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A Class of Compactly Supported Nonseparable Orthogonal Wavelets of $L^2(\mathbb{R}^n)$

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Abstract In this paper, we present a concrete method for constructing a class of compactly supported nonseparable orthogonal wavelet bases of $L^2(\mathbb{R}^n)$, $n \geq 2$. The orthogonal wavelets are associated with dilation matrix αI_n ($\alpha \geq 2$, $\alpha \in \mathbb{Z}$), where I_n is the identity matrix of order n. In the end, two examples are given to illustrate how to use our method to construct nonseparable orthogonal wavelet bases.

Keywords nonseparable; multivariate; orthogonal; compactly supported; stability.

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1. Introduction

Up to now, the theory and the design of 1-D compactly supported wavelet bases have been well studied. Recently, the wavelet research has primarily focused on multivariate wavelet bases. The most commonly used method is the tensor product of univariate wavelets. However, this construction leads to a separable wavelet which has a disadvantage of giving a particular importance to the horizontal and vertical directions. Separable wavelets are so special that they have very little design freedom, and separability imposes an unnecessary product structure on the plane which is artificial for natural images. Therefore, multivariate nonseparable wavelets have attracted the attention of many mathematicians and some researches on nonseparable wavelets have made great progress [1–14].

Unlike the separable wavelets, nonseparable wavelets are capable to detect structures that are not only horizontal, vertical or diagonal, but arbitrarily oriented. Hence they have a more isotropic treatment of an image [1]. Nonseparable wavelet bases have enough degrees of freedom to construct bases which have several properties simultaneously such as orthogonality, symmetry and compact support which is not possible for tensor-product scalar wavelets except for the Haar tensor. Although the theory and analysis of multivariate wavelet bases have been extensively studied, the design of n-D compactly supported nonseparable orthogonal wavelet basis is still a

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challenging problem. So far, the construction of nonseparable wavelets with dilation matrix $2I_n$ has been well studied and demonstrated to be useful there [2–10]. Particularly, in [6], the author has reviewed several methods for constructing bivariate compactly supported nonseparable orthogonal wavelets and shown some numerical experiments with nonseparable wavelets for image compression. In [7], the author has given a brief description of a fairly general method for constructing compactly supported nonseparable orthogonal wavelet bases of $L^2(\mathbb{R}^n)$. In [9] and [10], the author has studied the construction of nonseparable orthogonal and biorthogonal wavelets of $L^2(\mathbb{R}^n)$, $n \geq 2$, respectively. Currently, it also turns out that many researchers proceed to study the nonseparable wavelets with dilation matrix M especially the matrix M satisfying $M^2 = 2I$ (see [11–14]). Such dilation matrices make the MRA involve a unique wavelet which is easy to construct from the scaling function.

In this paper, by designing a set of special matrices D_i, $i = 1, \dots, n-1$, satisfying some conditions, we give the construction of n-D nonseparable orthogonal wavelets with dilation matrix αI_n, and provide a proof method that the constructed orthogonal wavelets are nonseparable. Finally, we give two examples.

2. Design of n-D nonseparable orthogonal wavelet filters

2.1. Definitions

In this section we first provide the reader with some definitions which will be used frequently in this work. In the following, we denote the point $(\omega_1, \omega_2, \dots, \omega_k) \in \mathbb{R}^k$, $\boldsymbol{\pi}_k = (\pi, \pi, \dots, \pi) \in \mathbb{R}^k$, and D_0 be the identity matrix I_n. Finally, if $D \in \mathbb{Z}^{n\times n}$ is a square matrix, then $D(\boldsymbol{\omega})$ will denote the product $D \cdot \boldsymbol{\omega}^T$.

Definition 1 *A ladder of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ is called a multiresolution analysis (MRA) if the following conditions hold:*

(i) $V_j \subset V_{j+1}$ *for* $j \in \mathbb{Z}$;

(ii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;

(iii) $f(\mathbf{x}) \in V_j \Longleftrightarrow f(\alpha\mathbf{x}) \in V_{j+1}$ $(\alpha \geq 2,\ \alpha \in \mathbb{Z})$;

(iv) There exists a function $\phi(\mathbf{x})$ *in* V_0 *such that the set* $\{\phi(\mathbf{x}\text{-}\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^n}$ *is a Riesz basis for* V_0.

Definition 2 *Define a set* $E_n = \{\frac{2i}{\alpha}\pi, i = 0, \dots, \alpha-1\}^n$ *and let* $\boldsymbol{\eta}_j$ *be an element of* E_n; *if* $\boldsymbol{\eta}_j^i = \frac{2i}{\alpha}\boldsymbol{\pi}_n + \boldsymbol{\eta}_j$, $i = 1, \dots, \alpha-1$, *then* $\boldsymbol{\eta}_j^i$ *are said to be symmetric of* $\boldsymbol{\eta}_j$ *in* E_n. *A subset* A *of* E_n *is said to be symmetric if* $\forall \boldsymbol{\eta}_j \in A$, $\exists \boldsymbol{\eta}_j^i \in A$ *mod* $(2\pi\mathbb{Z}^n)$, *where* $\boldsymbol{\eta}_j^i$, $i = 1, \dots, \alpha-1$, *are symmetric of* $\boldsymbol{\eta}_j$ *in* A.

Definition 3 *An n-D wavelet filter $H_n(\omega_1, \dots, \omega_n)$ is said to be separable if $H_n(\cdot)$ can be written in the following form:*

$$H_n(\omega_1, \dots, \omega_n) = \prod_{i=1}^{k} m_i(a_{i1}\omega_1 + \dots + a_{in}\omega_n),$$

for some integer $1 \le k \le n$. Here $m_i(\cdot)$ are some 1-D wavelet filters and $(a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n$.

