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# Additive Biderivations and Centralizing Maps on Nest Algebras

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**Abstract** Let $\mathcal{N}$ be a nest on a Banach space $X$, and Alg$\mathcal{N}$ be the associated nest algebra. It is shown that, if there exists a non-trivial element $N$ in $\mathcal{N}$ which is complemented in $X$ and $\dim N \neq 1$, then every additive biderivation from Alg$\mathcal{N}$ into itself is an inner biderivation. Based on this result, we give characterizations of centralizing (commuting) maps, cocentralizing derivations, and cocommuting generalized derivations on nest algebras.

**Keywords** biderivations; commuting maps; centralizing maps; nest algebra.

**MR(2010) Subject Classification** 47B47; 47L35

## 1. Introduction

Let $\mathcal{A}$ be an algebra (or a ring) with center $\mathcal{Z}$. Then $\mathcal{A}$ is a Lie algebra (Lie ring) under the Lie product $[A,B]=AB-BA$. Recall that a map $\Phi$ from $\mathcal{A}$ into itself is centralizing if $[\Phi(A),A]\in\mathcal{Z}$ for all $A\in\mathcal{A}$; is commuting if $[\Phi(A),A]=0$ for all $A\in\mathcal{A}$. The study of centralizing maps was initiated by a well-known theorem of Posner [1] which states that the existence of a nonzero centralizing derivation on a prime ring $\mathcal{R}$ implies that $\mathcal{R}$ is commutative. Brešar in [2] gave the structure of arbitrary centralizing additive maps on prime rings. For other results about centralizing maps, see [3–5] and the references therein.

The notion of additive commuting maps is closely connected with the notion of biderivations. Recall that a biadditive map $\delta:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is called a biderivation if it is a derivation in each argument; is called an inner biderivation if there exists $\lambda\in\mathcal{Z}$ such that $\delta(A,B)=\lambda[A,B]$ for all $A,B\in\mathcal{A}$. Zhang in [6] proved that every linear biderivation of nest algebras on a complex separable Hilbert space $H$ is an inner biderivation if and only if $\dim 0_+\neq 1$ or $\dim H_-^\perp\neq 1$. Every commuting additive map $\Phi:\mathcal{A}\to\mathcal{A}$ gives rise to a biderivation of $\mathcal{A}$. Namely, linearizing $[\Phi(A),A]=0, A\in\mathcal{A}$, we get $[\Phi(A),B]=[A,\Phi(B)]$ for all $A,B\in\mathcal{A}$, and hence the map $(A,B)\mapsto[\Phi(A),B]$ is a biderivation.

In [7], the author gave the concepts of skew-centralizing maps and skew-commuting maps. Recall that an additive map $\Phi:\mathcal{A}\to\mathcal{A}$ is skew-centralizing if $\Phi(A)A+A\Phi(A)\in\mathcal{Z}$ for all $A\in\mathcal{A}$; is skew-commuting if $\Phi(A)A+A\Phi(A)=0$ for every $A\in\mathcal{A}$. Moreover, the author [7]

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proved that there is no nonzero additive maps that are skew-commuting on ideals of prime rings with characteristic not 2. Later, Brešar in [8] gave the definitions of cocentralizing maps and cocommuting maps. Two additive maps $\Phi : \mathcal{A} \to \mathcal{A}$ and $\Psi : \mathcal{A} \to \mathcal{A}$ are said to be cocentralizing if $\Phi(A)A - A\Psi(A) \in \mathcal{Z}$ for all $A \in \mathcal{A}$; cocommuting if $\Phi(A)A = A\Psi(A)$ for all $A \in \mathcal{A}$.

The purpose of this paper is to consider additive biderivations, centralizing (skew-centralizing) maps, cocentralizing derivations and cocommuting generalized derivations on nest algebras.

