Negative $Z$-Homogeneous Derivations for Even Parts of Odd Hamiltonian Superalgebras

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Abstract In this paper we mainly study the negative $Z$-homogeneous derivations from the even part of the finite-dimensional odd Hamiltonian superalgebra $HO$ into the odd part of generalized Witt superalgebra $W$ over a field of prime characteristic $p > 3$. Using the generating set of $HO$, by means of calculating actions of derivations on the generating set, we first compute the derivations of $Z$-degree $-1$, then determine the derivations of $Z$-degree less than $-1$.

Keywords generalized Witt superalgebra; odd Hamiltonian superalgebra; derivation space.

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1. Introduction

The theory of Lie superalgebras has undergone a remarkable evolution in mathematics because of its important applications in physics. For example, Kac [1, 2] has classified the finite-dimensional simple Lie superalgebras and the infinite-dimensional simple linearly compact Lie superalgebras over algebraically closed fields of characteristic zero, respectively. For modular Lie superalgebras, as far as we know, [3] and [4] may be the earliest papers. We know that the derivation algebras were determined for the finite-dimensional modular Lie algebras of Cartan type [5–7]. In the super case, the superderivation algebras and outer superderivation algebras were also sufficiently studied for the finite-dimensional modular Lie superalgebras of Cartan type $W$, $S$, $H$, $K$, and $HO$ (see [8–12]). The derivations for the even part of the Lie superalgebras of Cartan type $W$, $S$ and $HO$ were studied in [13, 14].

2. Preliminaries

Throughout this paper the underlying field $\mathbb{F}$ is of characteristic $p > 3$. We write $\mathbb{N}$ for the positive integers, and $\mathbb{N}_0$ for the nonnegative integers. Fix $n \in \mathbb{N} \setminus \{1, 2\}$. Put $Y_0 := \{1, 2, \ldots, n\}$.

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$Y_1 := \{ n + 1, \ldots, 2n \}$ and $Y := Y_0 \cup Y_1$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, put $|\alpha| = \sum_{i=1}^n \alpha_i$. Fix

$$
\mathfrak{t} := (t_1, t_2, \ldots, t_n) \in \mathbb{N}_n \quad \text{and} \quad \pi := (\pi_1, \pi_2, \ldots, \pi_n),
$$

where $\pi_i := p^{i_1} - 1$ for $i \in Y_0$. Let $A := A (n; t) = \{ \alpha \in \mathbb{N}_0^n | \alpha_i \leq \pi_i, i \in Y_0 \}$. Following [7], let $O (n; \mathfrak{t})$ be the divided power algebra over $\mathbb{F}$ with $\mathbb{F}$-basis $\{ x^{(\alpha)} | \alpha \in A \}$. For $\varepsilon_i = (\delta_{i1}, \ldots, \delta_{in})$, write $x_i$ instead of $x^{(\varepsilon_i)}$ for $i = 1, \ldots, n$. Let $\Lambda (n)$ be the exterior algebra over $\mathbb{F}$ in $n$ variables $x_{n+1}, \ldots, x_{2n}$. Take the tensor product $O (n; \mathfrak{t}) \otimes_{\mathbb{F}} \Lambda (n)$. Then $O (n, n; \mathfrak{t})$ is an associative superalgebra with a $\mathbb{Z}_2$-grading induced by the trivial $\mathbb{Z}_2$-grading of $O (n; \mathfrak{t})$ and the natural $\mathbb{Z}_2$-grading of $\Lambda (n)$. For $g \in O (n; \mathfrak{t})$, $f \in \Lambda (n)$, write $gf$ for $g \otimes f$. Let

$$
\mathbb{B}_k := \{ (i_1, i_2, \ldots, i_k) \mid n + 1 \leq i_1 < i_2 < \cdots < i_k \leq 2n \}
$$

