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Fredholm Weighted Composition Operators on Hardy Space

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Abstract This paper characterizes bounded, Fredholm weighted composition operators on Hardy space. As an application, the Fredholm index of such operators is obtained.

Keywords Hardy space; weighted composition operator; Fredholmness.

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1. Introduction

Recently, weighted composition operators on various spaces of analytic functions have been studied intensively, see [1–6] and so on. Inspired by the study of bounded and invertible weighted composition operators on Hardy space $H^2(\mathbb{D})$ in [7], where $\mathbb{D}$ is the unit disk. In this paper, we characterize bounded Fredholm weighted composition operators on $H^2(\mathbb{D})$.

Recall $H^2(\mathbb{D})$ is the space of analytic function f on $\mathbb{D}$ with

$$\sup_{0<r<1}\int_0^{2\pi}|f(re^{i\theta})|^2\frac{\mathrm{d}\theta}{2\pi}<\infty.$$

It is well-known that $H^2(\mathbb{D})$ is a reproducing kernel function space with the reproducing kernel function

$$K_w(z)=\frac{1}{1-\bar{w}z},\quad z,w\in\mathbb{D}.$$

Let ψ and φ be analytic functions on $\mathbb{D}$ with $\varphi(\mathbb{D})\subset\mathbb{D}$. The weighted composition operator $C_{\psi,\varphi}$ is defined on $H^2(\mathbb{D})$ with symbols ψ,φ as

$$C_{\psi,\varphi}f=\psi\cdot f\circ\varphi,\quad f\in H^2(\mathbb{D}).$$

It is easy to see that $C_{\psi,\varphi}=M_\psi C_\varphi$, where M_ψ is the multiplication operator defined as

$$M_\psi f=\psi f,\quad f\in H^2(\mathbb{D}),$$

and C_φ is the composition operator defined by

$$C_\varphi f=f\circ\varphi,\quad f\in H^2(\mathbb{D}).$$

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By the Littlewood's subordination theorem, the composition operator C_φ is bounded on $H^2(\mathbb{D})$ (see [8]), but it is possible for $C_{\psi,\varphi}$ to be bounded with unbounded ψ (see [9]). In [7], the author gave the condition for $C_{\psi,\varphi}$ to be bounded and invertible. For the weighted composition operator $C_{\psi.\varphi}$ to be bounded and Fredholm, we have the following result.

Theorem 1.1 *Let ψ and φ be analytic functions on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then $C_{\psi,\varphi}$ is a bounded Fredholm operator on $H^2(\mathbb{D})$ if and only if ψ is bounded on $\mathbb{D}$ and bounded away from zero near the unit circle $\partial\mathbb{D}$, φ is an automorphism of $\mathbb{D}$.*

Theorem 1.2 *Let ψ and φ be analytic functions on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. If $C_{\psi,\varphi}$ is a bounded Fredholm operator on $H^2(\mathbb{D})$, then the index of $C_{\psi,\varphi}$ is the negative of the number of zeros of ψ in $\mathbb{D}$ counted according to their multiplicity.*

2. Proof of the main results

In the following, ψ and φ denote the analytic functions on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. It is easy to verify that $\psi \in H^2(\mathbb{D})$ if $C_{\psi,\varphi}$ is defined on $H^2(\mathbb{D})$.

Proposition 2.1 *Let $C_{\psi,\varphi}$ be a bounded Fredholm operator on $H^2(\mathbb{D})$. Then ψ is bounded away from the unit circle and φ is an inner function.*

Proof If $C_{\psi,\varphi}$ is bounded and Fredholm, then there exist a bounded operator T and a compact operator S such that

$$T(C_{\psi,\varphi})^* = I + S,$$

where I is the identity operator.

By $(C_{\psi,\varphi})^* K_w = \overline{\psi(w)}(C_\varphi)^* K_w = \overline{\psi(w)} K_{\varphi(w)}$ (see [1]), we have

$$\begin{aligned} \|T\|\ |\psi(w)|\ \frac{\|K_{\varphi(w)}\|}{\|K_w\|} &= \|T\|\ |\psi(w)|\ \|C_\varphi^* k_w\| \geq \|T(C_{\psi,\varphi})^* k_w\| \\ &\geq \|k_w\| - \|Sk_w\| = 1 - \|Sk_w\|, \end{aligned} \tag{1}$$

where $k_w = \frac{K_w}{\|K_w\|}$ is the normalization of reproducing kernel function K_w.

Since S is compact and k_w weakly converges to 0 as $|w| \to 1$, $\|Sk_w\| \to 0$ as $|w| \to 1$. It follows that there exists constant r, $0 < r < 1$, such that $\|Sk_w\| < \frac{1}{2}$ for all w with $r < |w| < 1$. Inequality (1) shows that

$$|\psi(w)| \geq \frac{1}{2\|T\|\ \|C_\varphi^* k_w\|}, \quad r < |w| < 1.$$

Note that C_φ is bounded, so $\|C_\varphi^* k_w\| \leq \|C_\varphi^*\|$ and hence

$$|\psi(w)| \geq m\ \big(= \frac{1}{2\|T\|\ \|C_\varphi^*\|}\big), \quad r < |w| < 1.$$

i.e., ψ is bounded away from the unit circle.

On the other hand, by inequality (1), we also have

$$\frac{\|T\||\psi(w)|}{\|K_w\|} \geq \frac{1}{2\|K_{\varphi(w)}\|}, \quad r < |w| < 1. \tag{2}$$

The following reasoning is the same as the proof of Lemma 2.0.3 in [7].