Let $X$ be a Banach space over the real or complex number field $\mathbb{F}$. As usual, $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on $X$. A nest $\mathcal{N}$ on $X$ is a chain of closed (under norm topology) subspaces of $X$ which is closed under the formation of arbitrary closed linear span (denoted by $\bigvee$) and intersection (denoted by $\bigwedge$), and which includes $\{0\}$ and $X$. The nest algebra associated to the nest $\mathcal{N}$, denoted by Alg$\mathcal{N}$, is a set

$$\mathrm{Alg}\mathcal{N} = \{T \in \mathcal{B}(X) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

It is clear that if $\mathcal{N}$ is trivial, then Alg$\mathcal{N} = \mathcal{B}(X)$. When $\mathcal{N} \neq \{0, X\}$, we say that $\mathcal{N}$ is non-trivial. Since $\mathcal{B}(X)$ is prime, we only consider the case $\mathcal{N} \neq \{0, X\}$ in this paper. Note that Alg$\mathcal{N}$ is not prime. It is easily proved that the commutant of Alg$\mathcal{N}$ coincides with $\mathbb{F}I$.

## 2. Additive biderivations and centralizing maps on nest algebras

In this section, we first discuss the additive biderivations on nest algebras. The following is our first main result.

**Theorem 2.1** *Let $\mathcal{N}$ be a nest on a Banach space $X$. If there exists a non-trivial element $N$ in $\mathcal{N}$ which is complemented in $X$ and* $\dim N \neq 1$, *then every additive biderivation from* Alg$\mathcal{N}$ *into itself is an inner biderivation.*

To prove Theorem 2.1, we need the following two lemmas.

**Lemma 2.2** ([9, Lemma 2.3]) *Let $\mathcal{R}$ be a ring and $\Phi : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be a biderivation. Then $\Phi(U, V)A[X, Y] = [U, V]A\Phi(X, Y)$ for all $A, U, V, X, Y \in \mathcal{R}$.*

**Lemma 2.3** ([9, Lemma 2.2]) *Let $\Omega$ be any set, $\mathcal{R}$ be a prime ring and $\mathcal{C}$ be the extended centroid of $\mathcal{R}$. If functions $f, h : \Omega \to \mathcal{R}$ satisfy $f(s)Xh(t) = h(s)Xf(t)$ for all $s, t \in \Omega, X \in \mathcal{R}$ and $f \neq 0$, then there exists an element $\lambda \in \mathcal{C}$ such that $h(s) = \lambda f(s)$ for all $s \in \Omega$.*

**Proof of Theorem 2.1** Assume that $\Phi$ is an additive biderivation of Alg$\mathcal{N}$. It follows from Lemma 2.2 that

$$\Phi(U, V)Z[A, B] = [U, V]Z\Phi(A, B) \tag{2.1}$$

for all $A, B, U, V, Z \in$ Alg$\mathcal{N}$. Next we will prove that $\Phi$ is an inner biderivation.

Since $N$ is complemented in $X$, there exists a non-trivial idempotent $E \in$ Alg$\mathcal{N}$ such that $EX = N$. It is clear that $E\mathcal{B}(X) \subseteq$ Alg$\mathcal{N}$. So by Eq. (2.1), we have

$$\Phi(U, V)EY[A, B] = [U, V]EY\Phi(A, B) \tag{2.2}$$

for all $A, B, U, V \in \mathrm{Alg}\,\mathcal{N}$ and $Y \in \mathcal{B}(X)$. Multiplying $E$ from the right side in Eq. (2.2), we get

$$\Phi(U,V)EY[A,B]E = [U,V]EY\Phi(A,B)E. \tag{2.3}$$

Now define two maps $f, h : \mathrm{Alg}\,\mathcal{N} \times \mathrm{Alg}\,\mathcal{N} \to \mathcal{B}(X)$ by

