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On a Pair of Hyperstandard Reciprocal Relations with Applications

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Abstract The method of non-standard analysis (NSA) is used to construct a pair of hyperstandard reciprocal formulas involving certain non-standard difference operators with real-number orders. Our main result consists of some extensions of earlier results appearing previously [5]. An essential meaning of the paper is to indicate the fact that only the basic idea of NSA is applicable to the construction of a unified pattern that may have certain applications to both the analysis and the number theory.

Keywords difference operator; hyper-real field; zero-monad; Riemann-Liouville integral; Möbius-type inversion; reciprocal relation.

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1. Introduction

The main purpose of this paper is to construct a pair of general reciprocal formulas involving non-standard difference operators and inverse operators. We will show that both the Fleck-type generalized Möbius inversion formulae and the reciprocity between multivariate integrals and derivatives with fractional orders are included as important consequences.

Throughout we will make use of the non-standard analysis (NSA) with a few brief explanations [6–9]. As usual, R, Z and N denote the set of ordinary real numbers, the set of integers and the natural number set (not including 0), respectively. Correspondingly, ${}^*R$, ${}^*Z$ and ${}^*N$ denote, respectively, the non-standard extensions of R, Z and N, containing non-standard elements. In particular, ${}^*N_\infty$ denotes the set consisting of all positive infinite integers.

Certainly, the concept of monad is important and usefull. Usually $m(0)$ is used to denote the zero-monad that consists of all infinitesimals (including 0) in ${}^*\mathbf{R}$. For any ordinary real number $\alpha \in R$, the set of all elements of the form $\alpha + \epsilon$ with $\epsilon \in m(0)$ is called the α-monad $m(\alpha)$. Note that a well-known basic theorem asserts that every finite element $x \in {}^*\mathbf{R}$ has a unique representation $x = x^0 + \epsilon$ with $x^0 \in R$ and $\epsilon \in m(0)$, where x^0 is called the standard part of x.

The operation "taking standard part", denoted by st, is also very useful. For every finite number $\delta \in {}^*\mathbf{R}$ we denote its standard part by $\mathrm{st}(\delta)$ or $(\delta)^0 \equiv \delta^0$. In particular, for $\omega \in {}^*N_\infty$ we define $\mathrm{st}(\pm\omega) = \pm\infty$. Generally, we may also denote $\mathrm{st}({}^*R) = R$, $\mathrm{st}({}^*Z) = Z$, $\mathrm{st}({}^*N) = N$, etc.

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All functions of a single variable considered in this paper, unless otherwise stated, are defined on some set $D \subset {}^*\mathbf{R}$ and taking values in ${}^*\mathbf{R}$, namely, they belong to the function class $\mathrm{Map}(D, {}^*R)$. For the multivariable case, say s variables the function class may be written as $\mathrm{Map}(\mathbf{D}^s, {}^*\mathbf{R}^s)$.

As was shown before [4]. the classical Möbius inversion formulae in Number Theory could be expressed as a discrete analogue of the Newton-Leibniz formula for integral calculus by using ordinary difference and inverse difference operators. Also, it has become a common knowledge that derivatives and definite integrals are just given by the standart parts of some related differences and inverse differences with infinitesimal increments in NSA, respectively. Actually, a connection with these two facts just leads to such a fruitful idea that utilizing some non-standard differences and inverse differences with either an infinitesimal increment or a finite increment $\delta \in {}^*R$ could build up a kind of doubly implicative model that would imply certain important pairs of inversion formulas in Analysis and Number Theory. Indeed, all what we could develop in this paper precisely follows the idea just mentioned above.

2. Preliminaries

Given $a, b \in {}^*R$ with $a < b$. The intervals (a, b) and $[a, b]$ are defined as usual, viz.

$$(a, b) \equiv \{x | a < x < b, x \in {}^*\mathbf{R}\}, \quad [a, b] \equiv \{x | a \leq x \leq b\}.$$

Suppose that $0 < \delta \in {}^*R$ and that $\lambda = \nu\delta$ ($\nu \in {}^*N$) is a positive real number in ${}^*R$. Then we denote

$$W_\lambda :\equiv \{k\delta | 0 \leq k \leq \nu \equiv \lambda/\delta\}$$

and call it a partition set of $[0, \lambda]$, with step-length δ. Frequently we have to use the set W_ω ($\omega \in {}^*\mathbf{N}_\infty$):

$$W_\omega \equiv \{k\delta | 0 \leq k \leq \omega/\delta\}, \quad \omega/\delta \in {}^*\mathbf{N}_\infty.$$

Of particular interest is the case $0 < \delta \in m(0)$, viz. $\delta^0 = 0$. In this case W_ω is called a fine partition of $[0, \omega]$.