2.2. Design of n-D Low-pass orthogonal wavelet filters

Let $\Phi(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$, be a compactly supported function satisfying the dilation equation

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} h(\mathbf{k}) \Phi(\alpha \mathbf{x}\text{-}\mathbf{k}) \ (\alpha \ge 2, \ \alpha \in \mathbb{Z}), \tag{2.1}$$

where $\sum_{\mathbf{k} \in \mathbb{Z}^n} h(\mathbf{k}) = \alpha^n$, and $h(\mathbf{k}) \neq 0$ for only finitely many $\mathbf{k} \in \mathbb{Z}^n$. By taking the Fourier transform on both sides of (2.1), we get

$$\widehat{\Phi}(\boldsymbol{\omega}) = \prod_{j=1}^{\infty} \mathcal{H}_0(\frac{\boldsymbol{\omega}}{\alpha^j}), \tag{2.2}$$

where $\mathcal{H}_0(\boldsymbol{\omega}) = \alpha^{-n} \sum_{\mathbf{k} \in \mathbb{Z}^n} h(\mathbf{k}) e^{-i\mathbf{k}^T \boldsymbol{\omega}}$ is called the mask symbol of the scaling function Φ. To construct compactly supported n-D orthogonal scaling functions, we start with the construction of n-D trigonometric polynomials $H_n(\boldsymbol{\omega})$ satisfying the following condition:

$$|H_n(\boldsymbol{\omega})|^2 + |H_n(\boldsymbol{\omega} + \frac{2}{\alpha}\boldsymbol{\pi}_n)|^2 + \cdots + |H_n(\boldsymbol{\omega} + \frac{2(\alpha-1)}{\alpha}\boldsymbol{\pi}_n)|^2 = 1, \ \forall \boldsymbol{\omega} \in \mathbb{R}^n. \tag{2.3}$$

Lemma 2.1 *Let $H_1(\omega)$ and $G_1^{(1)}(\omega), G_1^{(2)}(\omega), \ldots, G_1^{(\alpha-1)}(\omega)$ be a 1-D low-pass and $\alpha - 1$ high-pass orthogonal filters. Define the n-D filter $H_n(\omega_1, \ldots, \omega_n)$ by the following iterative process: $\forall\ 2 \le k \le n$, choose an integer $1 \le \ell_k \le k-1$,*

$$\begin{aligned}
P_{k-1}^{(i)}(\omega_1, \ldots, \omega_{k-1}) =& H_{k-1}(\alpha\omega_1 + \frac{2i}{\alpha}\pi, \ldots, \alpha\omega_{k-1} + \frac{2i}{\alpha}\pi), \ i = 0, \ldots, \alpha-1, \\
H_k(\omega_1, \ldots, \omega_k) =& P_{k-1}^{(0)}(\omega_1, \ldots, \omega_{k-1}) H_{\ell_k}(\omega_{k-\ell_k+1}, \ldots, \omega_k) + \\
& P_{k-1}^{(1)}(\omega_1, \ldots, \omega_{k-1}) G_{\ell_k}^{(1)}(\omega_{k-\ell_k+1}, \ldots, \omega_k) + \\
& \cdots + P_{k-1}^{(\alpha-1)}(\omega_1, \ldots, \omega_{k-1}) G_{\ell_k}^{(\alpha-1)}(\omega_{k-\ell_k+1}, \ldots, \omega_k),
\end{aligned}$$

where $G_\ell^{(j)}(\omega_1, \ldots, \omega_\ell), \ j = 1, \ldots, \alpha-1$, satisfy the following equation:

$$M_\ell M_\ell^* = I_\alpha, \tag{2.4}$$

$$M_\ell = \begin{pmatrix} H_\ell(\boldsymbol{\omega}) & H_\ell(\boldsymbol{\omega} + \frac{2}{\alpha}\boldsymbol{\pi}_\ell) & \cdots & H_\ell(\boldsymbol{\omega} + \frac{2(\alpha-1)}{\alpha}\boldsymbol{\pi}_\ell) \\ G_\ell^{(1)}(\boldsymbol{\omega}) & G_\ell^{(1)}(\boldsymbol{\omega} + \frac{2}{\alpha}\boldsymbol{\pi}_\ell) & \cdots & G_\ell^{(1)}(\boldsymbol{\omega} + \frac{2(\alpha-1)}{\alpha}\boldsymbol{\pi}_\ell) \\ \vdots & \vdots & \vdots & \vdots \\ G_\ell^{(\alpha-1)}(\boldsymbol{\omega}) & G_\ell^{(\alpha-1)}(\boldsymbol{\omega} + \frac{2}{\alpha}\boldsymbol{\pi}_\ell) & \cdots & G_\ell^{(\alpha-1)}(\boldsymbol{\omega} + \frac{2(\alpha-1)}{\alpha}\boldsymbol{\pi}_\ell) \end{pmatrix}, \quad \forall \boldsymbol{\omega} \in \mathbb{R}^\ell.$$

Then $H_n(0, \ldots, 0) = 1$. Moreover, $H_n(\omega_1, \ldots, \omega_n)$ satisfies the condition (2.3).

Proof The proof is carried out by induction. First, we check that the result of the Lemma holds for $k = 2$. In this case, $\ell_2 = 1$, $P_1^{(i)}(\omega_1) = H_1(\alpha\omega_1 + \frac{2i}{\alpha}\pi)$, $i = 0, \ldots, \alpha-1$. According to $H_1(0) = 1$ and (2.4), we get $G_1^{(j)}(0) = 0$, $j = 1, \ldots, \alpha-1$. Hence $H_2(0,0) = 1$. Since $P_1^{(i)}(\omega_1) = P_1^{(i)}(\omega_1 + \frac{2j}{\alpha}\pi)$, $i = 0, \ldots, \alpha-1$, $j = 1, \ldots, \alpha-1$, we get

$$\sum_{i=0}^{\alpha-1} |H_2(\omega_1 + \frac{2i}{\alpha}\pi, \omega_2 + \frac{2i}{\alpha}\pi)|^2$$