$$h(X,Y) = \Phi(X,Y)E \text{ and } f(X,Y) = [X,Y]E$$

for all $X, Y \in \mathrm{Alg}\,\mathcal{N}$. It follows from Eq. (2.3) that

$$h(U,V)Yf(A,B) = f(U,V)Yh(A,B) \tag{2.4}$$

for all $A, B, U, V \in \mathrm{Alg}\,\mathcal{N}$ and $Y \in \mathcal{B}(X)$. Note that $\dim N > 1$. There exist $A_0, B_0 \in \mathrm{Alg}\,\mathcal{N}$ such that $[A_0, B_0]E \neq 0$, and so $f \neq 0$. Hence by Eq. (2.4) and Lemma 2.3, there exists $\lambda \in \mathbb{F}$ such that $h(U,V) = \lambda f(U,V)$ for all $U, V \in \mathrm{Alg}\,\mathcal{N}$, that is,

$$\Phi(U,V)E = \lambda[U,V]E \text{ for all } U, V \in \mathrm{Alg}\,\mathcal{N}. \tag{2.5}$$

Combining Eq. (2.5) with Eq. (2.2), one obtains

$$[U,V]E\mathcal{B}(X)(\lambda[A,B] - \Phi(A,B)) = \{0\} \text{ for all } A, B, U, V \in \mathrm{Alg}\,\mathcal{N}.$$

Take $U = A_0$ and $V = B_0$ in the above equation. Since $[A_0, B_0]E \neq 0$ and $\mathcal{B}(X)$ is prime, it follows that $\Phi(A,B) = \lambda[A,B]$ for all $A, B \in \mathrm{Alg}\,\mathcal{N}$, that is, $\Phi$ is an inner biderivation. The proof of the theorem is completed. □

By Theorem 2.1, we can give the second main result in this section, which discusses centralizing additive maps on nest algebras.

**Theorem 2.4** *Let $\mathcal{N}$ be a nest on a Banach space $X$ over the real or complex number field $\mathbb{F}$. Suppose that $\Phi : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is a centralizing additive map (i.e., $[\Phi(A), A] \in \mathbb{F}I$, $\forall A$). If there exists a non-trivial element $N$ in $\mathcal{N}$ which is complemented in $X$ and $\dim N \neq 1$, then $\Phi(A) = \lambda A + \phi(A)I$ for all $A \in \mathrm{Alg}\,\mathcal{N}$, where $\lambda \in \mathbb{F}$ and $\phi : \mathrm{Alg}\,\mathcal{N} \to \mathbb{F}$ is an additive map.*

**Proof** Note that the unit $I$ cannot be a commutator $[A, B]$. Hence $\Phi$ is commuting, that is, $[\Phi(A), A] = 0$ for all $A \in \mathrm{Alg}\,\mathcal{N}$. Replacing $A$ by $A + B$ in $[\Phi(A), A] = 0$, we get $[\Phi(A), B] + [\Phi(B), A] = 0$, that is, $[\Phi(A), B] = [A, \Phi(B)]$ for all $A, B \in \mathrm{Alg}\,\mathcal{N}$. Let $\delta(A,B) = [\Phi(A), B]$ for each $A, B \in \mathrm{Alg}\,\mathcal{N}$. It is easy to prove that $\delta$ is an additive biderivation of $\mathrm{Alg}\,\mathcal{N}$. So by Theorem 2.1, there exists $\lambda \in \mathbb{F}$ such that $\delta(A,B) = \lambda[A,B]$ for all $A, B \in \mathrm{Alg}\,\mathcal{N}$. Hence $[\Phi(A), B] = \lambda[A,B]$, that is, $(\Phi(A) - \lambda A)B = B(\Phi(A) - \lambda A)$ for all $A, B \in \mathrm{Alg}\,\mathcal{N}$. Since $B$ is arbitrary, we have $\Phi(A) - \lambda A \in \mathbb{F}I$. Thus we can suppose that $\Phi(A) - \lambda A = \phi(A)I$, where $\phi : \mathrm{Alg}\,\mathcal{N} \to \mathbb{F}$ is a map. It is clear that $\phi$ is additive and $\Phi(A) = \lambda A + \phi(A)I$. The proof is completed. □

In [9], the author gave the following concept: a map $\Phi : \mathcal{A} \to \mathcal{A}$ is additive modulo $\mathcal{Z}$ if $\Phi(A + B) - \Phi(A) - \Phi(B) \in \mathcal{Z}$ for all $A, B \in \mathcal{A}$.

