^T = J_1.$$

Define $\Omega_1 = J_2Q^TJ_1X$ and $\Omega_2 = J_4Q_{[\perp]}^TJ_1X$. Then, considering the above equation, it is easy to verify that

$$\widetilde{Q}\begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix} = X.$$

As a result,

$$A + XY^T = \widetilde{Q}\begin{bmatrix} H \\ 0 \end{bmatrix} + \widetilde{Q}\begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix}V^TH = \widetilde{Q}CH, \tag{22}$$

where $C = \begin{bmatrix} I_n \\ 0 \end{bmatrix} + \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix}V^T$. From the fact that $\text{rank}(A + XY^T) = n$, we have that

$$(A + XY^T)^T(A + XY^T) = (\widetilde{Q}CH)^T(\widetilde{Q}CH) = H^T(\widetilde{Q}C)^T(\widetilde{Q}C)H$$

is positive definite, and so is $(\widetilde{Q}C)^T(\widetilde{Q}C)$. Note that

$$\begin{aligned}\widetilde{Q}C = \widetilde{Q}\left(\begin{bmatrix} I_n \\ 0 \end{bmatrix} + \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix}V^T\right) &= Q + Q\Omega_1V^T + Q_{[\perp]}\Omega_2V^T \\ &= Q + QJ_2Q^TJ_1XV^T + Q_{[\perp]}J_4Q_{[\perp]}^TJ_1XV^T = Q + XV^T.\end{aligned}$$

Then

$$\begin{aligned}(\widetilde{Q}C)^T(\widetilde{Q}C) &= (Q + XV^T)^T(Q + XV^T) \\ &= Q^TQ + Q^TXV^T + VX^TQ + VX^TXV^T = S(Q, W, V, X).\end{aligned} \tag{23}$$

Therefore, $S(Q, W, V, X)$ is positive definite. Now, using the above fact, we give the other forms of the upper bounds of $\|U\|_2$ and $\|U^{-1}\|_2$, and derive the upper bound of $\|U^{-T}Y\|_2$.

From (18), (20), and (22), we have $U = J_2P^TJ_1\widetilde{Q}CH$ and $U^{-1} = (\widetilde{Q}CH)^\dagger P$. Then

$$\|U\|_2 \le \|P\|_2\|\widetilde{Q}C\|_2\|H\|_2, \quad \|U^{-1}\|_2 \le \|P\|_2\|(\widetilde{Q}CH)^\dagger\|_2.$$

Considering (23), we get

$$\|U\|_2 \le (\rho(Q, W, V, X))^{1/2}\|P\|_2\|H\|_2. \tag{24}$$

Note that

$$\begin{aligned}\|(\widetilde{Q}CH)^\dagger\|_2 &= \lambda_{\max}^{1/2}\left((\widetilde{Q}CH)^\dagger((\widetilde{Q}CH)^\dagger)^T\right) = \lambda_{\max}^{1/2}\left(((\widetilde{Q}CH)^T(\widetilde{Q}CH))^\dagger\right) \\ &= \lambda_{\max}^{1/2}\left(((\widetilde{Q}CH)^T(\widetilde{Q}CH))^{-1}\right) = \lambda_{\max}^{1/2}\left(H^{-1}((\widetilde{Q}C)^T(\widetilde{Q}C))^{-1}H^{-T}\right) \\ &= \|H^{-1}((\widetilde{Q}C)^T(\widetilde{Q}C))^{-1}H^{-T}\|_2^{1/2} \le \|H^{-1}\|_2\|((\widetilde{Q}C)^T(\widetilde{Q}C))^{-1}\|_2^{1/2}.\end{aligned} \tag{25}$$

