Journal of Mathematical Research with Applications
Jan., 2013, Vol. 33, No. 1, pp. 101–110
DOI:10.3770/j.issn:2095-2651.2013.01.011
Http://jmre.dlut.edu.cn

Exponential Stability for Stochastic Volterra-Levin Equations

Li CHEN*, **Lianggen HU**
Department of Mathematics, Faculty of Science, Ningbo University, Zhejiang 315211*, P. R. China*

Abstract In this paper, we investigate the exponential stability in pth moment as well as the almost surely exponential stability of solutions of stochastic Volterra-Levin equations (SVLEs in short) by the use of fixed point theorem for $p \geq 2$. Our results extend and improve the corresponding results obtained in [3, 12], and the result in [12] is a special case of our results.

Keywords exponential stability; fixed point theorem; stochastic Volterra-Levin equation; pth moment.

MR(2010) Subject Classification 34K50; 60H10

1. Introduction

Mathematical model plays an important role in many branches of science and industry. For example, deterministic/stochastic differential equations and functional differential equations have been extensively used to model many of the phenomena arising in lots of areas such as finance, economics, biology, physics, medicine and so on. The existence and uniqueness of solutions for such equations have been extensively considered in many papers. Meanwhile, many papers have studied the properties of the solutions by the use of various methods [1–10, 12–19].

Lyapunov's direct method has been used to deal with those problems above for more than one hundred years, and many excellent results were obtained [4, 5, 14, 15]. However, numerous difficulties are encountered in the study of some special equations by utilizing Lyapunov's direct method, which has been a powerful tool in dealing with deterministic/stochastic differential equations and functional differential equations.

Recently, Becker [1], Burton [2], Furumochi [6], Raffoul [16] and Zhang [19] et. al. have overcome those difficulties to consider those differential equations by using fixed point theorem. On the other hand, Satio [17] and Serban [18] have investigated the stability of differential equations by the use of the fixed point theorem. To our best knowledge, there are few papers to study the stochastic differential equations by the use of the fixed point theorem [10, 13].

Received March 26, 2011; Accepted September 1, 2011
Supported by NSFC of Tianyuan Youth Foundation (Grant No. 11126125), K. C. Wong Magna Fund of Ningbo University and Ningbo Natural Science Foundation (Grant No. 2012A610031).
* Corresponding author
E-mail address: cllively@163.com (Li CHEN)

More recently, some researchers have been interested in the study of stochastic Volterra-Levin equation

$$\mathrm{d}x(t) = -\Big(\int_{t-L}^{t} h(s-t)g(x(s))\mathrm{d}s\Big)\mathrm{d}t + \sigma(t)\mathrm{d}B(t), \quad t \geq 0 \tag{1.1}$$

with the initial condition

$$x(t) = \psi(t) \in C([-L,0];\mathbb{R}), \quad -L \leq t \leq 0, \tag{1.2}$$

where $\sigma \in C([0,\infty);\mathbb{R})$, $h \in C([-L,0];\mathbb{R})$ and $g \in C(\mathbb{R};\mathbb{R})$.

In [3], Burton showed that the stochastic Volterra-Levin equation (1.1) has a unique continuous solution and the solution is almost surely stable under some additional conditions. In 2010, Luo [12] gave some conditions to ensure the exponential stability in mean square and the almost surely exponential stability of the stochastic Volterra-Levin equation (1.1) by using the fixed point theorem, which generalize the results in [3].

Motivated by the above papers, we investigate the exponential stability and the almost surely exponential stability of solutions in pth moment of the stochastic Volterra-Levin equation (1.1) by the use of the fixed point theorem for $p \geq 2$. As we will see, our results extend and improve the corresponding results in [3, 12] in two aspects: (1) We extend the exponential stability in mean square of the stochastic Volterra-Levin equation (1.1) to exponential stability in pth ($p \geq 2$) moment, that is, the result in [12] is a special case in our results; (2) We obtain the almost surely exponential stability of the stochastic Volterra-Levin equation (1.1) from the exponential stability in pth ($p \geq 2$) moment.

The paper is organized as follows: In Section 2, we give some lemmas, which will be required in the proof of our main theorems; In Section 3, we give two theorems to show that the stochastic Volterra-Levin equation (1.1) is exponentially stable in pth ($p \geq 2$) moment and almost surely exponentially stable in pth ($p \geq 2$) moment. Meanwhile, three examples are also given to illustrate our results.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and $\mathcal{F}_0$ contains all P-null sets), in which a standard Brownian motion $\{B(t), t \geq 0\}$ is defined. Let $L > 0$ and $C([-L,0];\mathbb{R})$ denote the all continuous functions φ from $[-L,0]$ to $\mathbb{R}$ with the supremum norm $\|\varphi\| = \sup_{-L\leq\theta\leq 0} |\varphi(\theta)|$. For an arbitrary interval $[a,b]$ of $\mathbb{R}$, we denote all continuous functions $\varphi : [a,b] \to \mathbb{R}$ by $C([a,b];\mathbb{R})$. For $p > 0$ and $t \geq 0$, let $L^p_{F_t}([-L,0];\mathbb{R})$ denote the family of all F_t-measurable function $\phi(\theta) \in C([-L,0];\mathbb{R})$ such that $\sup_{-L\leq\theta\leq 0} E|\phi(\theta)|^p < \infty$.

