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Negative $\mathbb{Z}$-Homogeneous Derivations for Even Parts of Odd Hamiltonian Superalgebras

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Abstract In this paper we mainly study the negative $\mathbb{Z}$-homogeneous derivations from the even part of the finite-dimensional odd Hamiltonian superalgebra HO into the odd part of generalized Witt superalgebra W over a field of prime characteristic $p > 3$. Using the generating set of $\mathcal{HO}$, by means of calculating actions of derivations on the generating set, we first compute the derivations of $\mathbb{Z}$-degree -1, then determine the derivations of $\mathbb{Z}$-degree less than -1.

Keywords generalized Witt superalgebra; odd Hamiltonian superalgebra; derivation space.

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1. Introduction

The theory of Lie superalgebras has undergone a remarkable evolution in mathematics because of its important applications in physics. For example, Kac [1, 2] has classified the finite-dimensional simple Lie superalgebras and the infinite-dimensional simple linearly compact Lie superalgebras over algebraically closed fields of characteristic zero, respectively. For modular Lie superalgebras, as far as we know, [3] and [4] may be the earliest papers. We know that the derivation algebras were determined for the finite-dimensional modular Lie algebras of Cartan type [5–7]. In the super case, the superderivation algebras and outer superderivation algebras were also sufficiently studied for the finite-dimensional modular Lie superalgebras of Cartan type W, S, H, K, and HO (see [8–12]). The derivations for the even part of the Lie superalgebras of Cartan type W, S and HO were studied in [13, 14].

2. Preliminaries

Throughout this paper the underlying field $\mathbb{F}$ is of characteristic $p > 3$. We write $\mathbb{N}$ for the positive integers, and $\mathbb{N}_0$ for the nonnegative integers. Fix $n \in \mathbb{N} \setminus \{1, 2\}$. Put $Y_0 := \{1, 2, \ldots, n\}$,

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$Y_1 := \{n+1,\dots,2n\}$ and $Y := Y_0 \cup Y_1$. For $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$, put $|\alpha| = \sum_{i=1}^n \alpha_i$. Fix

$$\underline{t} := (t_1, t_2, \dots, t_n) \in \mathbb{N}^n \quad \text{and} \quad \pi := (\pi_1, \pi_2, \dots, \pi_n),$$

where $\pi_i := p^{t_i} - 1$ for $i \in Y_0$. Let $\mathbb{A} := \mathbb{A}(n;\underline{t}) = \{\alpha \in \mathbb{N}_0^n \mid \alpha_i \le \pi_i, i \in Y_0\}$. Following [7], let $\mathcal{O}(n;\underline{t})$ be the divided power algebra over $\mathbb{F}$ with $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{A}\}$. For $\varepsilon_i = (\delta_{i1},\dots,\delta_{in})$, write x_i instead of $x^{(\varepsilon_i)}$ for $i = 1,\dots,n$. Let $\Lambda(n)$ be the exterior algebra over $\mathbb{F}$ in n variables $x_{n+1},\dots,x_{2n}$. Take the tensor product $\mathcal{O}(n,n;\underline{t}) = \mathcal{O}(n;\underline{t}) \otimes_{\mathbb{F}} \Lambda(n)$. Then $\mathcal{O}(n,n;\underline{t})$ is an associative superalgebra with a $\mathbb{Z}_2$-grading induced by the trivial $\mathbb{Z}_2$-grading of $\mathcal{O}(n;\underline{t})$ and the natural $\mathbb{Z}_2$-grading of $\Lambda(n)$. For $g \in \mathcal{O}(n;\underline{t})$, $f \in \Lambda(n)$, write gf for $g \otimes f$. Let

$$\mathbb{B}_k := \{\langle i_1, i_2, \dots, i_k\rangle \mid n+1 \le i_1 < i_2 < \dots < i_k \le 2n\}$$