**Corollary 2.5** *Let $\mathcal{N}$ be a nest on a Banach space $X$ over the real or complex number field $\mathbb{F}$. Assume that $\Phi$ is additive modulo $\mathbb{F}I$ and is commuting from $\mathrm{Alg}\,\mathcal{N}$ into itself. If there exists a non-trivial element $N$ in $\mathcal{N}$ which is complemented in $X$ and $\dim N \neq 1$, then there exist $\lambda \in \mathbb{F}$*

*and a map* $\phi : \mathrm{Alg}\,\mathcal{N} \to \mathbb{F}$ *such that* $\Phi(A) = \lambda A + \phi(A)I$ *for all* $A \in \mathrm{Alg}\,\mathcal{N}$.

**Proof** The proof is similar to that of Theorem 2.4. We omit it here. □

In addition, if $\Phi$ in Theorem 2.4 is a derivation (generalized derivation), we have the following theorems.

**Theorem 2.6** *Let* $\mathcal{N}$ *be a nest on a Banach space* $X$ *over the real or complex number field* $\mathbb{F}$. *Suppose that* $\Phi$ *is an additive centralizing derivation on* $\mathrm{Alg}\,\mathcal{N}$. *If there exists a non-trivial element* $N$ *in* $\mathcal{N}$ *which is complemented in* $X$ *and* $\dim N \neq 1$, *then* $\Phi \equiv 0$.

**Proof** By Theorem 2.4, we have $[\Phi(A), A] = 0$ for all $A \in \mathrm{Alg}\mathcal{N}$. Replacing $A$ by $A + B$ in $[\Phi(A), A] = 0$, we get

$$[\Phi(A), B] = [A, \Phi(B)], \quad \forall A, B \in \mathrm{Alg}\,\mathcal{N}. \tag{2.6}$$

In particular,

$$\begin{aligned}[\Phi(A), BA] =&[A, \Phi(BA)] = [A, \Phi(B)A + B\Phi(A)]\\ =&[A, \Phi(B)A] + [A, B\Phi(A)]]\\ =&[A, \Phi(B)]A + \Phi(B)[A, A] + B[\Phi(A), A] + [A, B]\Phi(A)\\ =&[A, \Phi(B)]A + [A, B]\Phi(A).\end{aligned} \tag{2.7}$$

On the other hand, using Eq. (2.6), we have

$$[\Phi(A), BA] = [\Phi(A), B]A + B[\Phi(A), A] = [\Phi(A), B]A = [A, \Phi(B)]A. \tag{2.8}$$

Combining Eq. (2.8) with Eq. (2.7), one gets $[A, B]\Phi(A) = 0$ for all $A, B \in \mathrm{Alg}\,\mathcal{N}$. Substituting $CB$ for $B$ in this equation, we have $0 = [A, CB]\Phi(A) = C[A, B]\Phi(A) + [A, C]B\Phi(A) = [A, C]B\Phi(A)$. That is,

$$[A, C]B\Phi(A) = 0, \quad \forall A, B, C \in \mathrm{Alg}\,\mathcal{N}. \tag{2.9}$$