be the set of $k$-tuples of strictly increasing integers between $n + 1$ and $2n$, and put $\mathbb{B} := \mathbb{B} (n) := \bigcup_{k=0}^n \mathbb{B}_k$, where $\mathbb{B}_0 := \emptyset$. Put $\mathbb{B}^0 := \{ u \in \mathbb{B} \mid |u| \text{ even} \}$ and $\mathbb{B}^1 := \{ u \in \mathbb{B} \mid |u| \text{ odd} \}$, where for $u = (i_1, i_2, \ldots, i_k) \in \mathbb{B}_k$, $|u| := k$, $|\emptyset| := 0$, $x^\emptyset := 1$. For $u = (i_1, i_2, \ldots, i_k) \in \mathbb{B}_k$, we set $x^u := x_{i_1} x_{i_2} \cdots x_{i_k}$; we also use $u$ to stand for the set $\{ i_1, i_2, \ldots, i_k \}$ if no confusion occurs.

Clearly, $\{ x^{(\alpha)} x^u \mid \alpha \in A, u \in \mathbb{B} \}$ constitutes an $\mathbb{F}$-basis of $O (n, n; \mathfrak{t})$. Let $\partial_1, \partial_2, \ldots, \partial_{2n}$ be the linear transformations of $O (n, n; \mathfrak{t})$ such that

$$
\partial_r (x^{(\alpha)} x^u) = \begin{cases} 
x^{(\alpha-\varepsilon_r)} x^u, & r \in Y_0 \\
x^{(\alpha)} \cdot \partial x^u / \partial x_r, & r \in Y_1.
\end{cases}
$$

Then $\partial_1, \partial_2, \ldots, \partial_{2n}$ are superderivations of the superalgebra $O (n, n; \mathfrak{t})$. Obviously, the parity $p(\partial_i) = \mu (i)$, where

$$
\mu (i) := \begin{cases} 
\mathbb{U} & i \in Y_0 \\
\mathbb{T} & i \in Y_1.
\end{cases}
$$

Let

$$
W (n, n; \mathfrak{t}) := \left\{ \sum_{r \in Y} f_r \partial_r \mid f_r \in O (n, n; \mathfrak{t}), r \in Y \right\}.
$$

Then $W (n, n; \mathfrak{t})$ is a finite-dimensional simple Lie superalgebra contained in the full superderivation algebra $\text{Der} \ O (n, n; \mathfrak{t})$ (see [15]). Note that $O (n, n; \mathfrak{t})$ is endowed with a natural $\mathbb{Z}$-grading structure $O (n, n; \mathfrak{t}) = \bigoplus_{\tau \in \mathbb{N}} O (n, n; \mathfrak{t})_{\tau}$, by putting

$$
O (n, n; \mathfrak{t})_{\tau} := \text{span}_{\mathbb{F}} \{ x^{(\alpha)} x^u \mid |\alpha| + |u| = \tau \}, \quad \tau := |\pi| + n.
$$

Obviously, $W (n, n; \mathfrak{t})$ is a free $O (n, n; \mathfrak{t})$-module with $O (n, n; \mathfrak{t})$-basis $\{ \partial_r \mid r \in Y \}$. Clearly, $W (n, n; \mathfrak{t})$ possesses a standard $\mathbb{F}$-basis $\{ x^{(\alpha)} x^u \partial_r \mid \alpha \in A, u \in \mathbb{B}, r \in Y \}$. Note that $W (n, n; \mathfrak{t})$ is naturally graded by $W (n, n; \mathfrak{t}) = \bigoplus_{\mu = 0}^\infty W (n, n; \mathfrak{t})_{\mu}$, where

$$
W (n, n; \mathfrak{t})_{i} := \text{span}_{\mathbb{F}} \{ f \partial_k \mid s \in Y, f \in O (n, n; \mathfrak{t})_{i+1} \}.
$$

Put

$$
i' := \begin{cases} 
i + n, & i \in Y_0 \\
i - n, & i \in Y_1.
\end{cases}
$$
Proof In view of Lemma 1 one may assume that $k, l, q$ and $s, r, t, n, n$ satisfy possibly different values of $i, j, p$.

\[ T_{H}(a) = \sum_{i \in Y} (-1)^{\nu(i)p(a)} \partial_{i}(a) \partial_{p} \quad \text{for all } a \in \mathcal{O}(n, n; \mathfrak{t}). \]

Then $T_{H}$ is odd and [11, Proposition 1]

\[ [T_{H}(a), T_{H}(b)] = T_{H}(T_{H}(a)(b)) \quad \text{for } a, b \in \mathcal{O}(n, n; \mathfrak{t}). \]

Put

\[ HO(n, n; \mathfrak{t}) := \{ T_{H}(a) \mid a \in \mathcal{O}(n, n; \mathfrak{t}) \}. \]

Then $HO(n, n; \mathfrak{t})$ is a finite-dimensional simple Lie superalgebra [2]. Following [11], we call this Lie superalgebra the odd Hamiltonian superalgebra.