be the set of k-tuples of strictly increasing integers between $n+1$ and $2n$, and put $\mathbb{B} := \mathbb{B}(n) := \bigcup_{k=0}^n \mathbb{B}_k$, where $\mathbb{B}_0 := \emptyset$. Put $\mathbb{B}^0 := \{u \in \mathbb{B} \mid |u|\,\text{even}\}$ and $\mathbb{B}^1 := \{u \in \mathbb{B} \mid |u|\,\text{odd}\}$, where for $u = \langle i_1, i_2, \dots, i_k\rangle \in \mathbb{B}_k$, $|u| := k$, $|\emptyset| := 0$, $x^{\emptyset} := 1$. For $u = \langle i_1, i_2, \dots, i_k\rangle \in \mathbb{B}_k$, we set $x^u := x_{i_1} x_{i_2} \cdots x_{i_k}$; we also use u to stand for the set $\{i_1, i_2, \dots, i_k\}$ if no confusion occurs. Clearly, $\left\{x^{(\alpha)} x^u \mid \alpha \in \mathbb{A}, u \in \mathbb{B}\right\}$ constitutes an $\mathbb{F}$-basis of $\mathcal{O}(n,n;\underline{t})$. Let $\partial_1, \partial_2, \dots, \partial_{2n}$ be the linear transformations of $\mathcal{O}(n,n;\underline{t})$ such that

$$\partial_r(x^{(\alpha)} x^u) = \begin{cases} x^{(\alpha-\varepsilon_r)} x^u, & r \in Y_0 \\ x^{(\alpha)} \cdot \partial x^u / \partial x_r, & r \in Y_1. \end{cases}$$

Then $\partial_1, \partial_2, \dots, \partial_{2n}$ are superderivations of the superalgebra $\mathcal{O}(n,n;\underline{t})$. Obviously, the parity $\mathrm{p}(\partial_i) = \mu(i)$, where

$$\mu(i) := \begin{cases} \bar{0}, & i \in Y_0 \\ \bar{1}, & i \in Y_1. \end{cases}$$

Let

$$W(n,n;\underline{t}) = \Big\{ \sum_{r \in Y} f_r \partial_r \mid f_r \in \mathcal{O}(n,n;\underline{t}), r \in Y \Big\}.$$

Then $W(n,n;\underline{t})$ is a finite-dimensional simple Lie superalgebra contained in the full superderivation algebra $\mathrm{Der}\,\mathcal{O}(n,n;\underline{t})$ (see [15]). Note that $\mathcal{O}(n,n;\underline{t})$ is endowed with a natural $\mathbb{Z}$-grading structure $\mathcal{O}(n,n;\underline{t}) = \bigoplus_{r=0}^{\xi} \mathcal{O}(n,n;\underline{t})_r$ by putting

$$\mathcal{O}(n,n;\underline{t})_r := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)} x^u \mid |\alpha| + |u| = r\}, \quad \xi := |\pi| + n.$$

Obviously, $W(n,n;\underline{t})$ is a free $\mathcal{O}(n,n;\underline{t})$-module with $\mathcal{O}(n,n;\underline{t})$-basis $\{\partial_r \mid r \in Y\}$. Clearly, $W(n,n;\underline{t})$ possesses a standard $\mathbb{F}$-basis $\{x^{(\alpha)} x^u \partial_r \mid \alpha \in \mathbb{A}, u \in \mathbb{B}, r \in Y\}$. Note that $W(n,n;\underline{t})$ is naturally graded by $W(n,n;\underline{t}) = \oplus_{i=-1}^{\xi-1} W(n,n;\underline{t})_i$, where

$$W(n,n;\underline{t})_i := \mathrm{span}_{\mathbb{F}}\{f \partial_s \mid s \in Y,\ f \in \mathcal{O}(n,n;\underline{t})_{i+1}\}.$$

Put

$$i' = \begin{cases} i+n, & i \in Y_0 \\ i-n, & i \in Y_1. \end{cases}$$

Define a linear mapping $\mathrm{T_H} : \mathcal{O}(n, n; \underline{t}) \to W(n, n; \underline{t})$ by means of

$$\mathrm{T_H}(a) := \sum_{i \in Y} (-1)^{\mu(i)\mathrm{p}(a)} \partial_i(a) \partial_{i'} \quad \text{for all } a \in \mathcal{O}(n, n; \underline{t}).$$