Since $N$ is complemented in $X$, there exists a non-trivial idempotent $E \in \mathcal{B}(X)$ such that $EX = N$. So $E\mathcal{B}(X) \subseteq \mathrm{Alg}\,\mathcal{N}$. Thus, by Eq. (2.9), we get $[A, C]ET\Phi(A) = 0$ for all $A, C \in \mathrm{Alg}\,\mathcal{N}$ and all $T \in \mathcal{B}(X)$. Since $\mathcal{B}(X)$ is prime, for every $A \in \mathrm{Alg}\,\mathcal{N}$, we have $[A, C]E = 0$ or $\Phi(A) = 0$. It is well known that a group cannot be the union of its two proper subgroups. Note that $\dim N > 1$. There exist $A_0, C_0 \in \mathrm{Alg}\,\mathcal{N}$ such that $[A_0, C_0]E \neq 0$. Hence $\mathrm{Alg}\,\mathcal{N} \neq \{A \in \mathrm{Alg}\,\mathcal{N} : [A, C]E = 0\}$, which implies that $\Phi(A) = 0$ for all $A \in \mathrm{Alg}\,\mathcal{N}$. The proof is completed. □

**Theorem 2.7** *Let* $\mathcal{N}$ *be a nest on a Banach space* $X$ *over the real or complex number field* $\mathbb{F}$. *Suppose that* $\Phi$ *is an additive centralizing generalized derivation on* $\mathrm{Alg}\,\mathcal{N}$. *If there exists a non-trivial element* $N$ *in* $\mathcal{N}$ *which is complemented in* $X$ *and* $\dim N \neq 1$, *then there exists* $\lambda \in \mathbb{F}$ *such that* $\Phi(A) = \lambda A$ *for all* $A \in \mathrm{Alg}\,\mathcal{N}$.

**Proof** Since $\Phi$ is a centralizing map on $\mathrm{Alg}\,\mathcal{N}$, by Theorem 2.4, $\Phi$ has the form $\Phi(A) = \lambda A + \phi(A)I$ for all $A \in \mathrm{Alg}\,\mathcal{N}$, where $\lambda \in \mathbb{F}$ and $\phi : \mathrm{Alg}\,\mathcal{N} \to \mathbb{F}$ is an additive map. Since $\Phi$ is an additive generalzied derivation, there exsits an additive derivation $\tau : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ such that

$\Phi(AB)=\Phi(A)B+A\tau(B)$ for all $A,B\in \mathrm{Alg}\mathcal{N}$. Thus $\lambda AB+\phi(AB)I=\Phi(AB)=\Phi(A)B+A\tau(B)=\lambda AB+\phi(A)B+A\tau(B)$, and so $A\tau(B)=-\phi(A)B+\phi(AB)I$ for all $A,B\in \mathrm{Alg}\mathcal{N}$. Taking $A=I$, we get $\tau(B)=-\phi(I)B+\phi(B)I$ for all $B\in \mathrm{Alg}\mathcal{N}$. It is easy to check that $\tau$ is a commuting derivation. By Theorem 2.6, $\tau=0$. Now we have proved that $\phi(I)A=\phi(A)I$ for all $A\in \mathrm{Alg}\mathcal{N}$. If $\phi(I)\neq 0$, then $A=\phi(I)^{-1}\phi(A)I$ for all $A\in \mathrm{Alg}\mathcal{N}$, which is impossible. So $\phi(I)=0$, and therefore, $\Phi(A)=\lambda A$ for all $A\in \mathrm{Alg}\mathcal{N}$. The proof of the theorem is completed. □

## 3. Cocentralizing derivations and cocommuting generalized derivations on nest algebras

In this section, we first consider the cocentralizing derivations on nest algebras.

**Theorem 3.1** *Let $\mathcal{N}$ be a nest on a Banach space $X$ over the real or complex number field $\mathbb{F}$. Suppose that $\delta$ and $\delta'$ are any two additive cocentralizing derivations on* $\mathrm{Alg}\mathcal{N}$. *If there exists a non-trivial element $N$ in $\mathcal{N}$ which is complemented in $X$ and* $\dim N\neq 1$, *then* $\delta=\delta'\equiv 0$.