Lemma 2.1 ([7]) *The following inequalities hold:*

(i) $(x+y)^p \leq (1+\epsilon)^{p-1}(x^p + \epsilon^{1-p}y^p)$, $\epsilon > 0$.

(ii) $(\sum_{i=1}^n x_i)^p \leq C_p \sum_{i=1}^n x_i^p$, $C_p = 1$ *when* $0 < p \leq 1$, $C_p = n^{(p-1)}$ *when* $p \geq 1$.

Lemma 2.2 ([11]) *If the functions $f(x)$ and $g(x)$ are nonnegative and continuous, then for any $p>1$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, the following inequality holds:*

$$\int_a^b f(x)g(x)\mathrm{d}x \le \Big(\int_a^b f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}.$$

Lemma 2.3 ([14]) *Let $\mathcal{M}^2((-\infty,0];\mathbb{R}^d)$ denote the family of the process $\{\xi(t)\}_{t\le 0}$ in $L^p((-\infty,0];\mathbb{R}^d)$ such that $E\int_{-\infty}^0|\xi(t)|^2<\infty$ a.s. If $p>2$, $g\in\mathcal{M}^2([t_0,T];\mathbb{R}^{d\times m})$ such that $E\int_{t_0}^T|g(s)|^p\mathrm{d}s<\infty$, then*

$$E\Big|\int_{t_0}^T g(s)\mathrm{d}B(s)\Big|^p \le (\frac{p(p-1)}{2})^{\frac{p}{2}}T^{\frac{p-2}{2}}E\int_{t_0}^T|g(s)|^p\mathrm{d}s.$$

Remark 2.1 If $p=2$, then the Itô isometry in [8] reduces to

$$E\Big|\int_{t_0}^T g(s)\mathrm{d}B(s)\Big|^2 = E\Big[\int_{t_0}^T g^2(s)\mathrm{d}s\Big].$$

Hence, Lemma 2.3 is satisfied for $p\ge 2$.

Lemma 2.4 ([9]) *Suppose constants $p\ge 1$ and $\lambda>0$. Let $X(t)=\int_0^t g(s)\mathrm{d}B(s)$. Then*

$$P\Big(\max_{0\le u\le t}|X(u)|\ge\lambda\Big)\le\frac{E|X(t)|^p}{\lambda^p},$$

for all $t\ge 0$.

Definition 2.1 ([7]) *The solution of the stochastic Volterra-Levin equation (1.1) with the initial condition (1.2) is said to be exponentially stable in pth moment if*

$$\limsup_{t\to\infty}\frac{1}{t}\ln(E|x(t)|^p)<0,\quad \text{for all}\ \ t\ge 0.$$

Definition 2.2 ([12]) *The solution of the stochastic Volterra-Levin equation (1.1) is said to be almost surely exponentially stable if there exist a constant $\lambda>0$ and a finite random variable β such that*

$$|x(t)|\le\beta e^{-\lambda t}\quad \text{a.s.}$$

for all $t\ge 0$, or equivalently if

$$\limsup_{t\to\infty}\frac{1}{t}\ln|x(t)|\le -\lambda\quad \text{a.s.}$$

3. Main results

Theorem 3.1 *Suppose that the following conditions hold:*

(1) *There exists a continuous function $g(x)$ such that*

$$xg(x)\ge 0, g(0)=0\ \text{and}\ \ \gamma:=\lim_{x\to 0}\frac{g(x)}{x}\ \text{exists;}\tag{3.1}$$

(2) *There exists a constant $\alpha>0$ such that*

$$\frac{g(x)}{x}\ge 2\alpha;\tag{3.2}$$

(3) There exists a constant $K > 0$ such that $|g(x) - g(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$, and there exists a continuous function $h(t)$ such that

$$2K \int_{-L}^{0} |h(s)s| \mathrm{d}s < 1 \text{ and } \int_{-L}^{0} h(s) \mathrm{d}s = 1; \tag{3.3}$$

(4) One of the following two conditions holds:

$$\int_{0}^{t} e^{2p\alpha s} |\sigma(s)|^p \mathrm{d}s < \infty, \ \ \text{for all } \ t \geq 0, \ p \geq 2; \tag{3.4}$$

or

$$\int_{0}^{\infty} e^{2p\alpha s} |\sigma(s)|^p \mathrm{d}s = \infty \ \ \text{and} \ \ e^{\alpha t} |\sigma(t)|^p \to 0 \ \text{ as } \ t \to \infty, \ \ p \geq 2. \tag{3.5}$$

Then the solution of the stochastic Volterra-Levin equation (1.1) with the initial condition (1.2) is exponentially stable in pth moment and the convergence rate is α.