Then $\mathrm{T_H}$ is odd and [11, Proposition 1]

$$[\mathrm{T_H}(a), \mathrm{T_H}(b)] = \mathrm{T_H}(\mathrm{T_H}(a)(b)) \quad \text{for } a, b \in \mathcal{O}(n, n; \underline{t}).$$

Put

$$HO(n, n; \underline{t}) := \{\mathrm{T_H}(a) \mid a \in \mathcal{O}(n, n; \underline{t})\}.$$

Then $HO(n, n; \underline{t})$ is a finite-dimensional simple Lie superalgebra [2]. Following [11], we call this Lie superalgebra the odd Hamiltonian superalgebra.

For convenience, in the sequel we shorten $W(n, n; \underline{t})$, $HO(n, n; \underline{t})$, to W, HO, and the even parts are simply denoted by $\mathcal{W}$, $\mathcal{HO}$, respectively.

Put $\mathcal{G} := \mathrm{span}_{\mathbb{F}}\{x^u \partial_r \mid \mathrm{p}(x^u \partial_r) = \bar{1}, r \in Y, u \in \mathbb{B}\}$. Clearly, $\mathcal{G}$ is a $\mathbb{Z}$-graded subspace of $W_{\bar{1}}$. The proof of the following lemma is standard.

Lemma 1 *Let $\phi \in \mathrm{Der}(\mathcal{HO}, W_{\bar{1}})$, $\phi(\mathcal{HO}_{-1}) = 0$ and $E \in \mathcal{HO}$. Then $[E, \mathcal{HO}_{-1}] \subseteq \ker \phi$ if and only if $\phi(E) \in \mathcal{G}$.*

Put

$$N := \{\mathrm{T_H}(x_k x_l x_q) \mid k, l, q \in Y_1\},$$

$$M := \{\mathrm{T_H}(x^{(q_i \varepsilon_i)} x_k) \mid i \in Y_0, 0 \leq q_i \leq \pi_i, k \in Y_1\}.$$

Lemma 2 ([14, Proposition 2.1]) *$\mathcal{HO}$ is generated by $M \cup N$.*

3. Negative $\mathbb{Z}$-homogeneous derivations

We first show that if a derivation $\phi \in \mathrm{Der}_{-1}(\mathcal{HO}, W_{\bar{1}})$ vanishes on $\mathcal{HO}_0$, then $\phi = 0$.

Lemma 3 *Let $\phi \in \mathrm{Der}_{-1}(\mathcal{HO}, W_{\bar{1}})$ satisfy $\phi(\mathcal{HO}_0) = 0$. Then $\phi(\mathrm{T_H}(x_k x_l x_q)) = 0$ for all k, l, $q \in Y_1$.*

Proof In view of Lemma 1 one may assume that $\phi(\mathrm{T_H}(x_k x_l x_q)) = \sum_{s \in Y_1,\, r \in Y_0} c_{sr} x_s \partial_r$, where $c_{sr} \in \mathbb{F}$. Direct computation shows that $[\mathrm{T_H}(x_{k'} x_k), \mathrm{T_H}(x_k x_l x_q)] = \mathrm{T_H}(x_k x_l x_q)$. Applying ϕ yields

$$\sum_{r \in Y_0} c_{kr} x_k \partial_r + \sum_{s \in Y_1} c_{sk'} x_s \partial_{k'} = \sum_{s \in Y_1,\, r \in Y_0} c_{sr} x_s \partial_r.$$

A comparison of coefficients shows that

$$c_{kk'} x_k + \sum_{s \in Y_1} c_{sk'} x_s = \sum_{s \in Y_1} c_{sk'} x_s; \quad c_{kr} x_k = \sum_{s \in Y_1} c_{sr} x_s \ \text{ for } r \in Y_0 \setminus k'.$$

It follows that $c_{kk'} = 0$, $c_{sr} = 0$ for $r \in Y_0 \setminus k'$, $s \in Y_1 \setminus k$. Thus

$$\phi(\mathrm{T_H}(x_k x_l x_q)) = \sum_{r \in Y_0 \setminus k'} c_{kr} x_k \partial_r + \sum_{s \in Y_1 \setminus k} c_{sk'} x_s \partial_{k'}.$$