For convenience, in the sequel we shorten $W(n, n; \mathfrak{t})$, $HO(n, n; \mathfrak{t})$, to $W$, $HO$, and the even parts are simply denoted by $W$, $HO$, respectively.

Put $G := \text{span}_{\mathbb{F}}\{ x^{u}\partial_{u} \mid p(x^{u}\partial_{u}) = T, r \in Y, u \in \mathbb{Z} \}$. Clearly, $G$ is a $\mathbb{Z}$-graded subspace of $W_{\mathfrak{t}}$.

The proof of the following lemma is standard.

Lemma 1 Let $\phi \in \text{Der}(HO, W_{\mathfrak{t}})$, $\phi(HO_{-1}) = 0$ and $E \in HO$. Then $[E, HO_{-1}] \subseteq \ker \phi$ if and only if $\phi(E) \in G$.

Put

\[ N := \{ T_{H}(x_{k}x_{l}x_{q}) \mid k, l, q \in Y_{1} \}, \]

\[ M := \{ T_{H}(x^{(q, c)}_{i}x_{k}) \mid i \in Y_{0}, 0 \leq q \leq \pi, k \in Y_{1} \}. \]

Lemma 2 ([14, Proposition 2.1]) $HO$ is generated by $M \cup N$.

3. Negative $\mathbb{Z}$-homogeneous derivations

We first show that if a derivation $\phi \in \text{Der}_{-1}(HO, W_{\mathfrak{t}})$ vanishes on $HO_{0}$, then $\phi = 0$.

Lemma 3 Let $\phi \in \text{Der}_{-1}(HO, W_{\mathfrak{t}})$ satisfy $\phi(HO_{0}) = 0$. Then $\phi(T_{H}(x_{k}x_{l}x_{q})) = 0$ for all $k, l, q \in Y_{1}$.

Proof In view of Lemma 1 one may assume that $\phi(T_{H}(x_{k}x_{l}x_{q})) = \sum_{s \in Y_{1}, r \in Y_{0}} c_{sr}x_{s}\partial_{r}$, where $c_{sr} \in \mathbb{F}$. Direct computation shows that $[T_{H}(x_{k}x_{l}x_{q}), T_{H}(x_{k}x_{l}x_{q})] = T_{H}(x_{k}x_{l}x_{q})$. Applying $\phi$ yields

\[ \sum_{r \in Y_{0}} c_{kr}x_{k}\partial_{r} + \sum_{s \in Y_{1}} c_{sk}x_{s}\partial_{k} = \sum_{s \in Y_{1}, r \in Y_{0}} c_{sr}x_{s}\partial_{r}. \]

A comparison of coefficients shows that

\[ c_{kk'}x_{k} + \sum_{s \in Y_{1}} c_{sk}x_{s} = \sum_{s \in Y_{1}} c_{sk}x_{s}; \quad c_{kr}x_{k} = \sum_{s \in Y_{1}} c_{sr}x_{s} \quad \text{for } r \in Y_{0} \setminus k'. \]

It follows that $c_{kk'} = 0$, $c_{sr} = 0$ for $r \in Y_{0} \setminus k'$, $s \in Y_{1} \setminus k$. Thus

\[ \phi(T_{H}(x_{k}x_{l}x_{q})) = \sum_{r \in Y_{0} \setminus k'} c_{kr}x_{k}\partial_{r} + \sum_{s \in Y_{1} \setminus k} c_{sk}x_{s}\partial_{k'}. \]
Note that \([T_H(x_k x_l x_q), T_H(x_{k'} x_l)] = 0\). Applying \(\phi\), we have
\[
- \sum_{s \in Y_1 \setminus k} c_{sk'} x_s \partial_r - \sum_{r \in Y_0 \setminus k'} c_{kr} x_l \partial_r = 0,
\]
and therefore,
\[
c_{sk'} = 0 \quad \text{for} \quad s \in Y_1 \setminus \{k, l\}; \quad c_{kr} = 0 \quad \text{for} \quad r \in Y_0 \setminus \{k', l'\}; \quad c_{kl} + c_{k'l'} = 0.
\]