Proof Define a continuous function $a : [0, \infty) \to [0, \infty)$ by

$$a(t) := \begin{cases} \dfrac{g(x(t))}{x(t)}, & \text{if } \ x(t) \neq 0, \\ \gamma, & \text{if } \ x(t) = 0. \end{cases}$$

The stochastic Volterra-Levin equation (1.1) can be written as

$$\mathrm{d}x(t) = -a(t)x(t)\mathrm{d}t + d\Big(\int_{-L}^{0} h(s) \int_{t+s}^{t} g(x(u)) \mathrm{d}u \mathrm{d}s\Big) + \sigma(t) \mathrm{d}B(t), \tag{3.6}$$

for $t \geq 0$. Multiplying $e^{\int_0^t a(u)\mathrm{d}u}$ in both sides of (3.6) and integrating from 0 to $t > 0$, we get

$$\begin{aligned} x(t) =& e^{-\int_0^t a(u)\mathrm{d}u} \Big(\psi(0) - \int_{-L}^{0} h(\omega) \int_{\omega}^{0} g(\psi(u)) \mathrm{d}u \mathrm{d}\omega\Big) + \\ & \int_{-L}^{0} h(\omega) \int_{t+\omega}^{t} g(x(u)) \mathrm{d}u \mathrm{d}\omega - \\ & \int_{0}^{t} e^{-\int_s^t a(u)\mathrm{d}u} a(s) \int_{-L}^{0} h(\omega) \int_{s+\omega}^{s} g(x(u)) \mathrm{d}u \mathrm{d}\omega \mathrm{d}s + \\ & \int_{0}^{t} e^{-\int_s^t a(u)\mathrm{d}u} \sigma(s) \mathrm{d}B(s) \end{aligned} \tag{3.7}$$

for any $t \geq 0$. It is easy to verify that (3.6) is equivalent to (3.7).

Let $(\mathcal{B}, \|\cdot\|)$ be the Banach space of all bounded and continuous pth $\mathcal{F}_0$-adapted process $\phi(t, \omega) : [-L, \infty) \times \Omega \to \mathbb{R}$ with the supremum norm $\|\phi\|_{\mathcal{B}} := \sup_{t \geq 0} E|\phi(t)|^p$ for $\phi \in \mathcal{B}$. Define a complete metric space $(\mathcal{S}, \rho)$ by

$$\mathcal{S} := \{\phi : [-L, \infty) \times \Omega \to \mathbb{R} \mid \phi \in \mathcal{B}, \phi(t) = \psi(t), \text{for } t \in [-L, 0], \ e^{\alpha t} E|\phi(t, \omega)|^p \to 0 \text{ as } t \to \infty\},$$

where ρ denotes the supremum metric: for $\phi_1, \phi_2 \in \mathcal{S}, \ \rho(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|$.

For a given continuous function $\psi : [-L, 0] \times \Omega \to \mathbb{R}$ and $x \in \mathcal{S}$, we define an operator $T : \mathcal{S} \to \mathcal{S}$ by

$$(Tx)(t) := \psi(t) \ \ \text{for } t \in [-L, 0];$$

and for $t > 0$,

$$
\begin{aligned}
(Tx)(t) :=& e^{-\int_0^t a(u)\mathrm{d}u}\Big(\psi(0) - \int_{-L}^0 h(\omega)\int_\omega^0 g(\psi(u))\mathrm{d}u\mathrm{d}\omega\Big) + \\
& \int_{-L}^0 h(\omega)\int_{t+\omega}^t g(x(u))\mathrm{d}u\mathrm{d}\omega - \\
& \int_0^t e^{-\int_s^t a(u)\mathrm{d}u} a(s)\int_{-L}^0 h(\omega)\int_{s+\omega}^s g(x(u))\mathrm{d}u\mathrm{d}\omega\mathrm{d}s + \\
& \int_0^t e^{-\int_s^t a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s) = \sum_{i=1}^4 \mu_i(t),
\end{aligned} \tag{3.8}
$$

where

$$
\begin{aligned}
\mu_1(t) &:= e^{-\int_0^t a(u)\mathrm{d}u}\Big(\psi(0) - \int_{-L}^0 (\omega)\int_\omega^0 g(\psi(u))\mathrm{d}u\mathrm{d}\omega\Big), \\
\mu_2(t) &:= \int_{-L}^0 h(\omega)\int_{t+\omega}^t g(x(u))\mathrm{d}u\mathrm{d}\omega, \\
\mu_3(t) &:= -\int_0^t e^{-\int_s^t a(u)\mathrm{d}u} a(s)\int_{-L}^0 h(\omega)\int_{s+\omega}^s g(x(u))\mathrm{d}u\mathrm{d}\omega\mathrm{d}s, \\
\mu_4(t) &:= \int_0^t e^{-\int_s^t a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s).
\end{aligned}
$$

Next, we divide the proof into three steps.

Step 1. We show that the operator T is pth continuous on $[0,\infty)$.