Note that $[\mathrm{T_H}(x_kx_lx_q), \mathrm{T_H}(x_{k'}x_l)] = 0$. Applying ϕ, we have

$$-\sum_{s\in Y_1\setminus k} c_{sk'}x_s\partial_{l'} - \sum_{r\in Y_0\setminus k'} c_{kr}x_l\partial_r = 0,$$

and therefore,

$$c_{sk'} = 0 \text{ for } s\in Y_1\setminus\{k,\, l\};\quad c_{kr} = 0 \text{ for } r\in Y_0\setminus\{k',\, l'\};\quad c_{lk'} + c_{kl'} = 0.$$

Hence,

$$\phi(\mathrm{T_H}(x_kx_lx_q)) = c_{kl'}x_k\partial_{l'} + c_{lk'}x_l\partial_{k'} = c_{kl'}x_k\partial_{l'} - c_{kl'}x_l\partial_{k'}.$$

Applying ϕ to $[\mathrm{T_H}(x_kx_lx_q),\, \mathrm{T_H}(x_{l'}x_q)] = 0$, one gets $-c_{kl'}x_k\partial_{q'} + c_{kl'}x_q\partial_{k'} = 0$. It follows that $c_{kl'} = 0$. Therefore, $\phi(\mathrm{T_H}(x_kx_lx_q)) = 0$.

Lemma 4 *Let* $\phi\in \mathrm{Der}_{-1}(\mathcal{HO}, W_{\bar 1})$ *satisfy* $\phi(\mathcal{HO}_0) = 0$. *Then* $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)) = 0$ *for all* $0\le a\le \pi_i$, $i\in Y_0$, $k\in Y_1$.

Proof *The proof is similar to that of [14, Lemma 4.2].*

By Lemmas 2, 3 and 4 we have the following proposition.

Proposition 1 *Let* $\phi\in \mathrm{Der}_{-1}(\mathcal{HO}, W_{\bar 1})$ *satisfy* $\phi(\mathcal{HO}_0) = 0$. *Then* $\phi = 0$.

Theorem 1 $\mathrm{Der}_{-1}(\mathcal{HO}, W_{\bar 1}) = \mathrm{ad}(W_{\bar 1})_{-1}$.

Proof Let $\phi\in \mathrm{Der}_{-1}(\mathcal{HO}, W_{\bar 1})$. By Lemma 1, assume that $\phi(\mathrm{T_H}(x_ix_k)) = \sum_{r\in Y_1} c_{ikr}\partial_r$, where $c_{ikr}\in\mathbb{F}$, $i\in Y_0$, $k\in Y_1$. Applying ϕ to $[\mathrm{T_H}(x_ix_k), \mathrm{T_H}(x_kx_{k'})] = -\mathrm{T_H}(x_ix_k)$, $i\in Y_0\setminus k'$, one gets $c_{ikk}\partial_k - c_{kk'k}\partial_{i'} = -\sum_{r\in Y_1} c_{ikr}\partial_r$. Consequently,

$$c_{ikk} = 0 \text{ for } k\in Y_1\setminus i';\quad c_{ikr} = 0, \text{ for } r\in Y_1\setminus\{k, i'\};\quad c_{iki'} = c_{kk'k}.$$

Therefore, $\phi(\mathrm{T_H}(x_ix_k)) = c_{iki'}\partial_{i'} = c_{kk'k}\partial_{i'}$. Put

$$\psi := \phi - \sum_{r\in Y_1} c_{rr'r}\mathrm{ad}\partial_r \text{ where } c_{rr'r}\in\mathbb{F}.$$

Then $\psi(\mathrm{T_H}(x_ix_k)) = 0$. For arbitrary $j'\in Y_1\setminus k$, $[\mathrm{T_H}(x_kx_{k'}),\, \mathrm{T_H}(x_jx_{j'})] = 0$. Applying ϕ yields that $c_{kk'j'}\partial_{j'} - c_{jj'k}\partial_k = 0$ and consequently, $c_{kk'j'} = 0$. Thus, $\phi(\mathrm{T_H}(x_kx_{k'})) = c_{kk'k}\partial_k$ and $\psi(\mathrm{T_H}(x_kx_{k'})) = 0$. Hence, $\psi(\mathcal{HO}_0) = 0$. By Proposition 1, $\psi = 0$; that is, $\phi = \sum_{r\in Y_1} c_{rr'r}\mathrm{ad}\partial_r \in \mathrm{ad}(W_{\bar 1})_{-1}$.