Hence,
\[
\phi(T_H(x_k x_l x_q)) = c_{kk'} x_k \partial_r + c_{k'l'} x_l \partial_r = c_{kk'} x_k \partial_r - c_{kl} x_l \partial_r.
\]

Applying \(\phi\) to \([T_H(x_k x_l x_q), T_H(x_{k'} x_l)] = 0\), one gets \(-c_{kl} x_k \partial_r + c_{k'l'} x_l \partial_r = 0\). It follows that \(c_{kl'} = 0\). Therefore, \(\phi(T_H(x_k x_l x_q)) = 0\).

**Lemma 4** Let \(\phi \in \text{Der}_{-1}(\mathcal{H}O, W_T)\) satisfy \(\phi(\mathcal{H}O_0) = 0\). Then \(\phi(T_H(x^{(a_{ij})} x_k)) = 0\) for all \(0 \leq a \leq \pi_i, i \in Y_0, k \in Y_1\).

**Proof** The proof is similar to that of [14, Lemma 4.2].

By Lemmas 2, 3 and 4 we have the following proposition.

**Proposition 1** Let \(\phi \in \text{Der}_{-1}(\mathcal{H}O, W_T)\) satisfy \(\phi(\mathcal{H}O_0) = 0\). Then \(\phi = 0\).

**Theorem 1** \(\text{Der}_{-1}(\mathcal{H}O, W_T) = \text{ad}(W_T)_{-1}\).

**Proof** Let \(\phi \in \text{Der}_{-1}(\mathcal{H}O, W_T)\). By Lemma 1, assume that \(\phi(T_H(x_i, x_k)) = \sum_{r \in Y_1} c_{ikr} \partial_r\), where \(c_{ikr} \in F, i \in Y_0, k \in Y_1\). Applying \(\phi\) to \([T_H(x_k x_k), T_H(x_{k'} x_{k'})] = -T_H(x_k x_k\{i'\}, i \in Y_0 \setminus k', \) one gets \(c_{ikk} \partial_r - c_{kkk'} \partial_r' = -\sum_{r \in Y_1} c_{ikr} \partial_r\). Consequently,
\[
c_{ikk} = 0 \quad \text{for} \quad k \in Y_1 \setminus \{i'\}; \quad c_{ikr} = 0, \quad \text{for} \quad r \in Y_1 \setminus \{k, i'\}; \quad c_{ikl} = c_{kk'l}.
\]

Therefore, \(\phi(T_H(x_k, x_k)) = c_{kk'} \partial_r = c_{kk'l} \partial_r\). Put
\[
\psi := \phi - \sum_{r \in Y_1} c_{rr'} \partial_r \quad \text{where} \quad c_{rr'} \in F.
\]

Then \(\psi(T_H(x_k, x_k)) = 0\). For arbitrary \(j' \in Y_1 \setminus k\), \([T_H(x_k x_{k'}), T_H(x_{j'} x_{j'})] = 0\). Applying \(\phi\) yields that \(c_{kk'j} \partial_{j'} = c_{kk'j} \partial_{j'} = 0\) and consequently, \(c_{kk'} = 0\). Thus, \(\phi(T_H(x_k x_{k'})) = c_{kk'} \partial_{k'}\) and \(\psi(T_H(x_k x_{k'})) = 0\). Hence, \(\psi(\mathcal{H}O_0) = 0\). By Proposition 1, \(\psi = 0\); that is, \(\phi = \sum_{r \in Y_1} c_{rr'} \partial_r \in \text{ad}(W_T)_{-1}\).