Let $x \in \mathcal{S}$, $t_1 \geq 0$ and $|\xi|$ be sufficiently small. Then from Lemma 2.1 and (3.8), we get that

$$
E|(Tx)(t_1+\xi) - (Tx)(t_1)|^p \leq 4^{(p-1)}\sum_{i=1}^4 E|\mu_i(t_1+\xi) - \mu_i(t_1)|^p.
$$

Clearly, it is easy to verify that $E|\mu_i(t_1+\xi) - \mu_i(t_1)|^p \to 0$ as $\xi \to 0$ $(i = 1,2,3)$.

On the other hand, Lemmas 2.1 and 2.3 imply

$$
\begin{aligned}
E|\mu_4(t_1+\xi) - \mu_4(t_1)|^p \leq& 2^{(p-1)}E\Big|\int_0^{t_1} e^{-\int_s^{t_1} a(u)\mathrm{d}u}\big(e^{-\int_{t_1}^{t_1+\xi} a(u)\mathrm{d}u} - 1\big)\sigma(s)\mathrm{d}B(s)\Big|^p + \\
& 2^{(p-1)}E\Big|\int_{t_1}^{t_1+\xi} e^{-\int_s^{t_1+\xi} a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s)\Big|^p \\
\leq& 2^{\frac{p}{2}-1}(p^2-p)^{\frac{p}{2}} t_1^{\frac{p-2}{2}} E\int_0^{t_1} e^{-p\int_s^{t_1} a(u)\mathrm{d}u}\Big|e^{-\int_{t_1}^{t_1+\xi} a(u)\mathrm{d}u} - 1\Big|^p |\sigma(s)|^p\mathrm{d}s + \\
& 2^{\frac{p}{2}-1}(p^2-p)^{\frac{p}{2}}(t_1+\xi)^{\frac{p-2}{2}} E\int_{t_1}^{t_1+\xi} e^{-p\int_s^{t_1+\xi} a(u)\mathrm{d}u}|\sigma(s)|^p\mathrm{d}s \\
\to& 0 \quad \text{as} \quad \xi \to 0.
\end{aligned}
$$

Therefore, the operator T is pth continuous on $[0,\infty)$.

Step 2. We claim that $T(\mathcal{S}) \subset \mathcal{S}$.

In fact, we only show that $e^{\alpha t}E|\mu_i(t)|^p \to 0$ as $t \to \infty$ $(i = 1,2,3,4)$. Obviously, it is easy to see that $e^{\alpha t}E|\mu_i(t)|^p \to 0$ as $t \to \infty$ $(i = 1,2)$.

From Lemma 2.2, we have

$$
\begin{aligned}
e^{\alpha t}E|\mu_3(t)|^p &\le e^{\alpha t}\Big(\int_0^t e^{-\int_s^t a(u)\mathrm{d}u}a(s)\mathrm{d}s\Big)^{\frac{p}{q}}E\int_0^t e^{-\int_s^t a(u)\mathrm{d}u}a(s)\Big|\int_{-L}^0 h(\omega)\int_{s+\omega}^s g(x(u))\mathrm{d}u\mathrm{d}\omega\Big|^p\mathrm{d}s\\
&\le e^{\alpha t}\Big(\int_{-L}^0 |h(\omega)|^q\mathrm{d}\omega\Big)^{\frac{p}{q}}E\int_0^t e^{-\int_s^t a(u)\mathrm{d}u}a(s)\int_{-L}^0\Big|\int_{s+\omega}^s g(x(u))\mathrm{d}u\mathrm{d}\omega\Big|^p\mathrm{d}s\\
&\le e^{\alpha t}\Big(\int_{-L}^0 |h(\omega)|^q\mathrm{d}\omega\Big)^{\frac{p}{q}}E\int_0^t e^{-\int_s^t a(u)\mathrm{d}u}a(s)\int_{-L}^0 (-\omega)^{\frac{p}{q}}\int_{s+\omega}^s |g(x(u))|^p\mathrm{d}u\mathrm{d}\omega\mathrm{d}s\\
&\le e^{\alpha t}\frac{(KL)^p}{p}\Big(\int_{-L}^0 |h(\omega)|^q\mathrm{d}\omega\Big)^{\frac{p}{q}}\int_0^t e^{-\int_s^t a(u)\mathrm{d}u}a(s)\int_{s-L}^s E|x(u)|^p\mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

Since $x\in\mathcal{S}$, there exists a constant $T_1>0$ such that $e^{\alpha s}E|x(s)|^p<\epsilon$ for any $\epsilon>0$ and $s\ge T_1-L$. Set $M:=\frac{(KL)^p}{p}(\int_{-L}^0|h(\omega)|^q\mathrm{d}\omega)^{\frac{p}{q}}$, then we find