$$
\begin{aligned}
&= |P_1^{(0)}(\omega_1)|^2\Big[\sum_{i=0}^{\alpha-1}|H_1(\omega_2+\frac{2i}{\alpha}\pi)|^2\Big]+|P_1^{(1)}(\omega_1)|^2\Big[\sum_{i=0}^{\alpha-1}|G_1^{(1)}(\omega_2+\frac{2i}{\alpha}\pi)|^2\Big]+\cdots+\\
&\quad |P_1^{(\alpha-1)}(\omega_1)|^2\Big[\sum_{i=0}^{\alpha-1}|G_1^{(\alpha-1)}(\omega_2+\frac{2i}{\alpha}\pi)|^2\Big]+\\
&\quad \sum_{i=0}^{\alpha-1}\sum_{j=1}^{\alpha-1}P_1^{(0)}(\omega_1)\overline{P_1^{(j)}}(\omega_1)H_1(\omega_2+\frac{2i}{\alpha}\pi)\overline{G_1^{(j)}}(\omega_2+\frac{2i}{\alpha}\pi)+\cdots+\\
&\quad P_1^{(\alpha-1)}(\omega_1)\overline{P_1^{(0)}}(\omega_1)G_1^{(\alpha-1)}(\omega_2+\frac{2i}{\alpha}\pi)\overline{H_1}(\omega_2+\frac{2i}{\alpha}\pi)+\\
&\quad \sum_{i=0}^{\alpha-1}\sum_{j=1}^{\alpha-2}P_1^{(\alpha-1)}(\omega_1)\overline{P_1^{(j)}}(\omega_1)G_1^{(\alpha-1)}(\omega_2+\frac{2i}{\alpha}\pi)\overline{G_1^{(j)}}(\omega_2+\frac{2i}{\alpha}\pi)\\
&= \sum_{i=0}^{\alpha-1}|P_1^{(i)}(\omega_1)|^2=\sum_{i=0}^{\alpha-1}|H_1(\alpha\omega_1+\frac{2i}{\alpha}\pi)|^2=1.
\end{aligned}
$$

Next, we assume that the result of the Lemma holds for all $2\le \ell\le k<n$. Then we need to check the result of the Lemma holds for $k+1$. For $2\le\ell\le k$, we have $H_\ell(0,\ldots,0)=1$ and $G_\ell^{(j)}(0,\ldots,0)=0,\ j=1,\ldots,\alpha-1$. Since $\ell_{k+1}\le k$,

$$
H_{k+1}(0,\ldots,0)=P_k^{(0)}(0,\ldots,0)H_{\ell_{k+1}}(0,\ldots,0)+\sum_{j=1}^{\alpha-1}P_k^{(j)}(0,\ldots,0)G_{\ell_{k+1}}^{(j)}(0,\ldots,0)=1.
$$

The induction hypothesis also implies that for $2\le\ell\le k$, $H_\ell,\ G_\ell^{(1)},\ldots,G_\ell^{(\alpha-1)}$ satisfy the equation (2.4). For simplicity, we denote $P_k^{(i)}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_k):=P_k^{(i)}(\omega_1+\frac{2i}{\alpha}\pi,\ldots,\omega_k+\frac{2i}{\alpha}\pi), i=0,\ldots,\alpha-1$. Similarly, H_{k+1} and $G_{\ell_{k+1}}^{(j)},\ j=1,\ldots,\alpha-1$, can be denoted as above. Since $P_k^{(i)}(\cdot)=P_k^{(i)}(\cdot+\frac{2j}{\alpha}\boldsymbol{\pi}_k),\ i=0,\ldots,\alpha-1,\ j=1,\ldots,\alpha-1$, by using the induction hypothesis, we get

$$
\begin{aligned}
&\sum_{i=0}^{\alpha-1}|H_{k+1}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{k+1})|^2\\
&\quad= |P_k^{(0)}(\cdot)|^2\Big[\sum_{i=0}^{\alpha-1}|H_{\ell_{k+1}}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{\ell_{k+1}})|^2\Big]+\sum_{i=0}^{\alpha-1}\sum_{j=1}^{\alpha-1}|P_k^{(j)}(\cdot)|^2|G_{\ell_{k+1}}^{(j)}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{\ell_{k+1}})|^2+\\
&\quad\quad \sum_{i=0}^{\alpha-1}\sum_{j=1}^{\alpha-1}P_k^{(0)}(\cdot)\overline{P_k^{(j)}}(\cdot)H_{\ell_{k+1}}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{\ell_{k+1}})\overline{G_{\ell_{k+1}}^{(j)}}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{\ell_{k+1}})+\cdots+\\
&\quad\quad P_k^{(\alpha-1)}(\cdot)\overline{P_k^{(0)}}(\cdot)\Big[\sum_{i=0}^{\alpha-1}G_{\ell_{k+1}}^{(\alpha-1)}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{\ell_{k+1}})\overline{H_{\ell_{k+1}}}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{\ell_{k+1}})\Big]+\\
&\quad\quad \sum_{i=0}^{\alpha-1}\sum_{j=1}^{\alpha-2}P_k^{(\alpha-1)}(\cdot)\overline{P_k^{(j)}}(\cdot)G_{\ell_{k+1}}^{(\alpha-1)}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{\ell_{k+1}})\overline{G_{\ell_{k+1}}^{(j)}}(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_{\ell_{k+1}})\\
&\quad= \sum_{j=0}^{\alpha-1}|P_k^{(j)}(\cdot)|^2=\sum_{i=0}^{\alpha-1}|H_k(\alpha\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_k)|^2=1.
\end{aligned}
$$

Then the induction hypothesis holds for $k+1$. Hence we get $\sum_{i=0}^{\alpha-1}|H_n(\cdot+\frac{2i}{\alpha}\boldsymbol{\pi}_n)|^2=1$. □

It is well known that to design a compactly supported orthogonal wavelet basis of $L^2(\mathbb{R}^n)$, it is necessary to construct one low-pass filter $\mathcal{H}_0$ and α^n-1 high-pass filters $\mathcal{H}_i,\ i=1,\ldots,\alpha^n-1$. Consequently, a set of special matrices is required in the design of $\mathcal{H}_0$.

For $k=1,\ldots,n-1$, we consider a set of matrices $D_k\in\mathbb{Z}^{n\times n}$ satisfying the following three conditions:

(c$_1$) $\forall\ \boldsymbol{\eta}_j\in E_n, \exists\ \boldsymbol{\eta}_j^i\in E_n$ such that $D_k(\boldsymbol{\eta}_j)=D_k(\boldsymbol{\eta}_j^i)$ mod $(2\pi\mathbb{Z}^n)$, where $\boldsymbol{\eta}_j^i=\frac{2i}{\alpha}\boldsymbol{\pi}_n+\boldsymbol{\eta}_j,\ i=1,\ldots,\alpha-1$;

(c$_2$) If $\boldsymbol{\eta}_{j'}\neq\boldsymbol{\eta}_j,\ \boldsymbol{\eta}_{j'}\neq\boldsymbol{\eta}_j^i$, then $D_k(\boldsymbol{\eta}_j)\neq D_k(\boldsymbol{\eta}_{j'})$ mod $(2\pi\mathbb{Z}^n)$;

(c$_3$) If $F_k=D_kD_{k-1}\cdots D_1(E_n)$ mod $(2\pi\mathbb{Z}^n)$, then F_k is a symmetric subset of E_n, i.e., $\forall\boldsymbol{\eta}\in F_k,\ \boldsymbol{\eta}^i\in F_k,\ i=1,\ldots,\alpha-1$.