Thus, considering (23), we get

$$\|U^{-1}\|_2 \le \|H^{-1}\|_2 \|P\|_2/(\delta(Q,W,V,X))^{1/2}. \tag{26}$$

Note that $U^{-T}Y = P^T((\widetilde{Q}CH)^{\dagger})^T Y$. Then

$$\|U^{-T}Y\|_2 \le \|P\|_2 \|((\widetilde{Q}CH)^{\dagger})^T Y\|_2. \tag{27}$$

Similarly to the induction of (25), we have

$$\|((\widetilde{Q}CH)^{\dagger})^T Y\|_2 = \|V^T((\widetilde{Q}C)^T(\widetilde{Q}C))^{-1}V\|_2^{1/2}.$$

Substituting the above equation into (27) and considering (23), we get

$$\|U^{-T}Y\|_2 \le \|P\|_2 \|V\|_2/(\delta(Q,W,V,X))^{1/2}. \tag{28}$$

In addition, from $A = QH$, it follows that

$$\|H\|_2 = \|J_1 Q^T J_1 A\|_2 \le \|Q\|_2 \|A\|_2,\ \|H^{-1}\|_2 = \|A^{\dagger} Q\|_2 \le \|A^{\dagger}\|_2 \|Q\|_2. \tag{29}$$

Thus, combining (17) with (19), (21), and (28), we get the inequality (5), and combining (17) with (24), (26), (28), and (29), we get the inequality (6).

Next, we prove the inequalities (7) and (8). From (10), we have

$$dP = (dA + dXY^T + XdY^T)U^{-1} - PdUU^{-1}. \tag{30}$$

Take $P_{[\perp]} \in \mathbb{R}^{m\times(m-n)}_{(m-n)}$ such that $\widetilde{P} = \left[P, P_{[\perp]}\right]$ satisfies (9). Premultiplying (30) by $P^T J_1$ and noting $P^T J_1 P = J_2$ and (14) gives

$$P^T J_1 dP = P^T J_1 (dA + dXY^T + XdY^T)U^{-1} - GRG^T. \tag{31}$$

Premultiplying (31) by G^T and postmultiplying it by G gives

$$G^T P^T J_1 dPG = G^T P^T J_1 (dA + dXY^T + XdY^T)U^{-1}G - R. \tag{32}$$

Meanwhile, from (12), it follows that

$$G^T P^T J_1 dPG + G^T dP^T J_1 PG = 0.$$

That is, $G^T P^T J_1 dPG$ is skew-symmetric. While, for any skew-symmetric matrix B, from the definition of the Frobenius norm, we have

$$\|B\|_F^2 = 2\|B_L\|_F^2,$$

where B_L denotes the lower triangular part of B. Applying the above fact to $G^T P^T J_1 dPG$ and noting the structure of R in (32) gives

$$\|G^T P^T J_1 dPG\|_F^2 = 2\|(G^T P^T J_1 dPG)_L\|_F^2 = 2\|[G^T P^T J_1 (dA + dXY^T + XdY^T)U^{-1}G]_L\|_F^2.$$

Since G is orthogonal and $\|B_L\|_F^2 \le \|B\|_F^2$ for any square matrix B, we have

$$\|P^T J_1 dP\|_F^2 \le 2\|P^T J_1 (dA + dXY^T + XdY^T)U^{-1}\|_F^2. \tag{33}$$

On the other hand, premultiplying (30) by $P_{[\perp]}^T J_1$ and noting $P_{[\perp]}^T J_1 P = 0$ gives

$$P_{[\perp]}^T J_1 dP = P_{[\perp]}^T J_1 (dA + dXY^T + XdY^T)U^{-1}.$$

Then

$$\|P_{[\perp]}^T J_1 dP\|_F^2 = \|P_{[\perp]}^T J_1 (dA + dXY^T + XdY^T)U^{-1}\|_F^2. \tag{34}$$

Note that

$$\|\widetilde{P}^T J_1 dP\|_F^2 = \|P^T J_1 dP\|_F^2 + \|P_{[\perp]}^T J_1 dP\|_F^2. \tag{35}$$