We shall consider functions of $\mathrm{Map}(W_\omega, {}^*R)$, denoted by f, g, etc. Since all operators to be employed are acting on functions of x within the interval $[0, \omega]$, we need to assume that

$$f(x) = g(x) = 0 \quad \text{for} \quad x < 0. \tag{2.1}$$

This may be called the "zero value condition" for f and g. Hereafter all functions to be used are assumed to satisfy the zero value condition (2.1).

Let us now introduce three basic definitions as follows.

Definition 2.1 *Given $f \in \mathrm{Map}(W_\omega, {}^*\mathbf{R})$ and any $\delta \in m(0)$ with $\delta > 0$. The hyperstandard backward difference operator $\overset{\delta}{\underset{x}{\Delta}}$ and divided difference operator $\overset{\delta}{\underset{x}{\Lambda}}$ are defined, respectively, via the relations*

$$\overset{\delta}{\underset{x}{\Delta}} f(x) = f(x) - f(x - \delta),$$

$$\overset{\delta}{\underset{x}{\Lambda}} f(x) = \delta^{-1} \overset{\delta}{\underset{x}{\Delta}} f(x) = \frac{1}{\delta}(f(x) - f(x-\delta)).$$

Definition 2.2 *Higher divided difference operators are defined by induction, namely*

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{1} = \overset{\delta}{\underset{x}{\Lambda}}, \quad \overset{\delta}{\underset{x}{\Lambda}}{}^{n} = \overset{\delta}{\underset{x}{\Lambda}}{}^{1} \overset{\delta}{\underset{x}{\Lambda}}{}^{n-1}, \quad n \geq 2,$$

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{0} = \mathbf{1}, \quad \mathbf{1}f(x) \equiv f(x), \quad \mathbf{1} \text{ being used as an identity operator.}$$

Definition 2.3 *For any given $\delta \in m(0)$ with $\delta > 0$, and any $g(x) \in \mathrm{Map}(W_\omega, {}^*R)$, the hyperstandard inverse difference operators (also called hyperstandard summation operator) $\overset{\delta}{\underset{x}{\Delta}}{}^{-1}$ and hyperstandard inverse divided difference operators $\overset{\delta}{\underset{x}{\Lambda}}{}^{-1}$ are defined respectively via the relations for $x = m\delta$ ($m \in N$):*

$$\overset{\delta}{\underset{x}{\Delta}}{}^{-1} g(x) = \sum_{j=0}^{m} g(j\delta) = \sum_{0 \leq j\delta \leq x} g(j\delta), \quad x = m\delta,$$

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{-1} g(x) = \delta \overset{\delta}{\underset{x}{\Delta}}{}^{-1} g(x) = \sum_{0 \leq j\delta \leq x} g(j\delta)\delta, \quad x = m\delta.$$

Moreover, higher inverse difference operators are defined inductively by the relations

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{-n} = \overset{\delta}{\underset{x}{\Lambda}}{}^{-1} \overset{\delta}{\underset{x}{\Lambda}}{}^{-(n-1)}, \quad n \geq 2.$$

Briefly one may write $\Lambda \equiv \delta^{-1}\Delta$ and $\Lambda^{-1} \equiv \delta\Delta^{-1}$, and it is not difficult to verify the following relation for f and g, under the zero value condition $f(x) = g(x) = 0$ for $x < 0$,

$$\Lambda^{-r}\Lambda^{r} = \Lambda^{r}\Lambda^{-r} = 1, \quad r \in N. \tag{2.2}$$

It is also evident that for the case $f(x)$ being differentiable for $0 < x \in {}^*R$ and $g(x)$ being integralbe on $[0, \lambda] \in {}^*R$, we have

$$\left(\overset{\delta}{\underset{x}{\Lambda}} f(x)\right)^0 = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x),$$

$$\left(\overset{\delta}{\underset{x}{\Lambda}}{}^{-1} g(x)\right)^0 = \left(\sum_{j=0}^{m} g(j\delta)\delta\right)^0 = \int_0^x g(t)\mathrm{d}t, \quad x = m\delta \in W_\omega,$$

where the differentiability and integrability are admitted in the sense of NSA.