Lemma 5 Let $\phi\in \mathrm{Der}_{-t}(\mathcal{HO}, W_{\bar 1})$ where $t>1$. If $\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_k)) = 0$ for all $i\in Y_0$, $k\in Y_1$, then $\phi = 0$.

Proof Similarly to the proof of [14, Lemma 4.5], one may show that $\phi(\mathrm{T_H}(x_kx_lx_q)) = 0$ for $k,\, l,\, q\in Y_1$. In the following we use induction on a to show that $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)) = 0$ for $i\in Y_0$, $k\in Y_1$. Similarly to the proof of [14, Lemma 4.5], one may show that in case $a\le t$ and $a-t\ge 2$, $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)) = 0$.

Case $a-t<2$. Clearly, $a-t=1$, that is, $|u|=1$. Thus

$$\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k))=\sum_{q\in Y_1,\, r\in Y_0}c_{qr}x_q\partial_r.$$

First consider the situation $k\neq i'$. Note that $[\mathrm{T_H}(x^{(a\varepsilon_i)}x_k),\, \mathrm{T_H}(x_ix_{i'})]=a\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)$. Applying ϕ, one gets

$$-\sum_{q\in Y_1}c_{qi}x_q\partial_i-\sum_{r\in Y_0}c_{i'r}x_{i'}\partial_r=a\sum_{q\in Y_1,\, r\in Y_0}c_{qr}x_q\partial_r.$$

A comparison of coefficients shows that

$$(a+1)\sum_{q\in Y_1}c_{qi}x_q+c_{i'i}x_{i'}=0;\quad a\sum_{q\in Y_1}c_{qr}x_q+c_{i'r}x_{i'}=0\ \text{ for }\ r\in Y_0\setminus i.$$

Consequently,

$$(a+2)c_{i'i}=0;\quad (a+1)c_{qi}=0\ \text{ for }\ q\in Y_1\setminus i';$$

$$(a+1)c_{i'r}=0\ \text{ for }\ r\in Y_0\setminus i;\quad ac_{qr}=0\ \text{ for }\ r\in Y_0\setminus i,\ q\in Y_1\setminus i'.$$

If $a\equiv 0\pmod p$, Similarly to the proof of [14, Lemma 4.5, the case $a\equiv 0\pmod p$], one may show that $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k))=0$.

If $a\not\equiv 0\pmod p$, the discussion is divided into the following three parts.

(i) Suppose $a\equiv -1\pmod p$. Then

$$c_{i'i}=0;\quad c_{qr}=0\ \text{ for }\ r\in Y_0\setminus i,\, q\in Y_1\setminus i'.$$

Thus

$$\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k))=\sum_{q\in Y_1\setminus i'}c_{qi}x_q\partial_i+\sum_{r\in Y_0\setminus i}c_{i'r}x_{i'}\partial_r.$$

Applying ϕ to $[\mathrm{T_H}(x^{(a\varepsilon_i)}x_k),\, \mathrm{T_H}(x_kx_{k'})]=-\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)$, we have

$$-c_{i'k'}x_{i'}\partial_{k'}-c_{ki}x_k\partial_i=-\sum_{q\in Y_1\setminus i'}c_{qi}x_q\partial_i-\sum_{r\in Y_0\setminus i}c_{i'r}x_{i'}\partial_r.$$

A comparison of coefficients yields

$$c_{ki}x_k=\sum_{q\in Y_1\setminus i'}c_{qi}x_q;\quad c_{i'r}x_{i'}=0\ \text{ for }\ r\in Y_0\setminus\{i,\, k'\}.$$