**Proof** By the assumption and the definition of cocentralizing maps, we have

$$\delta(A)A-A\delta'(A)\in \mathbb{F}I,\quad \forall A\in \mathrm{Alg}\mathcal{N}. \tag{3.1}$$

Replacing $A$ by $A+B$ in Eq. (3.1), we get

$$\delta(A)B+\delta(B)A-A\delta'(B)-B\delta'(A)\in \mathbb{F}I. \tag{3.2}$$

In particular, taking $B=I$ in Eq. (3.2), and noting that $\delta(I)=\delta'(I)=0$, we have

$$\delta(A)-\delta'(A)\in \mathbb{F}I,\quad \forall A\in \mathrm{Alg}\mathcal{N}. \tag{3.3}$$

Let $\Phi(A)=\delta(A)-\delta'(A)$ for all $A\in \mathrm{Alg}\mathcal{N}$. It is obvious that $\Phi$ is also an additive derivation of $\mathrm{Alg}\mathcal{N}$. Moreover, it follows from Eq. (3.3) that $\Phi$ is a centralizing derivation. So by Theorem 2.6, we get $\Phi\equiv 0$, that is, $\delta=\delta'$. Thus Eq. (3.1) becomes $\delta(A)A-A\delta(A)\in \mathbb{F}I$ for all $A\in \mathrm{Alg}\mathcal{N}$, which means that $\delta$ is an additive centralizing derivation from $\mathrm{Alg}\mathcal{N}$ into itself. By Theorem 2.6 again, it follows that $\delta=0$. The proof is completed. □

By Theorem 3.1, the following corollary is obvious.

**Corollary 3.2** *Let $\mathcal{N}$ be a nest on a Banach space $X$ over the real or complex number field $\mathbb{F}$. Suppose that $\delta$ is an additive skew-centralizing (skew-commuting) derivation on* $\mathrm{Alg}\mathcal{N}$. *If there exists a non-trivial element $N$ in $\mathcal{N}$ which is complemented in $X$ and* $\dim N\neq 1$, *then* $\delta\equiv 0$.

Now we consider additive cocommuting generalized derivations on nest algebras.

**Theorem 3.3** *Let $\mathcal{N}$ be a nest on a Banach space $X$ over the real or complex number field $\mathbb{F}$. Suppose that $\delta$ and $\delta'$ are any two additive cocommuting generalized derivations on* $\mathrm{Alg}\mathcal{N}$. *If there exists a non-trivial element $N$ in $\mathcal{N}$ which is complemented in $X$ and* $\dim N\neq 1$, *then there exists some $S\in \mathrm{Alg}\mathcal{N}$ such that $\delta(A)=AS$ and $\delta'(A)=SA$ for all $A\in \mathrm{Alg}\mathcal{N}$.*

We first prove the following lemma.

**Lemma 3.4** *Let $\mathcal{N}$ be a nest on a Banach space $X$ over the real or complex number field $\mathbb{F}$.*

*Suppose that* $\Phi : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ *is additive modulo* $\mathbb{F}I$. *If there exists a non-trivial element* $N$ *in* $\mathcal{N}$ *which is complemented in* $X$ *and* $\dim N \neq 1$, *and if* $[A, \Phi(BA) - \Phi(B)A] = 0$ *for all* $A, B \in \mathrm{Alg}\mathcal{N}$, *then there exist* $\lambda \in \mathbb{F}$ *and a map* $f : \mathrm{Alg}\mathcal{N} \to \mathbb{F}$ *such that*

$$\Phi(A) = (\lambda I + \Phi(I))A + f(A)I$$

*for all* $A \in \mathrm{Alg}\mathcal{N}$.