By Lemma 2.1, we prove the following theorem that provides us with the n-D low-pass orthogonal wavelet filters.

Theorem 2.2 *Let $H_n(\omega_1,\ldots,\omega_n)$ be the n-D filter of Lemma 2.1 and $D_1,D_2,\ldots,D_{n-1}$ be the matrices that satisfy the above three conditions $(c_1),(c_2)$ and (c_3). Define an n-D filter $\mathcal{H}_0$ by*

$$\mathcal{H}_0(\omega_1,\ldots,\omega_n)=\prod_{k=0}^{n-1}H_n\Big(D_k\cdots D_0(\omega_1,\ldots,\omega_n)\Big), \tag{2.5}$$

then $\mathcal{H}_0(0,\ldots,0)=1$. Moreover, $\mathcal{H}_0$ satisfies the following orthogonality condition

$$\sum_{i=0}^{\alpha^n-1}|\mathcal{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)|^2=1,\quad\forall\boldsymbol{\omega}\in\mathbb{R}^n, \tag{2.6}$$

where $\eta_i,\ i=0,\ldots,\alpha^n-1$, are the different points of the set $E_n=\{0,\frac{2}{\alpha}\pi,\ldots,\frac{2(\alpha-1)}{\alpha}\pi\}^n$.

Proof Since $H_n(0,\ldots,0)=1$, we get $\mathcal{H}_0(0,\ldots,0)=\prod_{k=0}^{n-1}H_n\Big(D_k\cdots D_0(0,\ldots,0)\Big)=1$. We first let $\boldsymbol{\eta}_i^s=\boldsymbol{\eta}_{s\cdot\alpha^{n-1}+i},\ i=0,\ldots,\alpha^{n-1}-1,\ s=1,\ldots,\alpha-1$. Then $D_1(\boldsymbol{\eta}_i)=D_1(\boldsymbol{\eta}_i^s)$ mod $(2\pi\mathbb{Z}^n),\ \forall\boldsymbol{\eta}_i\in E_n$, we deduce that

$$\begin{aligned}
&\sum_{i=0}^{\alpha^n-1}|\mathcal{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)|^2\\
&=\sum_{i=0}^{\alpha^n-1}|H_n(\boldsymbol{\omega}+\boldsymbol{\eta}_i)|^2\prod_{k=1}^{n-1}\Big|H_n\Big(D_k\cdots D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2\\
&=\sum_{i=0}^{\alpha^{n-1}-1}\Big[|H_n(\boldsymbol{\omega}+\boldsymbol{\eta}_i)|^2+\sum_{s=1}^{\alpha-1}|H_n(\boldsymbol{\omega}+\boldsymbol{\eta}_i^s)|^2\Big]\prod_{k=1}^{n-1}\Big|H_n\Big(D_k\cdots D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2\\
&=\sum_{i=0}^{\alpha^{n-1}-1}\prod_{k=1}^{n-1}\Big|H_n\Big(D_k\cdots D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2.
\end{aligned}$$

Again we let $D_1^s(\boldsymbol{\eta}_i)=D_1(\boldsymbol{\eta}_{s\cdot\alpha^{n-2}+i})$, where $D_1^s(\boldsymbol{\eta}_i)$ are symmetric of $D_1(\boldsymbol{\eta}_i)$, $i=0,\ldots,\ \alpha^{n-2}-$

$1,\ s=1,\dots,\alpha-1$. Then $D_2[D_1(\boldsymbol{\eta}_i)]=D_2[D_1^s(\boldsymbol{\eta}_i)] \bmod (2\pi\mathbb{Z}^n),\ \forall \boldsymbol{\eta}_i\in E_n$, we conclude that

$$\begin{aligned}
&\sum_{i=0}^{\alpha^{n-1}-1}\prod_{k=1}^{n-1}\Big|H_n\Big(D_k\cdots D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2\\
&=\sum_{i=0}^{\alpha^{n-1}-1}\Big|H_n\Big(D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2\prod_{k=2}^{n-1}\Big|H_n\Big(D_k\cdots D_2D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2\\
&=\sum_{i=0}^{\alpha^{n-2}-1}\Big[\Big|H_n\Big(D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2+\sum_{s=1}^{\alpha-1}\Big|H_n\Big(D_1^s(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2\Big]\times\prod_{k=2}^{n-1}\Big|H_n\Big(D_k\cdots D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2\\
&=\sum_{i=0}^{\alpha^{n-2}-1}\prod_{k=2}^{n-1}\Big|H_n\Big(D_k\cdots D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\Big)\Big|^2\\
&=\vdots\\
&=\Big|H_n\Big([D_{n-1}\cdots D_1](\boldsymbol{\omega}+\boldsymbol{\eta}_0)\Big)\Big|^2+\sum_{s=1}^{\alpha-1}\Big|H_n\Big([D_{n-1}\cdots D_1]^s(\boldsymbol{\omega}+\boldsymbol{\eta}_0)\Big)\Big|^2=1,
\end{aligned}$$

where $[D_{n-1}\cdots D_1]^s(\boldsymbol{\eta}_i),\ s=1,\dots,\alpha-1$, are symmetric of $[D_{n-1}\cdots D_1](\boldsymbol{\eta}_i)$. Hence (2.6) holds. $\square$

Theorem 2.3 *The wavelet filters $\mathcal{H}_0(\omega_1,\omega_2)$ given by Theorem 2.2 are nonseparable.*

Proof According to Theorem 2.2,

$$\mathcal{H}_0(\omega_1,\omega_2)=H_2(\omega_1,\omega_2)H_2\big(D_1(\omega_1,\omega_2)\big), \tag{2.7}$$

where $H_2(\omega_1,\omega_2)$ is defined as Lemma 2.1. To show that $\mathcal{H}_0(\omega_1,\omega_2)$ is nonseparable, it suffices to check that $H_2(\omega_1,\omega_2)$ is nonseparable. By Lemma 2.1, we get

$$H_2(\omega_1,\omega_2)=H_1(\alpha\omega_1)H_1(\omega_2)+\sum_{i=1}^{\alpha-1}H_1(\alpha\omega_1+\frac{2i}{\alpha}\pi)G_1^{(i)}(\omega_2), \tag{2.8}$$

where $H_1(\omega_1)$ is a 1-D orthogonal filter. We assume that $H_2(\omega_1,\omega_2)$ is separable.