Consequently,

$$c_{qi}=0\ \text{ for }\ q\in Y_1\setminus\{i',k\};\quad c_{i'r}=0\ \text{ for }\ r\in Y_0\setminus\{i,\, k'\}.$$

It follows that

$$\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k))=c_{i'k'}x_{i'}\partial_{k'}+c_{ki}x_k\partial_i.$$

Suppose

$$\phi(\mathrm{T_H}(x_ix_{l'}x_l))=\sum_{r\in Y_1}a_r\partial_r\ \text{ where }\ a_r\in\mathbb{F}.$$

For $l\in Y_1\setminus\{i',k\}$, one computes $[\mathrm{T_H}(x^{(a\varepsilon_i)}x_k),\, \mathrm{T_H}(x_ix_{l'}x_l)]=0$. Applying ϕ, one gets

$$c_{ki}x_kx_l\partial_l-c_{ki}x_kx_{l'}\partial_{l'}-c_{i'k'}x_{l'}x_l\partial_{k'}-a_k\mathrm{T_H}(x^{(a\varepsilon_i)})=0.$$

It follows that $c_{ki} = c_{i'k'} = 0$. Thus, $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)) = 0$.

(ii) Suppose $a \equiv -2 \pmod p$. Then $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)) = c_{i'i}x_{i'}\partial_i$. Applying ϕ to

$$[\mathrm{T_H}(x^{(a\varepsilon_i)}x_k),\ \mathrm{T_H}(x_kx_i)] = 0,$$

we have $-c_{i'i}x_{i'}\partial_{k'} - c_{i'i}x_k\partial_i = 0$. Then $c_{i'i} = 0$. Hence $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)) = 0$.

(iii) Suppose $a \not\equiv -1,\ -2 \pmod p$. Then it is clear that $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_k)) = 0$.

It remains to consider the situation $k = i'$. Direct computation yields $[\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'}),\ \mathrm{T_H}(x_ix_{i'})] = (a-1)\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'})$. Applying ϕ, one gets

$$-\sum_{q\in Y_1} c_{qi}x_q\partial_i - \sum_{r\in Y_0} c_{i'r}x_{i'}\partial_r = (a-1)\sum_{q\in Y_1,\ r\in Y_0} c_{qr}x_q\partial_r.$$

Then

$$a\sum_{q\in Y_1} c_{qi}x_q + c_{i'i}x_{i'} = 0; \quad (a-1)\sum_{q\in Y_1} c_{qr}x_q + c_{i'r}x_{i'} = 0 \ \text{ for } r \in Y_0 \setminus i.$$

Consequently,

$$(a+1)c_{i'i} = 0; \quad ac_{qi} = 0 \ \text{ for } q \in Y_1 \setminus i';$$

$$ac_{i'r} = 0 \ \text{ for } r \in Y_0 \setminus i; \quad (a-1)c_{qr} = 0 \ \text{ for } q \in Y_1 \setminus i',\ r \in Y_0 \setminus i.$$

We proceed in several steps. First suppose $a \equiv 0 \pmod p$. Then $c_{i'i} = 0$, $c_{qr} = 0$, $q \in Y_1 \setminus i'$, $r \in Y_0 \setminus i$. It follows that

$$\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'})) = \sum_{q\in Y_1\setminus i'} c_{qi}x_q\partial_i + \sum_{r\in Y_0\setminus i} c_{i'r}x_{i'}\partial_r.$$

For $j \in Y_0 \setminus i$, clearly, $[\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'}),\ \mathrm{T_H}(x_jx_{j'})] = 0$. Applying ϕ yields $-c_{i'j}x_{i'}\partial_j - c_{j'i}x_{j'}\partial_i = 0$ and then $c_{i'j} = c_{j'i} = 0$. Since j' is arbitrary, we obtain that $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'})) = 0$. Secondly, suppose $a \equiv 1 \pmod p$. Then

$$\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'})) = \sum_{q\in Y_1\setminus i',\ r\in Y_0\setminus i} c_{qr}x_q\partial_r.$$