Thus, substituting (33) and (34) into (35) gives

$$\begin{aligned}
\|\widetilde{P}^T J_1 dP\|_F^2 &\le 2\|P^T J_1 (dA + dXY^T + XdY^T)U^{-1}\|_F^2 + \|P_{[\perp]}^T J_1 (dA + dXY^T + XdY^T)U^{-1}\|_F^2 \\
&= \|P^T J_1 (dA + dXY^T + XdY^T)U^{-1}\|_F^2 + \|\widetilde{P}^T J_1 (dA + dXY^T + XdY^T)U^{-1}\|_F^2 \\
&\le 2\|\widetilde{P}^T J_1 (dA + dXY^T + XdY^T)U^{-1}\|_F^2.
\end{aligned}$$

Since $dP = J_1 \widetilde{P}^{-T} \widetilde{P}^T J_1 dP$, we have

$$\begin{aligned}
\|dP\|_F &\le \|\widetilde{P}^{-1}\|_F \|\widetilde{P}^T J_1 dP\|_F \\
&\le \sqrt{2}\kappa(\widetilde{P}) \left(\|U^{-1}\|_2(\|dA\|_F + \|X\|_2\|dY\|_F) + \|U^{-T}Y\|_2\|dX\|_F\right),
\end{aligned}$$

which together with (21) and (28) yields the inequality (7), and together with (26), (28), and (29) yields the inequality (8). □

Remark 2.1 If the hyperbolic matrix factorization in Theorem 2.2 is the hyperbolic QR factorization, the hypothesis that the eigenvalues of $J_2 dUU^{-1}$ are real is unnecessary. The reason is that, in this case, $J_2 dUU^{-1}$ itself is real upper triangular. Applying the technique from [14], we have

$$\|J_2 dUU^{-1} + (J_2 dUU^{-1})^T\|_F \ge \sqrt{2}\|J_2 dUU^{-1}\|_F,$$

which is the combination of (15) and (16). Hence, the Schur's triangulariation theorem is needless here, and nor is the hypothesis. Whereas, for hyperbolic polar factorization, the hypothesis is necessary. If not, the relation (14) cannot be derived, and nor can (16). In addition, if $J_2 dUU^{-1}$ is symmetric, of course its eigenvalues are real, we can obtain two smaller bounds. That is, the coefficient $\sqrt{2}$ in the bounds (5), (6), (7), and (8) will be replaced with 1.

Using Theorem 2.2, we can state the first order perturbation expansions for the updated hyperbolic factors, which extend the corresponding results in Theorem 3.2 in [10].

Theorem 2.3 *Assume that the conditions in Theorem 2.2 are satisfied. Moreover, let $\varepsilon_0 > 0$ be small enough so that the hyperbolic matrix factorizations*

$$A + \varepsilon E = (Q + R(\varepsilon))(H + S(\varepsilon)) \tag{36}$$

and

$$A + \varepsilon E + (X + \varepsilon F)(Y + \varepsilon G)^T = (P + Z(\varepsilon))(U + T(\varepsilon)) \tag{37}$$

always exist for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then

$$\begin{aligned}
\|T(\varepsilon)\|_F \le &\sqrt{2}\|P\|_2^3 \Big(\kappa(A + XY^T)(\|E\|_F + \|X\|_2\|G\|_F) + \\
&\frac{\|A + XY^T\|_2 \|V\|_2}{(\delta(Q, W, V, X))^{1/2}}\|F\|_F\Big)|\varepsilon| + O(\varepsilon^2),
\end{aligned} \tag{38}$$