First case: we prove that

$$H_2(\omega_1,\omega_2)=m_1(a_{11}\omega_1+a_{12}\omega_2)m_2(a_{21}\omega_1+a_{22}\omega_2) \tag{2.9}$$

is not possible, where $m_1(\cdot),\ m_2(\cdot)$ are two 1-D orthogonal filters, $(a_{11},a_{12}),(a_{21},a_{22})\in\mathbb{Z}^2$. Now we discuss the following cases:

(i) $a_{11}+a_{12}=0 \bmod (\alpha)$ and $a_{21}+a_{22}=1 \bmod (\alpha)$. For simplicity, we denote $m_1(A):=m_1(a_{11}\omega_1+a_{12}\omega_2)$, $m_2(B):=m_2(a_{21}\omega_1+a_{22}\omega_2)$. Since

$$\sum_{i=0}^{\alpha-1}|H_2(\omega_1+\frac{2i}{\alpha}\pi,\omega_2+\frac{2i}{\alpha}\pi)|^2=1, \tag{2.10}$$

by substituting (2.9) into (2.10), we get

$$\sum_{i=0}^{\alpha-1}\Big|m_1[A+\frac{2i}{\alpha}\pi(a_{11}+a_{12})]\Big|^2\Big|m_2[B+\frac{2i}{\alpha}\pi(a_{21}+a_{22})]\Big|^2$$

$$= |m_1(A)|^2\Big[\sum_{i=0}^{\alpha-1}|m_2(B+\frac{2i}{\alpha}\pi)|^2\Big] = |m_1(A)|^2\cdot 1 = 1,\ \forall(\omega_1,\omega_2)\in\mathbb{R}^2,$$

which is a contradiction. The same result holds in the following cases which are:

(ii) $a_{11}+a_{12}=0 \bmod (\alpha)$ and $a_{21}+a_{22}=j \bmod (\alpha),\ j=2,\ldots,\alpha-1$;

(iii) $a_{11}+a_{12}=i \bmod (\alpha),\ i=0,\ldots,\alpha-1$ and $a_{21}+a_{22}=0 \bmod (\alpha)$.

Next we consider the case:

(iv) $a_{11}+a_{12}=1 \bmod (\alpha)$ and $a_{21}+a_{22}=1 \bmod (\alpha)$. Similarly, by substituting (2.9) into (2.10), we get

$$\sum_{i=0}^{\alpha-1}\Big|m_1[A+\frac{2i}{\alpha}\pi(a_{11}+a_{12})]\Big|^2\Big|m_2[B+\frac{2i}{\alpha}\pi(a_{21}+a_{22})]\Big|^2$$

$$= \Big[\sum_{i=0}^{\alpha-2}|m_1(A+\frac{2i}{\alpha}\pi)|^2|m_2(B+\frac{2i}{\alpha}\pi)|^2\Big] + \Big[1-\sum_{i=0}^{\alpha-2}|m_1(A+\frac{2i}{\alpha}\pi)|^2\Big]\cdot\Big[1-\sum_{i=0}^{\alpha-2}|m_2(B+\frac{2i}{\alpha}\pi)|^2\Big]$$

$$= |m_1(A)|^2\Big[\sum_{i=0}^{\alpha-2}|m_2(B+\frac{2i}{\alpha}\pi)|^2-1\Big] + |m_2(B)|^2\Big[\sum_{i=0}^{\alpha-2}|m_1(A+\frac{2i}{\alpha}\pi)|^2-1\Big]+$$

$$|m_1(A+\frac{2}{\alpha}\pi)|^2\Big[\sum_{i=1}^{\alpha-2}|m_2(B+\frac{2i}{\alpha}\pi)|^2-1\Big] + |m_2(B+\frac{2}{\alpha}\pi)|^2\Big[\sum_{i=1}^{\alpha-2}|m_1(A+\frac{2i}{\alpha}\pi)|^2-1\Big]+$$

$$\cdots + |m_1(A+\frac{2(\alpha-2)}{\alpha}\pi)|^2\Big[|m_2(B+\frac{2(\alpha-2)}{\alpha}\pi)|^2-1\Big]+$$

$$|m_2(B+\frac{2(\alpha-2)}{\alpha}\pi)|^2\Big[|m_1(A+\frac{2(\alpha-2)}{\alpha}\pi)|^2-1\Big]+1=1.$$

Since $\sum_{i=k}^{\alpha-2}|m_1(A+\frac{2i}{\alpha}\pi)|^2$, $\sum_{i=k}^{\alpha-2}|m_1(B+\frac{2i}{\alpha}\pi)|^2<1$, $k=0,\ldots,\alpha-2$, which is a contradiction. The same result holds in the following cases which are:

(v) $a_{11}+a_{12}=i \bmod (\alpha)$ and $a_{21}+a_{22}=j \bmod (\alpha),\ i,j=1,\ldots,\alpha-1,(i,j)\neq(1,1)$. Hence we have proved that (2.9) is not possible.

Second case: we prove that

$$H_2(\omega_1,\omega_2)=m_0(a_1\omega_1+a_2\omega_2) \tag{2.11}$$

is also not possible, where $m_0(\cdot)$ is a 1-D orthogonal wavelet filter. Now we discuss the following three cases:

(i) $a_1+a_2=0 \bmod (\alpha)$, by substituting (2.11) into (2.10), we get $|m_0|^2(a_1\omega_1+a_2\omega_2)=1/\alpha,\forall(\omega_1,\omega_2)\in\mathbb{R}^2$, which is a contradiction with $m_0(0)=1$.