Since k_w weakly converges to 0 as $|w| \to 1$, $\langle \psi, k_w \rangle \to 0$ as $|w| \to 1$, that is

$$\frac{\psi(w)}{\|K_w\|} \to 0, \quad |w| \to 1.$$

It follows from (2) that $\|K_{\varphi(w)}\| = (\frac{1}{1-|\varphi(w)|^2})^{\frac{1}{2}} \to \infty$ and hence $|\varphi(w)| \to 1$ as $|w| \to 1$. i.e., φ is an inner function. □

Lemma 2.2 *If ψ is bounded away from the unit circle, then $\psi = BF$ with B a finite Blaschke product and $\frac{1}{F}$ bounded on $\mathbb{D}$.*

Proof Since ψ is bounded away from the unit circle, there exist constants m_1 $(m_1 > 0)$ and r $(0 < r < 1)$ such that

$$|\psi(w)| \geq m_1, \quad r \leq |w| < 1, \tag{3}$$

which implies that ψ has no zeros in the annulus $\{w | r \leq |w| < 1\}$.

Let B be the Blaschke products of the zeros of ψ. Then B is finite and $F = \frac{\psi}{B}$ has no zeros in $\mathbb{D}$. By (3), we have

$$|F(w)| \geq \frac{m_1}{|B(w)|} \geq m_1, \quad r \leq |w| < 1, \tag{4}$$

since $B(w)$ has no zeros in $\{w | r \leq |w| \leq 1\}$ and $|B(w)| \leq 1$.

Since F is continuous and has no zeros in $\{w | |w| \leq r\}$, there exists positive constant m_2 such that

$$|F(w)| \geq m_2, \quad |w| \leq r. \tag{5}$$

The boundedness of $\frac{1}{F}$ on $\mathbb{D}$ follows from (4) and (5). □

Lemma 2.3 *Let $C_{\psi,\varphi}$ be a bounded operator on $H^2(\mathbb{D})$, $\psi = BF$ with B an inner function. Then $C_{F,\varphi}$ is bounded.*

Proof Let M_B be the multiplication operator for the symbol B on $H^2(\mathbb{D})$. Then $C_{\psi,\varphi} = M_B C_{F,\varphi}$. Since B is an inner function, M_B is an isometry. For $f \in H^2(\mathbb{D})$,

$$\|C_{F.\varphi} f\| = \|C_{\psi,\varphi} f\|.$$

The boundedness of $C_{F,\varphi}$ follows from the boundedness of $C_{\psi,\varphi}$. □

Lemma 2.4 *Let F be an analytic function on $\mathbb{D}$ with zero free. If $C_{F,\varphi}$ is a bounded Fredholm operator on $H^2(\mathbb{D})$, then φ is univalent.*

Proof If $\varphi(a) = \varphi(b)$ for $a, b \in \mathbb{D}$, by a similar reasoning as Lemma 3.26 in [8], there exist infinite sets $\{a_n\}$ and $\{b_n\}$ in $\mathbb{D}$ which is disjoint such that $\varphi(a_n) = \varphi(b_n)$. Hence

$$(C_{F,\varphi})^* (\frac{K_{a_n}}{F(a_n)} - \frac{K_{b_n}}{F(b_n)}) = 0.$$

which contradicts that kernel of $(C_{F,\varphi})^*$ is finite dimensional. □

Corollary 2.5 *If $C_{\psi,\varphi}$ is a bounded Fredholm operator on $H^2(\mathbb{D})$, then φ is an automorphism of $\mathbb{D}$ and ψ is bounded.*

Proof By Proposition 2.1 and Lemma 2.2, $\psi = BF$ with B a finite Blaschke product and F zero free in $\mathbb{D}$. By Lemma 2.3, $C_{F,\varphi}$ is a bounded operator. Since $C_{\psi,\varphi} = M_B C_{F,\varphi}$ and M_B is a Fredholm operator, $C_{F,\varphi}$ is a Fredholm operator too. By Proposition 2.1 and Lemma 2.4, φ is a univalent inner function, and it follows from Corollary 3.28 in [8] that φ is an automorphism of $\mathbb{D}$.

Since $C_{\psi,\varphi}C_{\varphi^{-1}} = M_\psi$, M_ψ is a bounded multiplication operator on $H^2(\mathbb{D})$, which implies that ψ is bounded. □

Combining the results above, we give the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 If $C_{\psi,\varphi}$ is a bounded Fredholm operator on $H^2(\mathbb{D})$, by Proposition 2.1 and Corollary 2.5, ψ is bounded on $\mathbb{D}$ and bounded away from $\partial\mathbb{D}$, φ is an automorphism of $\mathbb{D}$.

On the other hand, if ψ is bounded on $\mathbb{D}$ and bounded away from $\partial\mathbb{D}$, then M_ψ is a Fredholm operator on $H^2(\mathbb{D})$ (see [10, Theorem 7.36]). Since φ is an automorphism of $\mathbb{D}$, C_φ is invertible. Hence $C_{\psi,\varphi} = M_\psi C_\varphi$ is a bounded Fredholm operator on $H^2(\mathbb{D})$. □

Proof of Theorem 1.2 If $C_{\psi,\varphi}$ is a bounded Fredholm operator on $H^2(\mathbb{D})$, by Theorem 1.1, we know that $C_{\psi,\varphi} = M_\psi C_\varphi$ with M_ψ a bounded Fredholm operator on $H^2(\mathbb{D})$ and C_φ a bounded invertible operator on $H^2(\mathbb{D})$. Hence

$$\text{Index } C_{\psi,\varphi} = \text{Index } M_\psi + \text{Index } C_\varphi = \text{Index } M_\psi.$$

It is well known that the index of M_ψ is the negative of number of zeros of ψ in $\mathbb{D}$ counted according to their multiplicity. □

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