**Lemma 5** Let \(\phi \in \text{Der}_{-t}(\mathcal{H}O, W_T)\) where \(t > 1\). If \(\phi(T_H(x^{(a_{ij})}) x_k) = 0\) for all \(i \in Y_0, k \in Y_1\), then \(\phi = 0\).

**Proof** Similarly to the proof of [14, Lemma 4.5], one may show that \(\phi(T_H(x_k x_l x_q)) = 0\) for \(k, l, q \in Y_1\). In the following we use induction on \(a\) to show that \(\phi(T_H(x^{(a_{ij})}) x_k) = 0\) for \(i \in Y_0, k \in Y_1\). Similarly to the proof of [14, Lemma 4.5], one may show that in case \(a \leq t\) and \(a - t \geq 2\), \(\phi(T_H(x^{(a_{ij})}) x_k) = 0\).
Case $a - t < 2$. Clearly, $a - t = 1$, that is, $|u| = 1$. Thus
\[ \phi(T_h(x^{(a_1)}x_k)) = \sum_{q \in Y_1, r \in Y_0} c_{qr}x_q \partial_r. \]
First consider the situation $k \neq i'$. Note that $[T_h(x^{(a_1)}x_k), T_h(x_i x_{i'})] = a T_h(x^{(a_1)}x_k)$. Applying $\phi$, one gets
\[ - \sum_{q \in Y_1} c_{qi}x_q \partial_i - \sum_{r \in Y_0} c_{i'r}x_{i'} \partial_r = a \sum_{q \in Y_1, r \in Y_0} c_{qr}x_q \partial_r. \]
A comparison of coefficients shows that
\[ (a + 1) \sum_{q \in Y_1} c_{qi}x_q + c_{i'i'}x_{i'} = 0; \quad a \sum_{q \in Y_1} c_{qi}x_q + c_{i'r}x_{i'} = 0 \text{ for } r \in Y_0 \setminus i. \]
Consequently,
\[ (a + 2)c_{i'i} = 0; \quad (a + 1)c_{qi} = 0 \text{ for } q \in Y_1 \setminus i'; \]
\[ (a + 1)c_{i'r} = 0 \text{ for } r \in Y_0 \setminus i; \quad ac_{qr} = 0 \text{ for } r \in Y_0 \setminus i, q \in Y_1 \setminus i'. \]
If $a \equiv 0 \pmod{p}$, Similarly to the proof of [14, Lemma 4.5, the case $a \equiv 0 \pmod{p}$], one may show that $\phi(T_h(x^{(a_1)}x_k)) = 0$.

If $a \not\equiv 0 \pmod{p}$, the discussion is divided into the following three parts.

(i) Suppose $a \equiv -1 \pmod{p}$. Then
\[ c_{i'i} = 0; \quad c_{q'i} = 0 \text{ for } r \in Y_0 \setminus i, q \in Y_1 \setminus i'. \]
Thus
\[ \phi(T_h(x^{(a_1)}x_k)) = \sum_{q \in Y_1 \setminus i'} c_{qi}x_q \partial_i + \sum_{r \in Y_0 \setminus i} c_{i'r}x_{i'} \partial_r. \]
Applying $\phi$ to $[T_h(x^{(a_1)}x_k), T_h(x_k x_{i'})] = -T_h(x^{(a_1)}x_k)$, we have
\[ -c_{i'k}x_{i'} \partial_{i'} - c_{ki}x_k \partial_i = - \sum_{q \in Y_1 \setminus i'} c_{qi}x_q \partial_i - \sum_{r \in Y_0 \setminus i} c_{i'r}x_{i'} \partial_r. \]
A comparison of coefficients yields
\[ c_{i'k}x_k = \sum_{q \in Y_1 \setminus i'} c_{qi}x_q; \quad c_{i'i'}x_{i'} = 0 \text{ for } r \in Y_0 \setminus \{i, k'\}. \]
Consequently,
\[ c_{qi} = 0 \text{ for } q \in Y_1 \setminus \{i', k\}; \quad c_{i'r} = 0 \text{ for } r \in Y_0 \setminus \{i, k'\}. \]
It follows that
\[ \phi(T_h(x^{(a_1)}x_k)) = c_{i'k}x_{i'} \partial_{i'} + c_{ki}x_k \partial_i. \]
Suppose
\[ \phi(T_h(x_i x_{i'} x_l)) = \sum_{r \in Y_1} a_r \partial_r \text{ where } a_r \in F. \]
For $l \in Y_1 \setminus \{i', k\}$, one computes $[T_h(x^{(a_1)}x_k), T_h(x_i x_{i'} x_l)] = 0$. Applying $\phi$, one gets
\[ c_{ki}x_k x_l \partial_l - c_k x_k x_{i'} x_l \partial_{i'} - c_{k'l} x_{i'} x_l \partial_{i'} - a_k T_h(x^{(a_1)}x_k) = 0. \]
It follows that $c_{ki} = c_{i'k'} = 0$. Thus, $\phi(T_H(x^{(a_{ki})}x_k)) = 0$.