(ii) $a_1+a_2=1 \bmod (\alpha)$. First, we assume that $a_1=1 \bmod (\alpha)$ and $a_2=0 \bmod (\alpha)$. According to (2.11) and (2.8), we get $H_2(\frac{2}{\alpha^2}\pi,0)=m_0(\frac{2}{\alpha^2}\pi a_1)=0$, $H_2(\frac{2}{\alpha^2}\pi,\frac{2i}{\alpha}\pi)=m_0(\frac{2}{\alpha^2}\pi a_1+\frac{2i}{\alpha}\pi a_2)=G_1^{(\alpha-1)}(\frac{2i}{\alpha}\pi),\ i=1,\ldots,\alpha-1$. On the other hand, since $a_2=0 \bmod (\alpha)$, we obtain $m_0(\frac{2}{\alpha^2}\pi a_1)=m_0(\frac{2}{\alpha^2}\pi a_1+\frac{2i}{\alpha}\pi a_2)=0,\ i=1,\ldots,\alpha-1$. Hence we get $G_1^{(\alpha-1)}(\frac{2i}{\alpha}\pi)=0,\ i=1,\ldots,\alpha-1$, which is a contradiction with $G_1^{(\alpha-1)}(0)=0$. Secondly, we assume that $a_1=0 \bmod (\alpha)$ and $a_2=1 \bmod (\alpha)$. According to (2.11) and (2.8), we get $H_2(\frac{2i}{\alpha}\pi,0)=m_0(\frac{2i}{\alpha}\pi a_1)=1,\ i=1,\ldots,\alpha-1$, since $a_1=0 \bmod (\alpha)$, we get $m_0(\frac{2i}{\alpha}\pi)=1,\ i=1,\ldots,\alpha-1$, which is a contradiction with $m_0(0)=1$.

(iii) $a_1 + a_2 = j \bmod (\alpha)$, $j = 1, \dots, \alpha - 1$. The proof is similar to (ii).

Collecting everything together, it follows that $\mathcal{H}_0(\omega_1, \omega_2)$ is nonseparable. $\square$

Similarly, the previous proof can be easily extended to the n-D case, then we get the following Corollary.

Corollary 2.4 *The wavelet filters $\mathcal{H}_0(\omega_1, \dots, \omega_n)$ given by Theorem 2.2 are nonseparable.*

2.3. Design of n-D High-pass orthogonal wavelet filters

As we have previously mentioned, the construction of the $\alpha^n - 1$ mother wavelets Ψ^i, $i = 1, \dots, \alpha^n - 1$, requires the construction of n-D high-pass filters $\mathcal{H}_i$, $i = 1, \dots, \alpha^n - 1$. These high-pass filters together with the previously defined filters $\mathcal{H}_0$ have to satisfy the equations

$$\sum_{j=0}^{\alpha^n-1} \mathcal{H}_i(\boldsymbol{\omega} + \boldsymbol{\eta}_j)\overline{\mathcal{H}_{i'}}(\boldsymbol{\omega} + \boldsymbol{\eta}_j) = \delta_{ii'}, \ \forall\ 0 \le i, i' \le \alpha^n - 1, \ \boldsymbol{\omega} \in \mathbb{R}^n, \tag{2.12}$$

where $\boldsymbol{\eta}_j$, $j = 0, \dots, \alpha^n - 1$, are the different points of the set $E_n = \{0, \frac{2}{\alpha}\pi, \dots, \frac{2(\alpha-1)}{\alpha}\pi\}^n$.

Theorem 2.5 *Let $H_0(\boldsymbol{\omega}) = H_n(\omega_1, \dots, \omega_n)$, $\boldsymbol{\omega} \in \mathbb{R}^n$, $H_n(\cdot)$ is the wavelet filter of Lemma 2.1. Let $D_1, D_2, \dots, D_{n-1}$ be the matrices that satisfy the three conditions $(c_1), (c_2)$ and (c_3). If $\mathcal{H}_i$ is the filter defined by:*

$$\mathcal{H}_i(\boldsymbol{\omega}) = \prod_{k=0}^{n-1} \Big[\varepsilon_k^i H_0(D_k \cdots D_0 \boldsymbol{\omega}) + \sum_{j=1}^{\alpha-1} \frac{1 - \varepsilon_k^i}{\sqrt{\alpha - 1}} G_0^{(j)}(D_k \cdots D_0 \boldsymbol{\omega})\Big], \tag{2.13}$$

where $G_0^{(j)}$, $j = 1, \dots, \alpha - 1$, together with H_0 satisfy the following equation

$$M_0 M_0^* = I_\alpha,$$

where $M_0 = \begin{pmatrix} H_0(\boldsymbol{\omega}) & H_0(\boldsymbol{\omega} + \frac{2}{\alpha}\boldsymbol{\pi}_n) & \cdots & H_0(\boldsymbol{\omega} + \frac{2(\alpha-1)}{\alpha}\boldsymbol{\pi}_n) \\ G_0^{(1)}(\boldsymbol{\omega}) & G_0^{(1)}(\boldsymbol{\omega} + \frac{2}{\alpha}\boldsymbol{\pi}_n) & \cdots & G_0^{(1)}(\boldsymbol{\omega} + \frac{2(\alpha-1)}{\alpha}\boldsymbol{\pi}_n) \\ \vdots & \vdots & \vdots & \vdots \\ G_0^{(\alpha-1)}(\boldsymbol{\omega}) & G_0^{(\alpha-1)}(\boldsymbol{\omega} + \frac{2}{\alpha}\boldsymbol{\pi}_n) & \cdots & G_0^{(\alpha-1)}(\boldsymbol{\omega} + \frac{2(\alpha-1)}{\alpha}\boldsymbol{\pi}_n) \end{pmatrix}$,
$(\varepsilon_0^i, \varepsilon_1^i, \dots, \varepsilon_n^i)_{i=1,\dots,\alpha^n-1}$ *are the different points of $\{0,1\}^n \backslash (0, \dots, 0)$. Then $\mathcal{H}_i$, $i = 1, \dots, \alpha^n - 1$, is a solution of (2.12).*

Proof First, we consider two integers $i, i' \in \{1, \dots, \alpha^n - 1\}$ such that $i \ne i'$ and we prove that

$$\sum_{j=0}^{\alpha^n-1} \mathcal{H}_i(\boldsymbol{\omega} + \boldsymbol{\eta}_j)\overline{\mathcal{H}_{i'}}(\boldsymbol{\omega} + \boldsymbol{\eta}_j) = 0.$$

Since $i \ne i'$, there exists $0 \le \ell \le n - 1$ such that $\varepsilon_k^i = \varepsilon_k^{i'}$, $\forall\ 0 \le k \le \ell - 1$ and $\varepsilon_\ell^i \ne \varepsilon_\ell^{i'}$. We assume that $\varepsilon_\ell^i = 1$ and $\varepsilon_\ell^{i'} = 0$.