For any $j \in Y_0 \setminus i$, it is easily seen that $[\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'}),\ \mathrm{T_H}(x_jx_{j'})] = 0$. Applying ϕ, one gets

$$-\sum_{q\in Y_1\setminus i'} c_{qj}x_q\partial_j - \sum_{r\in Y_0\setminus i} c_{j'r}x_{j'}\partial_r = 0.$$

Then

$$c_{qj} = 0 \ \text{ for } q \in Y_1 \setminus i'; \quad c_{j'r} = 0 \ \text{ for } r \in Y_0 \setminus \{i,\ j\}.$$

It follows that $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'})) = 0$. Thirdly, suppose $a \equiv -1 \pmod p$. Then

$$\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'})) = c_{i'i}x_{i'}\partial_i.$$

Note that for $l \in Y_1 \setminus i'$, $[\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'}),\ \mathrm{T_H}(x_ix_l)] = -\mathrm{T_H}(x^{(a\varepsilon_i)}x_l)$. Applying ϕ yields $c_{i'i} = 0$. Thus $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'})) = 0$. It remains to consider the case $a \not\equiv -1, 1, 0 \pmod p$, in which one sees immediately that $\phi(\mathrm{T_H}(x^{(a\varepsilon_i)}x_{i'})) = 0$.

Define $\Phi : \mathcal{HO} \to W_{\bar{1}}$ by means of $\mathrm{T_H}(f) \to \mathrm{T_H}(\sum_{i\in Y_0}\partial_i\partial_{i'}(f))$, where $f \in \mathcal{O}(n,n;\underline{t})_{\bar{1}}$. Since $\ker(\mathrm{T_H}) = \mathbb{F}$, Φ is well defined. The proof of the following lemma is standard.

Lemma 6 *$\Phi \in \mathrm{Der}(\mathcal{HO}, W_{\bar{1}})$ and $\mathrm{zd}(\Phi) = -2$.*

Theorem 2 *Suppose $t > 1$ is not any p-power. Then $\mathrm{Der}_{-t}(\mathcal{HO}, W_{\bar{1}}) = \mathbb{F}\,\Phi$.*

Proof Let $\phi \in \mathrm{Der}_{-t}(\mathcal{HO}, W_{\bar{1}})$. First suppose $t \not\equiv 0 \pmod p$. Since $\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_k)) \in (W_{\bar{1}})_{-1}$, assume that

$$\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_k)) = \sum_{r\in Y_1} a_r\partial_r \quad \text{where } a_r \in \mathbb{F}.$$

Note tat

$$[\mathrm{T_H}(x_ix_{i'}), \mathrm{T_H}(x^{(t\varepsilon_i)}x_k)] = (\delta_{k,i'} - t)\mathrm{T_H}(x^{(t\varepsilon_i)}x_k).$$

Applying ϕ, one gets

$$-a_{i'}\partial_{i'} = \Big[x_{i'}\partial_{i'} - x_i\partial_i, \sum_{r\in Y_1} a_r\partial_r\Big] = (\delta_{k,i'} - t)\sum_{r\in Y_1} a_r\partial_r.$$

If $k \neq i'$, similarly to the proof of [14, Proposition 4.6], one may show that $\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_k)) = 0$. If $k = i'$, then $(t-2)a_{i'} = 0$ and $(t-1)a_r = 0$, $r \in Y_1 \setminus i'$. If $t \equiv 1 \pmod p$, then $a_{i'} = 0$ and it follows that $\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_{i'})) = \sum_{r\in Y_1\setminus i'} a_r\partial_r$. For $j \in Y_0\setminus i$, we have $[\mathrm{T_H}(x^{(t\varepsilon_i)}x_{i'}),\ \mathrm{T_H}(x_jx_{j'})] = 0$. Applying ϕ, one gets $a_{j'} = 0$. Thus $\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_{i'})) = 0$. If $t \not\equiv 1 \pmod p$, then $a_r = 0$, $r \in Y_1 \setminus i'$. Here we proceed in two cases. First suppose $t \not\equiv 2 \pmod p$. Then $a_{i'} = 0$ and therefore, $\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_{i'})) = 0$. Let us consider the other case $t \equiv 2 \pmod p$. Clearly, $\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_{i'})) = a_{i'}\partial_{i'}$. Direct computation shows that

$$[\mathrm{T_H}(x^{((t-1)\varepsilon_i)}x_{i'}),\ \mathrm{T_H}(x^{(2\varepsilon_i)}x_{i'})] = \Big[\binom{t}{2} - t\Big]\mathrm{T_H}(x^{(t\varepsilon_i)}x_{i'}). \tag{1}$$