(ii) Suppose $a \equiv -2 \pmod{p}$. Then $\phi(T_H(x^{(a_{ki})}x_k)) = c_{i'x_{i'}\partial_i}$. Applying $\phi$ to

$$[T_H(x^{(a_{ki})}x_k), T_H(x_kx_i)] = 0,$$

we have $-c_{i'x_{i'}\partial_i} - c_{i'x_k\partial_i} = 0$. Then $c_{i'i} = 0$. Hence $\phi(T_H(x^{(a_{ki})}x_k)) = 0$.

(iii) Suppose $a \not\equiv -1, -2 \pmod{p}$. Then it is clear that $\phi(T_H(x^{(a_{ki})}x_k)) = 0$.

It remains to consider the situation $k = i'$. Direct computation yields $[T_H(x^{(a_{ki})}x_{i'}), T_H(x_{i'}x_{i'})] = (a - 1)T_H(x^{(a_{ki})}x_{i'})$. Applying $\phi$, one gets

$$- \sum_{q \in Y_1} c_{qi}x_q \partial_i - \sum_{r \in Y_0} c_{i'r}x_r \partial_r = (a - 1) \sum_{q \in Y_1, r \in Y_0} c_{qr}x_q \partial_r.$$

Then

$$a \sum_{q \in Y_1} c_{qi}x_q + c_{i'x_{i'}} = 0; \quad (a - 1) \sum_{q \in Y_1} c_{qr}x_q + c_{i'r}x_{i'} = 0 \text{ for } r \in Y_0 \setminus i.$$

Consequently,

$$(a + 1)c_{i'i} = 0; \quad ac_{qi} = 0 \text{ for } q \in Y_1 \setminus i';$$
$$ac_{i'r} = 0 \text{ for } r \in Y_0 \setminus i; \quad (a - 1)c_{qr} = 0 \text{ for } q \in Y_1 \setminus i', r \in Y_0 \setminus i.$$

We proceed in several steps. First suppose $a \equiv 0 \pmod{p}$. Then $c_{i'i} = 0$, $c_{qr} = 0$, $q \in Y_1 \setminus i'$, $r \in Y_0 \setminus i$. It follows that

$$\phi(T_H(x^{(a_{ki})}x_{i'})) = \sum_{q \in Y_1 \setminus i'} c_{qi}x_q \partial_i + \sum_{r \in Y_0 \setminus i} c_{i'r}x_r \partial_r.$$

For $j \in Y_0 \setminus i$, clearly, $[T_H(x^{(a_{ki})}x_{i'}), T_H(x_jx_{i'})] = 0$. Applying $\phi$ yields $-c_{j'i}x_{i'} \partial_j - c_{i'j}x_j \partial_i = 0$ and then $c_{j'i} = c_{i'j} = 0$. Since $j'$ is arbitrary, we obtain that $\phi(T_H(x^{(a_{ki})}x_{i'})) = 0$. Secondly, suppose $a \equiv 1 \pmod{p}$. Then

$$\phi(T_H(x^{(a_{ki})}x_{i'})) = \sum_{q \in Y_1 \setminus i'} \sum_{r \in Y_0 \setminus i} c_{qr}x_q \partial_r.$$