First case: $0 \le \ell < n - 1$. By using the factorization technique of the proof of Theorem 2.2, we get

$$\begin{aligned}
&\sum_{j=0}^{\alpha^n-1} \mathcal{H}_i(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\overline{\mathcal{H}_{i'}}(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\\
&= \sum_{j=0}^{\alpha^n-1} \Big[\varepsilon_0^i H_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_0^i}{\sqrt{\alpha-1}} G_0^{(j)}(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big]\times\\
&\quad \Big[\varepsilon_0^{i'}\overline{H_0}(\boldsymbol{\omega}+\boldsymbol{\eta}_j) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_0^{i'}}{\sqrt{\alpha-1}} \overline{G_0^{(j)}}(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big]\times\\
&\quad \prod_{k=1}^{n-1} \Big[\varepsilon_k^i H_0\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_k^i}{\sqrt{\alpha-1}} G_0^{(j)}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\times\\
&\quad \Big[\varepsilon_k^{i'}\overline{H_0}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_k^{i'}}{\sqrt{\alpha-1}} \overline{G_0^{(j)}}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\\
&= \sum_{j=0}^{\alpha^{n-1}-1} \Big\{\Big[\varepsilon_0^i H_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_0^i}{\sqrt{\alpha-1}} G_0^{(j)}(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big]\times\\
&\quad \Big[\varepsilon_0^{i'}\overline{H_0}(\boldsymbol{\omega}+\boldsymbol{\eta}_j) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_0^{i'}}{\sqrt{\alpha-1}} \overline{G_0^{(j)}}(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big]+\\
&\quad \sum_{s=1}^{\alpha-1} \Big[\varepsilon_0^i H_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j^s) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_0^i}{\sqrt{\alpha-1}} G_0^{(j)}(\boldsymbol{\omega}+\boldsymbol{\eta}_j^s)\Big]\times\\
&\quad \Big[\varepsilon_0^{i'}\overline{H_0}(\boldsymbol{\omega}+\boldsymbol{\eta}_j^s) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_0^{i'}}{\sqrt{\alpha-1}} \overline{G_0^{(j)}}(\boldsymbol{\omega}+\boldsymbol{\eta}_j^s)\Big]\Big\}\times\\
&\quad \prod_{k=1}^{n-1} \Big[\varepsilon_k^i H_0\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_k^i}{\sqrt{\alpha-1}} G_0^{(j)}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\times\\
&\quad \Big[\varepsilon_k^{i'}\overline{H_0}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_k^{i'}}{\sqrt{\alpha-1}} \overline{G_0^{(j)}}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\\
&= \sum_{j=0}^{\alpha^{n-1}-1} \prod_{k=1}^{n-1} \Big[\varepsilon_k^i H_0\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_k^i}{\sqrt{\alpha-1}} G_0^{(j)}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\times\\
&\quad \Big[\varepsilon_k^{i'}\overline{H_0}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_k^{i'}}{\sqrt{\alpha-1}} \overline{G_0^{(j)}}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\\
&= \vdots\\
&= \sum_{j=0}^{\alpha^{n-\ell}-1} \prod_{k=\ell}^{n-1} \Big[\varepsilon_k^i H_0\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_k^i}{\sqrt{\alpha-1}} G_0^{(j)}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\times\\
&\quad \Big[\varepsilon_k^{i'}\overline{H_0}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big) + \sum_{j=1}^{\alpha-1} \frac{1-\varepsilon_k^{i'}}{\sqrt{\alpha-1}} \overline{G_0^{(j)}}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\alpha^{n-\ell-1}-1} \Big\{ H_0\Big([D_\ell\cdots D_0](\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big[\sum_{j=1}^{\alpha-1}\frac{1}{\sqrt{\alpha-1}}\overline{G_0^{(j)}}\Big([D_\ell\cdots D_0](\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]+\\
&\quad \sum_{s=1}^{\alpha-1} H_0\Big([D_\ell\cdots D_0]^s(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\times\Big[\sum_{j=1}^{\alpha-1}\frac{1}{\sqrt{\alpha-1}}\overline{G_0^{(j)}}\Big([D_\ell\cdots D_0]^s(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\Big\}\times\\
&\quad \prod_{k=\ell+1}^{n-1}\Big[\varepsilon_k^i H_0\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)+\sum_{j=1}^{\alpha-1}\frac{1-\varepsilon_k^i}{\sqrt{\alpha-1}}G_0^{(j)}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\times\\
&\quad \Big[\varepsilon_k^{i'}\overline{H_0}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)+\sum_{j=1}^{\alpha-1}\frac{1-\varepsilon_k^{i'}}{\sqrt{\alpha-1}}\overline{G_0^{(j)}}\Big(D_k\cdots D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\Big)\Big]\\
&= 0,
\end{aligned}
$$

where $[D_\ell\cdots D_0]^s(\boldsymbol{\eta}_j)$ are symmetric of $[D_\ell\cdots D_0](\boldsymbol{\eta}_j),\ s=1,\ldots,\alpha-1,\ \ell=0,\ldots,n-2.$

Second case: $\ell=n-1$. It can be easily proved that

$$
\begin{aligned}
&\sum_{j=0}^{\alpha^n-1}\mathcal{H}_i(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\overline{\mathcal{H}_{i'}}(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\\
&= H_0\Big([D_{n-1}\cdots D_0](\boldsymbol{\omega}+\boldsymbol{\eta}_0)\Big)\Big[\sum_{j=1}^{\alpha-1}\frac{1}{\sqrt{\alpha-1}}\overline{G_0^{(j)}}\Big([D_{n-1}\cdots D_0](\boldsymbol{\omega}+\boldsymbol{\eta}_0)\Big)\Big]+\\
&\quad \sum_{s=1}^{\alpha-1}H_0\Big([D_{n-1}\cdots D_0]^s(\boldsymbol{\omega}+\boldsymbol{\eta}_0)\Big)\times\Big[\sum_{j=1}^{\alpha-1}\frac{1}{\sqrt{\alpha-1}}\overline{G_0^{(j)}}\Big([D_{n-1}\cdots D_0]^s(\boldsymbol{\omega}+\boldsymbol{\eta}_0)\Big)\Big]\\
&= 0.
\end{aligned}
$$