$$
\begin{aligned}
e^{\alpha t}E|\mu_3(t)|^p \le& Me^{\alpha t}\Big\{\int_0^{T_1} e^{-\int_s^t a(u)\mathrm{d}u}a(s)\int_{s-L}^s E|x(u)|^p\mathrm{d}u\mathrm{d}s+\\
&\int_{T_1}^t e^{-\int_s^t a(u)\mathrm{d}u}a(s)\int_{s-L}^s e^{-\alpha u}e^{\alpha u}E|x(u)|^p\mathrm{d}u\mathrm{d}s\Big\}\\
\le& Me^{\alpha t}\Big\{LE\big(\sup_{-L\le s\le T_1}|x(s)|^p\big)\int_0^{T_1} e^{-2\alpha(t-s)}a(s)\mathrm{d}s+\\
&\epsilon\int_{T_1}^t e^{-\alpha(t-s)}e^{-\frac{1}{2}\int_s^t a(u)\mathrm{d}u}a(s)\frac{(e^{\alpha L}-1)e^{-\alpha s}}{\alpha}\mathrm{d}s\Big\}\\
=& M\Big\{e^{-\alpha t}LE\big(\sup_{-L\le s\le T_1}|x(s)|^p\big)\int_0^{T_1} e^{2\alpha s}a(s)\mathrm{d}s+\frac{2(e^{\alpha L-1})}{\alpha}\epsilon\Big\}\\
\to& 0\ \text{ as }\ t\to\infty. \qquad (3.9)
\end{aligned}
$$

On the other hand, condition (3.4) or (3.5) implies

$$
\begin{aligned}
e^{\alpha t}E|\mu_4(t)|^p \le& e^{\alpha t}\big(\frac{p(p-1)}{2}\big)^{\frac{p}{2}}t^{\frac{p-2}{2}}\int_0^t e^{-p\int_s^t a(u)\mathrm{d}u}|\sigma(s)|^p\mathrm{d}s\\
\le&\big(\frac{p(p-1)}{2}\big)^{\frac{p}{2}}\frac{t^{\frac{p-2}{2}}}{e^{\alpha t}}e^{-(2p-2)\alpha t}\int_0^t e^{2p\alpha s}|\sigma(s)|^p\mathrm{d}s\\
\to& 0\ \text{ as }\ t\to\infty. \qquad (3.10)
\end{aligned}
$$

Consequently, (3.9) and (3.10) imply $T(\mathcal{S})\subset\mathcal{S}$.

Step 3. We show that T is a contractive mapping.

For any $x,y\in\mathcal{S}$, we have

$$
\begin{aligned}
&E\sup_{s\in[0,t]}|(Tx)(s)-(Ty)(s)|^p\\
&\le E\sup_{s\in[0,t]}\Big\{\int_{-L}^0|h(v)|\int_{s+v}^s|g(x(u))-g(y(u))|\mathrm{d}u\mathrm{d}v+\\
&\quad\int_0^s e^{-\int_v^s a(\tau)\mathrm{d}\tau}a(v)\int_{-L}^0|h(\tau)|\int_{v+\tau}^v|g(x(u))-g(y(u))|\mathrm{d}u\mathrm{d}\tau\mathrm{d}v\Big\}^p\\
&\le E\sup_{s\in[0,t]}\Big\{K\int_{-L}^0|h(v)|\int_{s+v}^s|x(u)-y(u)|\mathrm{d}u\mathrm{d}v+
\end{aligned}
$$

$$K\int_0^s e^{-\int_v^s a(\tau)\mathrm{d}\tau}a(v)\int_{-L}^0|h(\tau)|\int_{v+\tau}^v|x(u)-y(u)|\mathrm{d}u\mathrm{d}\tau\mathrm{d}v\Big\}^p$$
$$\leq E\sup_{s\in[0,t]}|x(s)-y(s)|^p\Big(2K\int_{-L}^0|h(s)s|\mathrm{d}s\Big)^p.$$

Together with (3.3), we conclude that T is a contractive mapping. Hence, from the Banach fixed point theorem, we get that there is a unique fixed point $x(t)$ which is the solution of the stochastic Volterra-Levin equation (1.1) with the initial condition (1.2) in $\mathcal{S}$ and $e^{\alpha t}E|x(t)|^p\to 0$ as $t\to\infty$. We conclude that the stochastic Volterra-Levin equation (1.1) with the initial condition (1.2) is exponentially stable in pth moment and the convergence rate is α. This completes the proof. □

Theorem 3.2 *Suppose that the same assumptions as in Theorem 3.1 hold. Then the solution of the stochastic Volterra-Levin equation (1.1) with the initial condition (1.2) is almost surely exponentially stable.*

Proof Let N be a sufficiently large positive integer. We get that for $N\leq t\leq N+1$,

$$\begin{aligned}x(t)=&e^{-\int_N^t a(u)\mathrm{d}u}\Big(x(N)-\int_{-L}^0 h(\omega)\int_{N+\omega}^N g(x(u))\mathrm{d}u\mathrm{d}\omega\Big)+\\&\int_{-L}^0 h(\omega)\int_{t+\omega}^t g(x(u))\mathrm{d}u\mathrm{d}\omega-\\&\int_N^t e^{-\int_s^t a(u)\mathrm{d}u}a(s)\int_{-L}^0 h(\omega)\int_{s+\omega}^s g(x(u))\mathrm{d}u\mathrm{d}\omega\mathrm{d}s+\\&\int_N^t e^{-\int_s^t a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s).\end{aligned}$$