Proposition 2.1 *Let f and g be functions of the class $\mathrm{Map}(W_\omega, {}^*R)$. Then for any given $n \in N$ and $x = m\delta \in W_\omega$, there hold the following expressions*

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{-n} g(x) = \left(\delta \overset{\delta}{\underset{x}{\Delta}}{}^{-1}\right)^n g(x) = \sum_{0 \leq t \leq x/\delta} \binom{-n}{t} (-1)^t g(x - t\delta)\delta^n, \tag{2.3}$$

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{n} f(x) = \left(\delta^{-1} \overset{\delta}{\underset{x}{\Delta}}\right)^n f(x) = \sum_{0 \leq t \leq x/\delta} \binom{n}{t} (-1)^t f(x - t\delta)\delta^{-n}, \tag{2.4}$$

where the summations involved in (2.3)-(2.4) both extend to all integers within the interval $[0, x/\delta] \equiv [0, m]$.

Actually, for $\delta = 1$ the equalities (2.3)-(2.4) have already been mentioned previously [4]. As may be observed, a direct verifiction of (2.3)-(2.4) could be done exactly in the same way as that for the case $\delta = 1$. So the details may be omitted here.

Remark 1 The explicit formulas (2.3)-(2.4) could be used to verify the reciprocal relations

$$f(x) = \overset{\delta}{\underset{x}{\Lambda}}{}^{-n} g(x) \Longleftrightarrow g(x) = \overset{\delta}{\underset{x}{\Lambda}}{}^{n} f(x). \tag{2.5}$$

These are equivalent to the following equalities

$$f(x) = \overset{\delta}{\underset{x}{\Lambda}}{}^{-n} \overset{\delta}{\underset{x}{\Lambda}}{}^{n} f(x), \quad g(x) = \overset{\delta}{\underset{x}{\Lambda}}{}^{n} \overset{\delta}{\underset{x}{\Lambda}}{}^{-n} g(x). \tag{2.6}$$

Remark 2 Making use of some non-standard computations, one can find that, for $0 < \delta \in m(0)$, taking standard parts of the both sides of (2.3)-(2.4) and of (2.5) will lead to the reciprocal relations

$$(f(x))^0 = \Big(\overset{\delta}{\underset{x}{\Lambda}}{}^{-n} g(x) \Big)^0 = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} g(t) \mathrm{d}t, \tag{2.7}$$

$$(g(x))^0 = \Big(\overset{\delta}{\underset{x}{\Lambda}}{}^{n} f(x) \Big)^0 = (\frac{\mathrm{d}}{\mathrm{d}t})^n f(x), \tag{2.8}$$

where $f(x)$ is assumed to have the derivative of order n, and $g(t)$ is assumed to be continuous.

Note that $g(t)$ and $f(x)$ appearing on the RHS (right-hand sides) of (2.7)-(2.8) are ordinary (standard) real functions, while $g(x)$ and $f(x)$ on the LHS (left-hand sides) are certainly their non-standard extensions in ${}^*R$, respectively. However for brevity we always use the same notations such as f, g, etc. This is actually a convention used frequently in NSA.

3. Operators with real-number orders

Recall the fact that the positive integer n appearing in the integration on the RHS of (2.7) could be replaced by any real number $r > 1$, so that we have the well-known Riemann-Liouville integral that leads to a classic definition for the integration of order r (see [10, Chap.2, §8]). Naturally, this suggests us to introduce the following two definitions for the non-standard difference operators and inverse operators.

Definition 3.1 *Let g and f be any functions belonging to* $\mathrm{Map}(W_\omega, {}^*R)$ *and satisfying the zero value condition (2.1). Then for any given $\delta \in {}^*R$ with $\delta > 0$ and any real number $r \geq 1$ we define*

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{-r} g(x) := \sum_{0 \leq t \leq x/\delta} \binom{-r}{t} (-1)^t g(x - t\delta) \delta^r, \tag{3.1}$$

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{r} f(x) := \sum_{0 \leq t \leq x/\delta} \binom{r}{t} (-1)^t f(x - t\delta) \delta^{-r}, \tag{3.2}$$

where $x \in W_\omega \equiv \{x = m\delta | 0 \leq m \leq \omega/\delta\}$, $\omega/\delta \in {}^*N_\infty$.