Finally for the case $i=i'$, it is similar to the proof of Theorem 2.2 and we conclude that $\sum_{j=0}^{\alpha^n-1}\mathcal{H}_i(\boldsymbol{\omega}+\boldsymbol{\eta}_j)\overline{\mathcal{H}_i}(\boldsymbol{\omega}+\boldsymbol{\eta}_j)=1$. $\Box$

Remark It is well known that condition (2.12) does not ensure that $\mathcal{H}_i, i=1,\ldots,\alpha^n-1$, generate an orthogonal wavelet basis of $L^2(\mathbb{R}^n)$. In fact, we need to study the stability of the wavelet functions $\Psi_{j,k}^i$ generated by $\mathcal{H}_i$. The stability of $\Psi_{j,k}^i$ can be similarly done as [9].

3. Example

Example 3.1 For the case $\alpha=2$. Let $D_1=\begin{pmatrix} a & x\\ b & y\end{pmatrix}$, a,b,x,y be odd integers. Then it is easy to check that D_1 satisfies the three conditions $(c_1),(c_2)$ and (c_3). We choose $D_1=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. Let $H_1(\omega)=\frac{1}{2}(1+z)$ and $G_1^{(1)}(\omega)=\frac{1}{2}(z-1),\ z=e^{-i\omega},\ \omega\in\mathbb{R}$, be a 1-D low-pass and a high-pass orthogonal filters (see [15]). By Lemma 2.1, we get the 2-D filter $H_2(\omega_1,\omega_2)=\frac{1}{2}(z_1^2+z_2)$, $z_1=e^{-i\omega_1},\ z_2=e^{-i\omega_2}$, satisfying the condition (2.3). Applying Theorem 2.2, we conclude that the 2-D nonseparable wavelet filter $\mathcal{H}_0$ satisfying the orthogonality condition (2.6) by

$$\mathcal{H}_0(\omega_1,\omega_2)=H_2(\omega_1,\omega_2)H_2\Big(D_1(\omega_1,\omega_2)\Big)=\frac{1}{4}(z_1^2+z_2)(z_1^2z_2^2+z_1z_2^{-1}).$$

Furthermore, let $G_2^{(1)}(\omega_1,\omega_2)=e^{-i\omega_1}\overline{H_2}(\omega_1+\pi,\omega_2+\pi)$ be the corresponding high-pass filters of $H_2(\omega_1,\omega_2)$. Then it is easy to see that $H_2(\omega_1,\omega_2), G_2^{(1)}(\omega_1,\omega_2)$ satisfy the Eq. (2.4). According to Theorem 2.5, the corresponding high-pass filter $\mathcal{H}_i$, $i=1,2,3$, can be obtained via (2.13).

Since the scaling function ϕ generated by $H_1(\omega)$ satisfies $\sum_{k\in\mathbb{Z}}|\hat{\phi}(\omega+2\pi k)|^2=1$, it can be seen that the translates of ϕ are stable. It follows from [9] that the translates of Φ generated by $\mathcal{H}_0$ are stable and $\mathcal{H}_0$ generates a stable orthogonal wavelet basis of $L^2(\mathbb{R}^2)$.

Example 3.2 For the case $\alpha=3$. Let $D_1=\begin{pmatrix} a & x \\ b & y\end{pmatrix}$, $a+x=0 \bmod (3)$, and $b+y=0 \bmod (3)$. Then D_1 satisfies the condition (c_1). Furthermore, let $a=b\neq 0 \bmod (3)$ or $x=y\neq 0 \bmod (3)$, it is easy to check that D_1 satisfies (c_2) and (c_3). Choose $D_1=\begin{pmatrix} 2 & 1 \\ 2 & 4\end{pmatrix}$, then D_1 satisfies the conditions (c_1), (c_2) and (c_3). Let $H_1(\omega)=\frac{1}{3}(1+z+z^2)$, $G_1^{(1)}(\omega)=-\frac{\sqrt{2}}{6}+\frac{\sqrt{2}}{3}z-\frac{\sqrt{2}}{6}z^2$, $G_1^{(2)}(\omega)=-\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}z^2$, $z=e^{-i\omega}$, $\omega\in\mathbb{R}$ be a 1-D low-pass filter and two high-pass orthogonal filters, respectively [16]. By Lemma 2.1, we get the 2-D filter

$$\begin{aligned}H_2(\omega_1,\omega_2)=&\frac{1}{9}\Big[1-\frac{\sqrt{2}+\sqrt{6}}{2}+(1+\sqrt{2})z_2+(1+\frac{\sqrt{6}-\sqrt{2}}{2})z_2^2+\\
&(1+\frac{\sqrt{2}+\sqrt{6}}{4}+\frac{\sqrt{6}-3\sqrt{2}}{4}i)z_1^3+(1-\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i)z_1^3z_2+\\
&(1+\frac{\sqrt{2}-\sqrt{6}}{4}+\frac{\sqrt{6}+3\sqrt{2}}{4}i)z_1^3z_2^2+(1+\frac{\sqrt{2}-\sqrt{6}}{4}-\frac{\sqrt{6}+3\sqrt{2}}{4}i)z_1^6+\\
&(1-\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i)z_1^6z_2+(1+\frac{\sqrt{2}+\sqrt{6}}{4}+\frac{3\sqrt{2}-\sqrt{6}}{4}i)z_1^6z_2^2\Big]\end{aligned}$$

satisfying the condition (2.3), where $z_1=e^{-i\omega_1}$, $z_2=e^{-i\omega_2}$. Applying Theorem 2.2, we conclude that the 2-D nonseparable wavelet filter $\mathcal{H}_0$ given by

$$\mathcal{H}_0(\omega_1,\omega_2)=H_2(\omega_1,\omega_2)H_2(2\omega_1+\omega_2,2\omega_1+4\omega_2)$$

satisfies the orthogonality condition (2.6). According to [9], $\mathcal{H}_0$ generates a stable orthogonal wavelet basis of $L^2(\mathbb{R}^2)$.

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