Since $\phi(\mathrm{T_H}(x^{((t-1)\varepsilon_i)}x_{i'})) = 0$, assume that

$$\phi(\mathrm{T_H}(x^{(2\varepsilon_i)}x_{i'})) = \sum_{r\in Y_1} b_r\partial_r \quad \text{where } b_r \in \mathbb{F}.$$

Then applying ϕ to (1) yields

$$\Big[\mathrm{T_H}(x^{((t-1)\varepsilon_i)}x_{i'}),\ \sum_{r\in Y_1} b_r\partial_r\Big] = \Big[\binom{t}{2} - t\Big]\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_{i'})).$$

Consequently, $-b_{i'}x^{((t-2)\varepsilon_i)}\partial_{i'} = \Big[\binom{t}{2} - t\Big]a_{i'}\partial_{i'}$. If $t \neq 2$, since $t - \binom{t}{2} \not\equiv 0 \pmod p$, we have $a_{i'} = 0$. Then $\phi(\mathrm{T_H}(x^{(t\varepsilon_i)}x_{i'})) = 0$. By Lemma 5, $\phi = 0$. If $t = 2$, then $\phi(\mathrm{T_H}(x^{(2\varepsilon_i)}x_{i'})) = a_{i'}\partial_{i'}$. Similarly, we have $\phi(\mathrm{T_H}(x^{(2\varepsilon_{k'})}x_k)) = b_k\partial_k$ for $i' \neq k$, where $b_k \in \mathbb{F}$. One may assume that

$$\phi(\mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_{i'})) = \sum_{r\in Y_1} c_r\partial_r \quad \text{where } c_r \in \mathbb{F}.$$

Note that

$$[\mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_{i'}),\ \mathrm{T_H}(x_kx_{k'})] = \mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_{i'}).$$

Applying ϕ, one can get $\phi(\mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_{i'})) = c_k\partial_k$. Similarly, $\phi(\mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_k)) = d_{i'}\partial_{i'}$, where $d_{i'} \in \mathbb{F}$. Applying ϕ to $[\mathrm{T_H}(x^{(2\varepsilon_i)}x_{i'}),\ \mathrm{T_H}(x_{i'}x_{k'})] = \mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_{i'})$, we have $a_{i'} = c_k$. Applying ϕ to $[\mathrm{T_H}(x^{(2\varepsilon_{k'})}x_k),\ \mathrm{T_H}(x_ix_k)] = \mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_k)$, we have $b_k = d_{i'}$. Note that

$$[\mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_{i'}),\ \mathrm{T_H}(x_ix_k)] = 2\mathrm{T_H}(x^{(2\varepsilon_i)}x_{i'}) - \mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{k'})}x_k).$$

Applying ϕ yields $a_{i'} = b_k$ for all $i' \neq k$. Putting $\lambda := a_{i'} = b_k$, one gets $\phi(\mathrm{T_H}(x^{(2\varepsilon_i)}x_{i'})) = \lambda\partial_{i'}$. Put $\varphi := \phi - \lambda\Phi$. Then $\varphi(\mathrm{T_H}(x^{(2\varepsilon_i)}x_{i'})) = 0$. By Lemmas 5, 6, $\varphi = 0$.

It remains to consider the case $t \equiv 0 \pmod p$, in which just as in the proof of [14, Proposition 4.6, the case $t \equiv 0 \pmod p$, p. 29], one may prove $\phi = 0$.

Theorem 3 *Let* $t = p^r$ *for some* $r \in \mathbb{N}$. *Then* $\mathrm{Der}_{-t}(\mathcal{HO}, W_{\bar{1}}) = 0$.

Proof The proof is similar to the one of [14, Proposition 4.7].

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