Definition 3.2 *With the same conditions as stated in Definition 3.1, we define the following* *for the case* $0 < \delta \in m(0)$ *and* $r \geq 1$*:*

$$D^{-r}g(x) \equiv (\frac{\mathrm{d}}{\mathrm{d}x})^{-r}g(x) := \mathrm{st}\Big(\overset{\delta}{\underset{x}{\Lambda}}{}^{-r}g(x)\Big), \tag{3.3}$$

$$D^{r}f(x) \equiv (\frac{\mathrm{d}}{\mathrm{d}x})^{r}f(x) := \mathrm{st}(\Lambda^{r}f(x)), \tag{3.4}$$

where $D^{-r}g(x)$ *and* $D^{r}f(x)$ *stand for the* r*-th order inverse derivative of* $g(x)$ *and* r*-th order derivative of* $f(x)$*, respectively; and it is assumed that the RHS of (3.3) and of (3.4) give definite values in* R*.*

In accordance with the above definition, we may say that $f(x)$ and $g(x)$ are r-th order differentiable and r-th order integrable, respectively, in ${}^*R$.

Proposition 3.1 *For a given real number* $r \geq 1$*, let* $g(x)$ *be* r*-th order integrable. Then* $D^{-r}g(x)$ *can be expressed in the form of Riemann-Liouville integral*

$$D^{-r}g(x) = \int_0^x \frac{(x-t)^{r-1}}{\Gamma(r)}g(t)\mathrm{d}t \tag{3.5}$$

where $g(t)$ *as a standard function belongs to the class* $\mathrm{Map}(R_+, R)$ *with* R_+ *denoting the set of non-negative real numbers.*

Proof We have to compute the RHS of (3.3) by using NSA techniques. In the first place, let us verify the following result: For $x, y \in W_\omega$ with $(x-y)^0 > 0$, and $\delta \in m(0)$ with $\delta > 0$, we have, for $r > 1$,

$$\mathrm{st}\left(\binom{(x-y)/\delta + r - 1}{(x-y)/\delta}\delta^{r-1}\right) = \frac{(x-y)^{r-1}}{\Gamma(r)}. \tag{3.6}$$

Recall that for any real number $z > 0$, we have the classical formula for the gama function $\Gamma(z)$:

$$\lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n!n^z} = \frac{1}{\Gamma(z)}. \tag{3.7}$$

Let us denote $n = (x-y)/\delta$ and rewrite the LHS of (3.6) in the form

$$\mathrm{st}\Big(\frac{(n+r-1)(n+r-2)\cdots r\cdot(r-1)\cdot((x-y)/\delta)^{r-1}}{n!\cdot(r-1)\cdot n^{r-1}}\cdot\delta^{r-1}\Big).$$

Taking $z = r-1$ and noticing that $n \in {}^*N_\infty$ (for $0 < \delta \in m(0)$), we see that the above expression just gives the standard real value $(x-y)^{r-1}/(r-1)\Gamma(r-1) = (x-y)^{r-1}/\Gamma(r)$. Thus (3.6) is verified.

Denote $x = m\delta$ so that $m = x/\delta$, and we may compute $D^{-r}g(x)$ as follows

$$\begin{aligned}
\mathrm{st}\Big(\overset{\delta}{\underset{x}{\Lambda}}{}^{-r}g(x)\Big) &= \mathrm{st}\sum_{0\leq t\leq m}\binom{-r}{t}(-1)^t g(m\delta - t\delta)\delta^r \\
&= \mathrm{st}\sum_{0\leq t\leq m}\binom{r+t-1}{t}g((m-t)\delta)\delta^r \\
&= \mathrm{st}\sum_{0\leq t\leq m}\binom{r+m-t-1}{m-t}g(t\delta)\delta^r \\
&= \mathrm{st}\sum_{0\leq t\delta=y\leq x}\binom{(x-y)/\delta+r-1}{(x-y)/\delta}\delta^{r-1}\cdot g(y)\delta, \quad y = t\delta,
\end{aligned}$$

where the last summation also extends to all non-negative integers t such that $0 \le t \le x/\delta = m$. Now making use of (3.6), we see that the standart part of the last summation precisely gives the value

$$\int_0^x \frac{(x-y)^{r-1}}{\Gamma(r)} g(y)\mathrm{d}y$$

in accordance with the concept of integration in NSA. Moreover, (3.5) is obviously true for the case $r = 1$. Hence the proposition holds for all $r \ge 1$. □

Ovbiously, (3.5) is an extension of the equality (2.7).