$$\begin{aligned}
\|T(\varepsilon)\|_F \leq & \sqrt{2}(\frac{\rho(Q,W,V,X)}{\delta(Q,W,V,X)})^{1/2}\|P\|_2^3(\kappa(A)\|Q\|_2^2(\|E\|_F + \|X\|_2\|G\|_F)+ \\
& \|Q\|_2\|A\|_2\|V\|_2\|F\|_F)|\varepsilon| + O(\varepsilon^2), \qquad (39) \\
\|Z(\varepsilon)\|_F \leq & \sqrt{2}\kappa(\widetilde{P})\|P\|_2\Big(\|(A+XY^T)^\dagger\|_2(\|E\|_F + \|X\|_2\|G\|_F)+ \\
& \frac{\|V\|_2}{(\delta(Q,W,V,X))^{1/2}}\|F\|_F\Big)|\varepsilon| + O(\varepsilon^2), \qquad (40) \\
\|Z(\varepsilon)\|_F \leq & \frac{\sqrt{2}\kappa(\widetilde{P})\|P\|_2}{(\delta(Q,W,V,X))^{1/2}}\big(\|A^\dagger\|_2\|Q\|_2(\|E\|_F + \|X\|_2\|G\|_F)+ \\
& \|V\|_2\|F\|_F\big)|\varepsilon| + O(\varepsilon^2), \qquad (41)
\end{aligned}$$

where $\widetilde{P} = \big[P, P_{[\perp]}\big]$ *satisfies (9).*

Proof For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, let

$$\begin{aligned}
& A(\varepsilon) = A + \varepsilon E,\ Q(\varepsilon) = Q + R(\varepsilon),\ H(\varepsilon) = H + S(\varepsilon), \\
& P(\varepsilon) = P + Z(\varepsilon),\ U(\varepsilon) = U + T(\varepsilon),\ X(\varepsilon) = X + \varepsilon F,\ Y(\varepsilon) = Y + \varepsilon G, \\
& W(\varepsilon) = Q^T(\varepsilon)X(\varepsilon),\ V(\varepsilon) = H^{-T}(\varepsilon)Y(\varepsilon),\ \widetilde{P}(\varepsilon) = [P(\varepsilon),\ P_{[\perp]}(\varepsilon)]. \qquad (42)
\end{aligned}$$

Then when $\varepsilon \to 0$, the following facts hold

$$\begin{aligned}
& \kappa(A(\varepsilon)) = \kappa(A) + \mathrm{O}(\varepsilon),\ \kappa(A(\varepsilon) + X(\varepsilon)Y^T(\varepsilon)) = \kappa(A + XY^T) + \mathrm{O}(\varepsilon), \\
& \|A(\varepsilon) + X(\varepsilon)Y^T(\varepsilon)\|_2 = \|A + XY^T\|_2 + \mathrm{O}(\varepsilon), \\
& \|(A(\varepsilon) + X(\varepsilon)Y^T(\varepsilon))^\dagger\|_2 = \|(A + XY^T)^\dagger\|_2 + \mathrm{O}(\varepsilon), \\
& \|A^\dagger(\varepsilon)\|_2 = \|A^\dagger\|_2 + \mathrm{O}(\varepsilon), \|Q(\varepsilon)\|_2 = \|Q\|_2 + \mathrm{O}(\varepsilon), \\
& \|P(\varepsilon)\|_2 = \|P\|_2 + \mathrm{O}(\varepsilon),\ \|\widetilde{P}(\varepsilon)\|_2 = \|\widetilde{P}\|_2 + \mathrm{O}(\varepsilon),\ \|V(\varepsilon)\|_2 = \|V\|_2 + \mathrm{O}(\varepsilon), \\
& \|\delta(Q(\varepsilon), W(\varepsilon), V(\varepsilon), X(\varepsilon))\|_2 = \|\delta(Q,W,V,X)\|_2 + \mathrm{O}(\varepsilon), \\
& \|\rho(Q(\varepsilon), W(\varepsilon), V(\varepsilon), X(\varepsilon))\|_2 = \|\rho(Q,W,V,X)\|_2 + \mathrm{O}(\varepsilon). \qquad (43)
\end{aligned}$$