Then, for any given $\epsilon_N>0$, we find

$$\begin{aligned}&P\Big\{\sup_{N\leq t\leq N+1}|\,x(t)\,|>\epsilon_N\Big\}\\&\leq P\Big\{\sup_{N\leq t\leq N+1}e^{-\int_N^t a(u)\mathrm{d}u}\Big|x(N)-\int_{-L}^0 h(\omega)\int_{N+\omega}^N g(x(u))\mathrm{d}u\mathrm{d}\omega\Big|>\epsilon_N/4\Big\}+\\&\quad P\Big\{\sup_{N\leq t\leq N+1}\Big|\int_{-L}^0 h(\omega)\int_{t+\omega}^t g(x(u))\mathrm{d}u\mathrm{d}\omega\Big|>\epsilon_N/4\Big\}+\\&\quad P\Big\{\sup_{N\leq t\leq N+1}\Big|\int_N^t e^{-\int_s^t a(u)\mathrm{d}u}a(s)\int_{-L}^0 h(\omega)\int_{s+\omega}^s g(x(u))\mathrm{d}u\mathrm{d}\omega\mathrm{d}s\Big|>\epsilon_N/4\Big\}+\\&\quad P\Big\{\sup_{N\leq t\leq N+1}\Big|\int_N^t e^{-\int_s^t a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s)\Big|>\epsilon_N/4\Big\}\\&:=O_1+O_2+O_3+O_4,\end{aligned}$$

where

$$O_1:=P\Big\{\sup_{N\leq t\leq N+1}e^{-\int_N^t a(u)\mathrm{d}u}\Big|x(N)-\int_{-L}^0 h(\omega)\int_{N+\omega}^N g(x(u))\mathrm{d}u\mathrm{d}\omega\Big|>\epsilon_N/4\Big\},$$
$$O_2:=P\Big\{\sup_{N\leq t\leq N+1}\Big|\int_{-L}^0 h(\omega)\int_{t+\omega}^t g(x(u))\mathrm{d}u\mathrm{d}\omega\Big|>\epsilon_N/4\Big\},$$

$$O_3 := P\Big\{ \sup_{N\le t\le N+1} \Big| \int_N^t e^{-\int_s^t a(u)\mathrm{d}u} a(s) \int_{-L}^0 h(\omega) \int_{s+\omega}^s g(x(u))\mathrm{d}u\mathrm{d}\omega\mathrm{d}s \Big| > \epsilon_N/4 \Big\},$$
$$O_4 := P\Big\{ \sup_{N\le t\le N+1} \Big| \int_N^t e^{-\int_s^t a(u)\mathrm{d}u} \sigma(s)\mathrm{d}B(s) \Big| > \epsilon_N/4 \Big\}.$$

Theorem 3.1 implies that there exists a constant $C > 0$ such that $E|x(t)|^p \le Ce^{-\alpha t}$ for all $t \ge 0$. Hence, we have

$$\begin{aligned} O_1 &\le (4/\epsilon_N)^p E \sup_{N\le t\le N+1} e^{-p\int_N^t a(u)\mathrm{d}u} \Big| x(N) - \int_{-L}^0 h(\omega) \int_{N+\omega}^N g(x(u))\mathrm{d}u\mathrm{d}\omega \Big|^p \\ &\le (4/\epsilon_N)^p 2^{p-1} \Big\{ E|x(N)|^p + E\Big| \int_{-L}^0 h(\omega) \int_{N+\omega}^N g(x(u))\mathrm{d}u\mathrm{d}\omega \Big|^p \Big\} \\ &\le (4/\epsilon_N)^p 2^{p-1} \Big\{ Ce^{-\alpha N} + \Big(\int_{-L}^0 |h(\omega)|^q \mathrm{d}\omega \Big)^{\frac{p}{q}} E \int_{-L}^0 \Big| \int_{N+\omega}^N g(x(u))\mathrm{d}u \Big|^p \mathrm{d}\omega \Big\} \\ &\le (4/\epsilon_N)^p 2^{p-1} \Big\{ Ce^{-\alpha N} + \Big(\int_{-L}^0 |h(\omega)|^q \mathrm{d}\omega \Big)^{\frac{p}{q}} E \int_{-L}^0 (-\omega)^{\frac{p}{q}} \int_{N+\omega}^N |g(x(u))|^p \mathrm{d}u\mathrm{d}\omega \Big\} \\ &\le (4/\epsilon_N)^p 2^{p-1} \Big\{ Ce^{-\alpha N} + \frac{(KL)^p}{p} \Big(\int_{-L}^0 |h(\omega)|^q \mathrm{d}\omega \Big)^{\frac{p}{q}} \int_{N-L}^N E|x(u)|^p \mathrm{d}u \Big\} \\ &\le (4/\epsilon_N)^p 2^{p-1} C \Big\{ 1 + \frac{(KL)^p}{p} \frac{e^{\alpha L-1}}{\alpha} \Big(\int_{-L}^0 |h(\omega)|^q \mathrm{d}\omega \Big)^{\frac{p}{q}} \Big\} e^{-\alpha N}. \end{aligned} \tag{3.11}$$