4. General reciprocity theorems and consequences

The object of this section is to present a pair of theorems and three corollaries.

Theorem 4.1 *For given $r \in R_+$ with $r \ge 1$, we have a pair of reciprocal relations as follows*

$$f(x) = \overset{\delta}{\underset{x}{\Lambda}}{}^{-r} g(x), \tag{4.1}$$

$$g(x) = \overset{\delta}{\underset{x}{\Lambda}}{}^{r} f(x), \tag{4.2}$$

where the operators involved are defined by (3.1)-(3.2).

Proof To show that (4.1)⇒ (4.2), we may do the computations, in accordance with (3.1) and (3.2) with $x = m\delta \in W_\omega$, as follows

$$\begin{aligned}
\overset{\delta}{\underset{x}{\Lambda}}{}^{r} f(x) &= \overset{\delta}{\underset{x}{\Lambda}}{}^{r} \Big(\overset{\delta}{\underset{x}{\Lambda}}{}^{-r} g(x) \Big) \\
&= \sum_{0 \le t \le x/\delta} \binom{-r}{t} (-1)^t \delta^r \overset{\delta}{\underset{x}{\Lambda}}{}^{r} g(x - t\delta) \\
&= \sum_{0 \le t \le x/\delta} \binom{-r}{t} (-1)^t \delta^r \sum_{0 \le j \le (x - t\delta)/\delta} \binom{r}{j} (-1)^j \delta^{-r} g(x - t\delta - j\delta) \\
&= \sum_{0 \le s \le x/\delta} (-1)^s g(x - s\delta) \sum_{t+j=s} \binom{-r}{t} \binom{r}{j} \\
&= \sum_{0 \le s \le m} (-1)^s g(x - s\delta) \binom{0}{s} = g(x).
\end{aligned}$$

In a similar manner, the implication (4.2)⇒ (4.1) can be verified. Hence we have the reciprocity (4.1)⇔(4.2). □

Briefly, the reciprocity may be written as

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{r} \overset{\delta}{\underset{x}{\Lambda}}{}^{-r} = \overset{\delta}{\underset{x}{\Lambda}}{}^{-r} \overset{\delta}{\underset{x}{\Lambda}}{}^{r} = 1 \text{ (identity operator).} \tag{4.3}$$

It is worth noticing that Theorem 4.1 can be naturally extended to the case involving multivariate functions $f((x))$ and $g((x))$ of the class $\mathrm{Map}(W_\omega^s, {}^*R)$ with $(x) \equiv (x_1, \ldots, x_s) \in W_\omega^s$ $(s \in N)$. In what follows it is always assumed that $f((x))$ and $g((x))$ satisfy the zero value condition, namely $f((x)) = g((x)) = 0$ whenever there is some $x_i < 0$ $(1 \le i \le s)$.

Theorem 4.2 *Given $\delta \in {}^*R$ with $\delta > 0$. Then for every s-type $(r_1, \ldots, r_s) \in R_+^s$ with each $r_i \geq 1$ $(1 \leq i \leq s)$ there hold the reciprocal relations*

$$f((x)) = \Big(\prod_{i=1}^{s} \overset{\delta}{\underset{x_i}{\Lambda}}{}^{-r_i}\Big) g((x)), \tag{4.4}$$

$$g((x)) = \Big(\prod_{i=1}^{s} \overset{\delta}{\underset{x_i}{\Lambda}}{}^{r_i}\Big) f((x)), \tag{4.5}$$

where both $f((x))$ and $g((x))$ belong to $\mathrm{Map}(W_\omega^s, {}^*R)$.