**Proof** For each $A \in \mathrm{Alg}\mathcal{N}$, we define $\Psi(A) = \Phi(A) - \Phi(I)A$. It is clear that $[\Psi(A), A] = 0$ for all $A \in \mathrm{Alg}\mathcal{N}$. For any $B \in \mathrm{Alg}\mathcal{N}$, we have

$$\begin{aligned}\Psi(A+B) - \Psi(A) - \Psi(B) =& \Phi(A+B) - \Phi(I)(A+B) - \Phi(A) + \Phi(I)A - \Phi(B) + \Phi(I)B \\ =& \Phi(A+B) - \Phi(A) - \Phi(B) \in \mathbb{F}I.\end{aligned}$$

So $\Psi$ is additive modulo $\mathbb{F}I$. Replacing $A$ by $A+B$ in $[\Psi(A), A] = 0$, we have that the map $\delta(A, B) = [\Psi(A), B]$ is an additive biderivation of Alg$\mathcal{N}$. By Theorem 2.1, there exists $\lambda \in \mathbb{F}$ such that $[\Psi(A), B] = \lambda[A, B]$ for all $A, B \in \mathrm{Alg}\mathcal{N}$, that is, $[\Psi(A) - \lambda A, B] = 0$ for all $A, B \in \mathrm{Alg}\mathcal{N}$. Since $(\mathrm{Alg}\mathcal{N})' = \mathbb{F}I$, there exists a map $f : \mathrm{Alg}\mathcal{N} \to \mathbb{F}$ such that $\Psi(A) - \lambda A = f(A)I$. Hence $\Phi(A) = \Phi(A) - \Phi(I)A = \lambda A + f(A)I$, that is, $\Phi(A) = (\lambda I + \Phi(I))A + f(A)I$ for all $A \in \mathrm{Alg}\mathcal{N}$. The proof is completed. $\Box$

**Proof of Theorem 3.3** Linearizing $\delta(A)A - A\delta'(A) = 0$, we get

$$\delta(A)B + \delta(B)A - A\delta'(B)A - B\delta'(A) = 0 \tag{3.4}$$

for all $A, B \in \mathrm{Alg}\mathcal{N}$. It is easy to verify that

$$A[X, -\delta'(BX) + \delta'(B)X] + B[X, -\delta'(AX) + \delta'(A)X] = 0 \tag{3.5}$$

for all $A, B, X \in \mathrm{Alg}\mathcal{N}$. For each $X \in \mathrm{Alg}\mathcal{N}$, let $\Phi(B) = [X, -\delta'(BX) + \delta'(B)X]$. It follows from Eq. (3.5) that

$$A\Phi(B) + B\Phi(A) = 0 \quad \forall A, B \in \mathrm{Alg}\mathcal{N}. \tag{3.6}$$

Taking $A = B = I$ in Eq. (3.6), we get $\Phi(I) = 0$. Taking $B = I$ in Eq. (3.6) again, we have $\Phi(A) = -A\Phi(I) = 0$. Thus, we get that $[X, -\delta'(AX) + \delta'(A)X] = 0$ for all $A, X \in \mathrm{Alg}\mathcal{N}$. By Lemma 3.4, there exist $\lambda \in \mathbb{F}$ and an additive map $f : \mathrm{Alg}\mathcal{N} \to \mathbb{F}$ such that $-\delta'(X) = -SX + f(X)I$ for all $X \in \mathrm{Alg}\mathcal{N}$, where $-S = \lambda I - \delta'(I)$. Next using the similar proof to that of Theorem 3.1, one can easily check that $f = 0$. Hence $\delta'(X) = SX$ for all $X \in \mathrm{Alg}\mathcal{N}$.

Now Eq. (3.4) reduces to $\delta(A)B + \delta(B)A - ASBA - BSA = 0$, that is,

$$(\delta(A) - AS)B + (\delta(B) - BS)A = 0, \quad \forall A, B \in \mathrm{Alg}\mathcal{N}. \tag{3.7}$$

Let $\Psi(A) = \delta(A) - AS$. Eq. (3.7) becomes $\Psi(A)B + \Psi(B)A = 0$. Taking $A = B = I$ in Eq. (3.7), we get $\Psi(I) = 0$. Taking $B = I$ in Eq. (3.7) again, we have $\Psi(A) = -A\Psi(I) = 0$. Thus we have proved that $\delta(A) = AS$ for all $A \in \mathrm{Alg}\mathcal{N}$, completing the proof. $\Box$

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