For any $j \in Y_0 \setminus i$, it is easily seen that $[T_H(x^{(a_{ki})}x_{i'}), T_H(x_jx_{i'})] = 0$. Applying $\phi$, one gets

$$- \sum_{q \in Y_1 \setminus i'} c_{iq}x_q \partial_j - \sum_{r \in Y_0 \setminus i} c_{j'r}x_r \partial_r = 0.$$

Then

$$c_{iq} = 0 \text{ for } q \in Y_1 \setminus i'; \quad c_{j'r} = 0 \text{ for } r \in Y_0 \setminus \{i, j\}.$$

It follows that $\phi(T_H(x^{(a_{ki})}x_{i'})) = 0$. Thirdly, suppose $a \equiv -1 \pmod{p}$. Then

$$\phi(T_H(x^{(a_{ki})}x_{i'})) = c_{i'i}x_{i'} \partial_i.$$

Note that for $l \in Y_1 \setminus i'$, $[T_H(x^{(a_{ki})}x_{i'}), T_H(x_lx_i)] = -T_H(x^{(a_{ki})}x_i)$. Applying $\phi$ yields $c_{i'i} = 0$. Thus $\phi(T_H(x^{(a_{ki})}x_{i'})) = 0$. It remains to consider the case $a \not\equiv -1, 1, 0 \pmod{p}$, in which one sees immediately that $\phi(T_H(x^{(a_{ki})}x_{i'})) = 0$.

Define $\Phi : \mathcal{H}\mathcal{O} \to W_T$ by means of $T_H(f) \to T_H(\sum_{i \in Y_0} \partial_i \partial_{i'}(f))$, where $f \in \mathcal{O}(n, n; \mathbb{F})_T$. Since $\ker(T_H) = \mathbb{F}$, $\Phi$ is well defined. The proof of the following lemma is standard.
Lemma 6 \( \Phi \in \text{Der}(\mathcal{H}C, W_T) \) and \( zd(\Phi) = -2 \).

Theorem 2 Suppose \( t > 1 \) is not any \( p \)-power. Then \( \text{Der}_-(\mathcal{H}C, W_T) = F \Phi \).

Proof Let \( \phi \in \text{Der}_-(\mathcal{H}C, W_T) \). First suppose \( t \not\equiv 0 \pmod{p} \). Since \( \phi(T_H(x^{(t\varepsilon)})x_k)) \in (W_T)^{-1} \), assume that
\[
\phi(T_H(x^{(t\varepsilon)})x_k)) = \sum_{r \in Y_1} a_r \partial_r \quad \text{where } a_r \in F.
\]
Note that \( \phi(T_H(x^{(t\varepsilon)})x_k)) = (\varepsilon + t)T_H(x^{(t\varepsilon)})x_k) \).

Applying \( \phi \), one gets
\[
-a_r \partial_r = [x_r \partial_r - x_i \partial_i, \sum_{r \in Y_1} a_r \partial_r] = (\delta_{k,k'} - t) \sum_{r \in Y_1} a_r \partial_r.
\]
If \( k \neq k' \), similarly to the proof of [14, Proposition 4.6], one may show that \( \phi(T_H(x^{(t\varepsilon)})x_k)) = 0 \).