Note that both f and g involve s independent variables, so that the reciprocity between (4.4) and (4.5) can be verified by repeated applications of (4.3). Clearly Theorem 4.2 is an extension of Theorem 1 in [5], wherein r_i $(i = 1, \ldots, s)$ are restricted to be positive integers.

As immediate applications of Theorem 4.2, we shall now show that two particular choices of δ, viz. (i) $\delta \in m(0)$ with $\delta > 0$ and (ii) $\delta = 1 \in R$, will lead to two kinds of remarkable consequences, respectively.

Indeed, letting $0 < \delta \in m(0)$, taking the standard parts of the both sides of (4.4) and of (4.5) with recalling Definition 3.2, and making use of Proposition 3.1 for the cases $r = r_i$ $(i = 1, \ldots, s)$, we see that Theorem 4.2 implies the following

Corollary 4.3 *There hold the reciprocal relations via (4.4)-(4.5) with $F((x)) = \mathrm{st}\, f((x))$ and $G((x)) = \mathrm{st}\, g((x))$:*

$$F((x)) = \int_0^{x_1} \cdots \int_0^{x_s} \Big(\prod_{i=1}^{s} (x_i - t_i)^{r_i - 1}/\Gamma(r_i)\Big) G((t)) \mathrm{d}((t)), \tag{4.6}$$

$$G((x)) = (\frac{\partial}{\partial x_1})^{r_1} \cdots (\frac{\partial}{\partial x_s})^{r_s} F((x)), \tag{4.7}$$

where each real number $r_i \geq 1$ $(i = 1, \ldots, s)$, $\mathrm{d}(t) = \mathrm{d}t_1 \cdots \mathrm{d}t_s$, and $F((x))$ and $G((x))$ are assumed to satisfy certain differentiability and integrability conditions, respectively.

Certainly, the reciprocity between (4.6) and (4.7) is a well-known result in the multivariate calculus, and its deduction from (4.4)-(4.5) is a demonstration in the sense of NSA. Also, by a comparison of (4.6) with (3.5) we may rewrite the RHS of (4.6) in terms of the inverse differential operators, namely we have

$$F((x)) = (\frac{\partial}{\partial x_1})^{-r_1} \cdots (\frac{\partial}{\partial x_s})^{-r_s} G((x)). \tag{4.8}$$

Still worth noticing is the fact that Corollary 4.3 or the reciprocity between (4.7) and (4.8) may be viewed as a continuous analogue of Fleck-type generalization of Möbius inversion formulae. Obviously, for the choice (ii), $\delta = 1$, we have $W_\omega = \{m | 0 \leq m \leq \omega\}$. For the need in the number theory, we shall only consider functions defined on the set $N = W_\omega \backslash \{0, \omega\} = \{1, 2, 3, \ldots\}$. Denote by $S \equiv \{p\} \equiv \{p_i\}$ the sequence of all prime numbers in the increasing ordering:

$$2 = p_1 < p_2 < p_3 < \cdots$$

For positive integers n and d with $d|n$ (i.e., d is a divisor of n), write

$$n = p_1^{x_1} \cdots p_s^{x_s}, \quad d = p_1^{t_1} \cdots p_s^{t_s}, \quad s \in N$$

which may be rewritten as

$$n = \prod_{p|n} p^{\partial_p(n)}, \quad d = \prod_{p|n} p^{\partial_p(d)} \tag{4.9}$$

where $\partial_p(n)$ denotes the highest index k of p such that p^k divides n. The divisibility relation $d|n$ just means that $\partial_p(d) \le \partial_p(n)$, and this may also be denoted by

$$(t) \equiv (t_1, \ldots, t_s) \le (x) \equiv (x_1, \ldots, x_s).$$

We also denote $(x) - (t) \equiv (x_1 - t_1, \ldots, x_s - t_s)$, $(0) \equiv (0, \ldots, 0)$.

Note that for $\delta = 1$, we have $\overset{\delta}{\underset{x}{\Lambda}} \equiv \underset{x}{\Delta}$ (the ordinary backward difference operator in the calculus of finite differences), so that Theorem 4.2 could be specialized to give the following consequence.