Thus, from (42), (43), and the inequality (5), we have

$$\begin{aligned}
\|T(\varepsilon)\|_F = & \|U(\varepsilon) - U(0)\|_F = \|\int_0^\varepsilon \mathrm{d}U(\tau)\|_F \leq \int_0^{|\varepsilon|} \|\mathrm{d}U(\tau)\|_F \\
\leq & \sqrt{2}\int_0^{|\varepsilon|} \|P\|_2^3\Big(\kappa(A(\tau) + X(\tau)Y^T(\tau))(\|E\|_F + \|X(\tau)\|_2\|G\|_F)+ \\
& \frac{\|A(\tau) + X(\tau)Y^T(\tau)\|_2\|V(\tau)\|_2}{(\delta(Q(\tau), W(\tau), V(\tau), X(\tau)))^{1/2}}\|F\|_F\Big)\mathrm{d}\tau \\
\leq & \sqrt{2}\|P\|_2^3\Big(\kappa(A + XY^T)(\|E\|_F + \|X\|_2\|G\|_F)+ \\
& \frac{\|A + XY^T\|_2\|V\|_2}{(\delta(Q,W,V,X))^{1/2}}\|F\|_F\Big)|\varepsilon| + \mathrm{O}(\varepsilon^2).
\end{aligned}$$

Using the same method, combining (42) and (43) with the inequalities (6), (7), and (8), we can obtain the inequalities (39), (40), and (41), respectively. □

Remark 2.2 When the signature matrices such as J_1, J_2, J_3, and J_4 reduce to the corresponding identity matrices, the results in Theorems 2.2 and 2.3 will reduce to the corresponding ones for the updating problem of the classical matrix factorizations [10].

References

[1] N. J. HIGHAM. *J-orthogonal matrices: properties and generation*. SIAM Rev., 2003, **45**(3): 504–519.

[2] A. BUNSE-GERSTNER. *An analysis of the HR algorithm for computing the eigenvalues of a matrix*. Linear Algebra Appl., 1981, **35**: 155–173.

[3] M. BERHANU. *The Polynomial Eigenvalue Problem*. PhD Thesis, School of Mathematics, University of Manchester, 2005.

[4] S. SINGER. *Indefinite QR factorization*. BIT, 2006, **46**(1): 141–161.

[5] S. SINGER, S. SINGER. *Rounding-error and perturbation bounds for the indefinite QR factorization*. Linear Algebra Appl., 2000, **309**(1-3): 103–119.

[6] Y. BOLSHAKOV, C. V. M. VAN DER MEE, A. C. M. RAN, et al. *Extension of isometries in fnite-dimensional hyperbolic scalar product spaces and polar decompositions*. SIAM J. Matrix Anal. Appl., 1997, **18**(3): 752–774.

[7] Y. BOLSHAKOV, C. V. M. VAN DER MEE, A. C. M. RAN, et al. *Polar decompositions in finite dimensional hyperbolic scalar product spaces: general theory*. Linear Algebra Appl., 1997, **261**: 91–141.

[8] U. KINTZEL. *Procrustes problems in finite dimensional indefinite scalar product spaces*. Linear Algebra Appl., 2005, **402**: 1–28.

[9] J. DANIEL, W. B. GRAGG, L. KAUFMAN, G. W. STEWART. *Reorthogonalization and stable algorithms for updating the Gram-Schmidt QR factorization*. Math. Comp., 1976, **30**(136): 772–795.

[10] Jiguang SUN. *Perturbation analysis of the Cholesky downdating and QR updating problems*. SIAM J. Matrix Anal. Appl., 1995, **16**(3): 760–775.

[11] G. H. GOLUB, C. F. VAN LOAN. *Matrix Computation*. 3rd ed., The Johns Hopkins University Press, Baltimore, 1996.

[12] Ximing FANG. *The generalized Cholesky factorization and perturbation of the real symmetric matrices*. J. Zhaoqing Univ., 2008, **29** (5): 14–16. (in Chinese)

[13] R. A. HORN. C. R. JOHNSON. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[14] G. W. STEWART. *Perturbation bounds for the QR factorization of a matrix*. SIAM J. Numer. Anal., 1977, **14**(3): 509–518.