Similarly, we get that

$$O_2 \le (4/\epsilon_N)^p \frac{(KL)^p}{p} \frac{C(e^{\alpha L-1})}{\alpha} \Big(\int_{-L}^0 |h(\omega)|^q \mathrm{d}\omega \Big)^{\frac{p}{q}} e^{-\alpha N}, \tag{3.12}$$

and

$$O_3 \le (4/\epsilon_N)^p \frac{(KL)^p}{p} \frac{C(e^{\alpha L-1})}{\alpha} \Big(\int_{-L}^0 |h(\omega)|^q \mathrm{d}\omega \Big)^{\frac{p}{q}} e^{-\alpha N}. \tag{3.13}$$

On the other hand, it follows from Lemmas 2.3 and 2.4 that

$$\begin{aligned} O_4 &\le (4/\epsilon_N)^p E \sup_{N\le t\le N+1} \Big| \int_N^t e^{-\int_s^t a(u)\mathrm{d}u} \sigma(s)\mathrm{d}B(s) \Big|^p \\ &\le (4/\epsilon_N)^p \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} t^{\frac{p-2}{2}} \sup_{N\le t\le N+1} \int_N^t e^{-p\int_s^t a(u)\mathrm{d}u} |\sigma(s)|^p \mathrm{d}s \\ &\le (4/\epsilon_N)^p \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} \frac{(N+1)^{\frac{p-2}{2}}}{e^{\alpha N}} \frac{\int_N^{N+1} e^{2p\alpha s} |\sigma(s)|^p \mathrm{d}s}{e^{(2p-2)\alpha N}} e^{-\alpha N}. \end{aligned} \tag{3.14}$$

Since N is sufficiently large, we get from (3.4) or (3.5) that there exists a constant $L_1 > 0$ such that

$$\frac{\int_N^{N+1} e^{2p\alpha s} |\sigma(s)|^p \mathrm{d}s}{e^{(2p-2)\alpha N}} < L_1.$$

Together with (3.14), we obtain

$$O_4 \le (4/\epsilon_N)^p \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} L_1 e^{-\alpha N}. \tag{3.15}$$

Therefore, from (3.11)–(3.13) and (3.15), we find

$$P\Big\{ \sup_{N\le t\le N+1} |x(t)| > \epsilon_N \Big\} \le (D/{\epsilon_N}^p) e^{-\alpha N},$$

where

$$D := 4^p\Big\{2^{p-1}C + 2^{p-1}\frac{C(KL)^p}{p}\frac{(e^{\alpha L}-1)}{\alpha}\Big(\int_{-L}^{0}|h(\omega)|^q \mathrm{d}\omega\Big)^{\frac{p}{q}} + \frac{2C(KL)^p}{p}\frac{(e^{\alpha L}-1)}{\alpha}\Big(\int_{-L}^{0}h^q(\omega)\mathrm{d}\omega\Big)^{\frac{p}{q}} + \big(\frac{p(p-1)}{2}\big)^{\frac{p}{2}}L_1\Big\}e^{-\alpha N}.$$

Choosing $\epsilon_N = D^{\frac{1}{p}}e^{-\frac{\alpha N}{2p}}$, we get from the Borel-Cantelli Lemma that there exists a random time $T(\omega) > 0$ such that

$$|\, x(t) \,| \le D^{\frac{1}{p}}e^{\frac{\alpha}{2p}}e^{-\frac{\alpha t}{2p}}, \quad \text{a.s.} \quad \text{for} \quad t > T(\omega).$$

This completes the proof. □

Remark 3.1 In this paper, we investigate that the solutions of the stochastic Volterra-Levin equations are exponentially stable in pth moment and almost surely exponentially stable. If we only take $p = 2$ in Theorem 3.1, then we get the same results as in [12]. Therefore, we generalize and extend some results in recent references.

Remark 3.2 Adopting the similar method as in Theorems 3.1 and 3.2, we can consider the case in n-dimensional space, but we take the norm $|\cdot|$ which is the Euclidean norm in R^n.

Next, we give two examples to illustrate our results.

Example 3.1 Consider the following equation:

$$\mathrm{d}x(t) = -\Big(\int_{t-L}^{t}h(s-t)g(x(s))\mathrm{d}s\Big)\mathrm{d}t + \sigma(t)\mathrm{d}B(t), \quad t > 0, \tag{3.16}$$

where $g(x) = 2x$, $h(t) = t^2$ and $\sigma(t) = e^{-\frac{1}{2}\alpha t}$.

Clearly,

$$\int_0^t e^{4\alpha s}|\sigma(s)|^2\mathrm{d}s = \frac{1}{3\alpha}(e^{3\alpha t}-1)$$

and

$$e^{\alpha t}|\sigma(t)|^2 = 1.$$

We can see that the conditions (i) and (ii) of Theorem 2.1 in [12] fail.