Corollary 4.4 *For every $s \in N$ and every s-tuple $(r_1, \ldots, r_s) \in \mathbf{R}_+^s$ with each $r_i \ge 1$ $(1 \le i \le s)$, there always hold the reciprocal relations*

$$f((x)) = \Big(\prod_{i=1}^{s} \underset{x}{\Delta}^{-r_i}\Big) g((x)) = \sum_{(0)\le(t)\le(x)} \mu_{(-r)}((t)) g((x)-(t)), \tag{4.10}$$

$$g((x)) = \Big(\prod_{i=1}^{s} \underset{x}{\Delta}^{r_i}\Big) f((x)) = \sum_{(0)\le(t)\le(x)} \mu_{(r)}((t)) f((x)-(t)), \tag{4.11}$$

where $f((x)) \in \mathrm{Map}(N^s, R)$, $g((x)) \in \mathrm{Map}(N^s, R)$, and $(0) \le (t) \le (x)$ stands for the summation condition, and $\mu_{(-r)}((t))$ and $\mu_{(r)}((t))$ are generalized Möbius functions given by the following expressions (3.1)-(3.2)

$$\mu_{(-r)}((t)) = \prod_{i=1}^{s} \binom{-r_i}{t_i} (-1)^{t_i}, \quad \mu_{(r)}((t)) = \prod_{i=1}^{s} \binom{r_i}{t_i} (-1)^{t_i}. \tag{4.12}$$

Let $F(n)$ and $G(n)$ be functions of the class $\mathrm{Map}(N, R)$, and $f((x))$ and $g((x))$ be of the class $\mathrm{Map}(N^s, R)$ with s being an unspecified non-negative integer. Evidently, there is a simple one-to-one mapping between the two classes of functions, namely

$$\begin{aligned}
&F(n) \equiv F(p_1^{x_1} \cdots p_s^{x_s}) \longleftrightarrow f((x)) \equiv f(x_1, \ldots, x_s), \quad G(n) \longleftrightarrow g((x)), \\
&f((x)-(t)) \equiv f(x_1 - t_1, \ldots, x_s - t_s) \longleftrightarrow F(n/d), \\
&g((x)-(t)) \equiv g(x_1 - t_1, \ldots, x_s - t_s) \longleftrightarrow G(n/d),
\end{aligned}$$

and $d|n \leftrightarrow \partial_p(d) \le \partial_p(n) \leftrightarrow (t) \le (x)$. Thus it is clear that Corollary 4.4 is precisely equivalent to the following number theoretic proposition.

Corollary 4.5 *For any given function $r(p) \in \mathrm{Map}(S, R)$ with $S \equiv \{p\}$, there always hold the reciprocal relations*

$$F(n) = \sum_{d|n} \mu_{(-r)}(d) G(n/d), \tag{4.13}$$

$$G(n) = \sum_{d|n} \mu_{(r)}(d) F(n/d), \tag{4.14}$$

where $\mu_{(-r)}(d)$ and $\mu_{(r)}(d)$ are given by the following expressions

$$\mu_{(-r)}(d) = \prod_{p|d} \binom{-r(p)}{\partial_p(d)} (-1)^{\partial_p(d)}, \quad \mu_{(r)}(d) = \prod_{p|d} \binom{r(p)}{\partial_p(d)} (-1)^{\partial_p(d)}.$$

Note that the reciprocal pair (4.13)-(4.14) is known as the Fleck-type extension of Möbius inversion formulae [1–3]. The simplest case $r(p) \equiv 1$ gives $\mu_{(-1)}(d) \equiv 1$, and $\mu_{(1)}(d) \equiv \mu(d)$ becomes the ordinary Möbius function, so that (4.13)-(4.14) reduce to the classical inversion formulae. Also, the equivalence between (4.10)⇔(4.11) and (4.13)⇔(4.14) implies that the reciprocal pair (4.13)-(4.14) is a discrete analogue of the pair (4.6)-(4.7). Conversely, (4.6)-(4.7) may be regarded as a continuous analogue of (4.13)-(4.14).

Remark The fact that the two reciprocal pairs (4.6)-(4.7) and (4.13)-(4.14) from the two different mathematical subjects, have the single same source (Theorem 4.2 with (4.4)⇔(4.5)) and are certain analogues of each other, may be recognized as a kind of mathematical phenomenon. Surely, such phenomenon could be hardly found without the aid of NSA, and may help to strengthen the belief of Kurt Gödel that the NSA should be the mathematical analysis in the 21st century.

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