If \( k = k' \), then \( (t - 2)a_r = 0 \) and \( (t - 1)a_r = 0 \), \( r \in Y_1 \backslash k' \). If \( t \equiv 1 \pmod{p} \), then \( a_r = 0 \) and it follows that \( \phi(T_H(x^{(t\varepsilon)})x_k)) = \sum_{r \in Y_1 \backslash k} a_r \partial_r \). For \( j \in Y_1 \backslash i \), we have \( [T_H(x^{(t\varepsilon)})x_k), T_H(x_j x_{k'})] = 0 \). Applying \( \phi \), one gets \( a_r = 0 \). Thus \( \phi(T_H(x^{(t\varepsilon)})x_k)) = 0 \). If \( t \not\equiv 1 \pmod{p} \), then \( a_r = 0 \), \( r \in Y_1 \backslash k' \). Here we proceed in two cases. First suppose \( t \neq 2 \pmod{p} \). Then \( a_r = 0 \) and therefore, \( \phi(T_H(x^{(t\varepsilon)})x_k)) = 0 \). Let us consider the other case \( t \equiv 2 \pmod{p} \). Clearly, \( \phi(T_H(x^{(t\varepsilon)})x_k)) = a_r \partial_r \). Direct computation shows that
\[
[T_H(x^{(t-1\varepsilon)})x_k), T_H(x^{(2t\varepsilon)})x_k)] = \left(\frac{t}{2}\right) - t \phi(T_H(x^{(t\varepsilon)})x_k)).
\]
Since \( \phi(T_H(x^{(t-1\varepsilon)})x_k)) = 0 \), assume that
\[
\phi(T_H(x^{(2t\varepsilon)})x_k)) = \sum_{r \in Y_1} b_r \partial_r \quad \text{where } b_r \in F.
\]
Then applying \( \phi \) to (1) yields
\[
[T_H(x^{(t-1\varepsilon)})x_k), \sum_{r \in Y_1} b_r \partial_r] = \left(\frac{t}{2}\right) - t \phi(T_H(x^{(t\varepsilon)})x_k)).
\]
Consequently, \( -b_r x^{(t-2\varepsilon)} \partial_r = \left(\frac{t}{2} - t\right) a_r \partial_r \). If \( t \not\equiv 2 \pmod{p} \), we have \( a_r = 0 \). Then \( \phi(T_H(x^{(t\varepsilon)})x_k)) = 0 \). By Lemma 5, \( \phi = 0 \). If \( t = 2 \), then \( \phi(T_H(x^{(2\varepsilon)})x_k)) = a_r \partial_r \). Similarly, we have \( \phi(T_H(x^{(2\varepsilon)})x_k)) = b_k \partial_k \) for \( k' \neq k \), where \( b_k \in F \). One may assume that
\[
\phi(T_H(x^{(c_i + 2\varepsilon')})x_k)) = \sum_{r \in Y_1} c_r \partial_r \quad \text{where } c_r \in F.
\]
Note that
\[
[T_H(x^{(c_i + 2\varepsilon')})x_k), T_H(x_k x_{k'})] = T_H(x^{(c_i + 2\varepsilon')})x_k).
\]
Applying \( \phi \), one can get \( \phi(T_H(x^{(c_i + 2\varepsilon')})x_k)) = c_k \partial_k \). Similarly, \( \phi(T_H(x^{(c_i + 2\varepsilon')})x_k)) = d_r \partial_r \), where \( d_r \in F \). Applying \( \phi \) to \( [T_H(x^{(c_i + 2\varepsilon')})x_k), T_H(x_k x_{k'})] = T_H(x^{(c_i + 2\varepsilon')})x_k) \), we have \( a_r = c_k \).
Applying \( \phi \) to \( [T_H(x^{(c_i + 2\varepsilon')})x_k), T_H(x_k x_k)] = T_H(x^{(c_i + 2\varepsilon')})x_k) \), we have \( b_k = d_r \). Not that
\[
[T_H(x^{(c_i + 2\varepsilon')})x_k), T_H(x_k x_k)] = \phi(T_H(x^{(2\varepsilon')})x_k) - 2\phi(T_H(x^{(2\varepsilon')})x_k) - \phi(T_H(x^{(c_i + 2\varepsilon')})x_k).
\]
Applying $\phi$ yields $a_{i'} = b_k$ for all $i' \neq k$. Putting $\lambda := a_{i'} = b_k$, one gets $\phi(T_H(x^{(2k)}_{i'} x_{i'})) = \lambda \partial_{i'}$.

Put $\varphi := \phi - \lambda \Phi$. Then $\varphi(T_H(x^{(2k)}_{i'} x_{i'})) = 0$. By Lemmas 5, 6, $\varphi = 0$.

It remains to consider the case $t \equiv 0 \pmod{p}$, in which just as in the proof of [14, Proposition 4.6, p. 29], one may prove $\phi = 0$.

**Theorem 3** Let $t = p^r$ for some $r \in \mathbb{N}$. Then $\text{Der}_{-1}(\mathfrak{h}_C, W_\tau) = 0$.

**Proof** The proof is similar to the one of [14, Proposition 4.7].

**References**