On the other hand, it is obvious that $xg(x) \ge 0$, $g(0) = 0$, and $\lim_{x\to 0}\frac{g(x)}{x} = 2$, which implies that the condition (1) in Theorem 3.1 holds. For any constant $\alpha \in (0,1)$, we can see that $\frac{g(x)}{x} \ge 2\alpha$, that is, the condition (2) in Theorem 3.1 holds. Let $K = 2$. Then the condition (3) in Theorem 3.1 holds if $L = 3^{\frac{1}{3}}$. If $p = 3$,

$$\int_0^\infty e^{6\alpha s}|\sigma(s)|^3\mathrm{d}s = \infty,$$

and

$$e^{\alpha t}|\sigma(t)|^3 = e^{-\alpha t} \to 0 \;\; \text{as} \;\; t \to \infty.$$

Then the condition (4) in Theorem 3.1 holds. From Theorem 3.1, the solution of (3.16) is exponentially stable in 3rd moment and the convergence rate is α. Again from Theorem 3.2, the solution of (3.16) is almost surely exponentially stable.

Example 3.2 Consider the following equation:

$$\mathrm{d}x(t) = -\Big(\int_{t-L}^{t} h(s-t)g(x(s))\mathrm{d}s\Big)\mathrm{d}t + \sigma(t)\mathrm{d}B(t), \quad t > 0, \tag{3.17}$$

where $g(x) = 2x$, $h(t) = t^2$ and $\sigma(t) = e^{-3\alpha t}$.

Following the notation in Example 3.1, it is obvious that the conditions (1)–(3) in Theorem 3.1 hold. For any $p \geq 2$,

$$\int_0^t e^{2p\alpha s}|\sigma(s)|^p \mathrm{d}s = \int_0^t e^{-p\alpha s}\mathrm{d}s \leq \frac{1}{\alpha p}, \quad t > 0.$$

Hence the condition (4) in Theorem 3.1 holds. From Theorem 3.1, the solution of (3.17) is exponentially stable in pth moment and the convergence rate is α. Again from Theorem 3.2, the solution of (3.17) is almost surely exponentially stable.

Acknowledgements The authors would like to thank the referees very much for helpful comments and suggestions.

References

[1] L. C. BECKER, T. A. BURTON. *Stability, fixed points and inverses of delays.* Proc. Roy. Soc. Edinburgh Sect. A, 2006, **136**(2): 245–275.

[2] T. A. BURTON. *Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem.* Nonlinear Stud., 2002, **9**(2): 181–190.

[3] T. A. BURTON. *Stability by Fixed Point Theory for Functional Differential Equations.* Dover Publications, Inc., Mineola, NY, 2006.

[4] K. D. DO, J. PAN. *Nonlinear control of an active heave compensation system.* Ocean Engineering, 2008, **35**: 558–571.

[5] K. D. DO, Z. P. JIANG, J. PAN. *Global partial-state feedback and output-feedback tracking controllers for underactuated ships.* Systems Control Lett., 2005, **54**(10): 1015–1036.

[6] T. FURUMOCHI. *Stabilities in FDEs by Schauder's theorem.* Nonlinear Anal., 2005, **63**: 217–224.

[7] Shigeng HU. *Stochastic Differential Equation.* Science Publishing, Beijing, 2008.

[8] B. K. KSENDAL. *Stochastic Differential Equations.* Springer Berlin Heidelberg, 2006.

[9] Yuanlie LIN. *Applied Stochastic Process.* Tsinghua University Press, Beijing, 2005.

[10] Dezhi LIU, Guiyuan YANG, Wei ZHANG. *The stability of neutral stochastic delay differential equations with Poisson jumps by fixed points.* J. Comput. Appl. Math., 2011, **235**(10): 3115–3120.

[11] Yulian LIU. *Counsellong Book of Mathematical Analysis.* Higher Education Publisher, Beijing, 2003.

[12] Jiaowan LUO. *Fixed points and exponential stability for stochastic Volterra-Levin equations.* J. Comput. Appl. Math., 2010, **234**(3): 934–940.

[13] Jiaowan LUO. *Fixed points and stability of neutral stochastic delay differential equations.* J. Math. Anal. Appl., 2007, **334**(1): 431–440.

[14] Xuerong MAO. *Stochastic Differential Equations and Their Applications.* Horwood Publishing Limited, Chichester, 1997.

[15] Xuerong MAO. *Exponential Stability of Stochastic Differential Equations.* Marcel Dekker, Inc., New York, 1994.

[16] Y. N. RAFFOUL. *Stability in neutral nonlinear differential equations with functional delays using fixed-point theory.* Math. Comput. Modelling, 2004, **40**(7-8): 691–700.

[17] S. SAITO. *Global stability of solutions for quasilinear ordinary differential systems.* Math. Japon., 1989, **34**(5): 821–829.

[18] M. A. SERBAN. *Global asymptotic stability for some difference equations via fixed point technique.* Semin. Fixed Point Theory Cluj-Napoca, 2001, **2**: 87–95.

[19] Bo ZHANG. *Fixed points and stability in differential equations with variable delays.* Nonlinear Anal., 2005, **63**: 233–242.