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Positive Solutions of Singular Sturm-Liouville Boundary Value Problems with Positive Green Function

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Abstract The existence and multiplicity of positive solutions are studied for a singular Sturm-Liouville boundary value problem with positive Green function, where the nonlinearity may be super-strongly singular with respect to the space variable. By constructing suitable control functions, the a priori bound of solution is exactly estimated. By applying the Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type, several existence results are proved.

Keywords singular ordinary differential equation; Sturm-Liouville boundary value problem; positive solution; existence and multiplicity.

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1. Introduction

In this paper, we consider the existence and multiplicity of positive solutions for the nonlinear Sturm-Liouville boundary value problem

$$\text{(P)}\quad \begin{cases} (p(t)u'(t))' + h(t)f(t,u(t)) + g(t,u(t)) = 0, \quad 0 < t < 1, \\ au(0) - bp(0)u'(0) = 0, \quad cu(1) + dp(1)u'(1) = 0, \end{cases}$$

where $p : [0,1] \to (0,+\infty)$ is a continuous function, a, b, c, d are four nonnegative constants such that $da + ac + cb > 0$.

We need the following definitions. For other boundary value problems, the definitions are analogous.

Let $G(t,s)$ be the Green function of the problem (P). $G(t,s)$ is called positive if $\min_{0\le t,s\le 1} G(t,s) > 0$, nonnegative if $\min_{0\le t,s\le 1} G(t,s) \ge 0$. $g(t,u)$ is called super-strongly singular at $u = 0$ if $\lim_{u\to+0} u^k g(t,u) = +\infty$ for any $0 < t < 1$ and any positive integer k. $u^* \in C[0,1]$ is called positive solution of (P) if $u^*(t)$ satisfies (P) and $u^*(t) > 0$, $\forall 0 \le t \le 1$.

The problem (P) arises quite naturally in a variety of mathematical models. For example, the paper [4] considered its applications to the nonlinear diffusion theory generated by nonlinear sources. For the recent existence results of (P) (see [3, 5, 7, 9, 10, 13] and the references therein).

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However, all of these results are obtained for the problem (P) with nonnegative Green function $G(t,s)$.

It is well known that some periodic or Neumann boundary value problems have positive Green function. The positivity guarantees the existence of positive solutions when the nonlinearity is super-strongly singular at the space variable $u=0$ (see [6, 8, 11, 12, 15]).

When $bd>0$, the problem (P) has a positive Green function $G(t,s)$, see Section 2. Motivated by above-mentioned papers, the aim of this paper is to study the problem (P) under the following assumptions:

(H1) $b>0,\ d>0$.

(H2) $h:(0,1)\to[0,+\infty)$ is continuous and $0<\int_0^1 h(t)\mathrm{d}t<+\infty$.

(H3) $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.

(H4) $g:(0,1)\times(0,+\infty)\to[0,+\infty)$ is continuous.

(H5) For every pair of positive numbers $r_2>r_1>0$, there exists a nonnegative function $j_{r_1}^{r_2}\in C(0,1)\cap L^1[0,1]$ such that $g(t,u)\le j_{r_1}^{r_2}(t)$ for any $(t,u)\in(0,1)\times[\sigma r_1,r_2]$, where

$$\sigma=\min\Big\{\frac{b}{b+a\int_0^1\frac{\mathrm{d}t}{p(t)}},\frac{d}{d+c\int_0^1\frac{\mathrm{d}t}{p(t)}}\Big\}.$$

The assumption (H2) allows $h(t)$ to be singular at $t=0,\ t=1$. (H4) and (H5) show that $g(t,u)$ may be singular at $t=0,t=1$ for any $u\in[0,+\infty)$, and at $u=0$ for any $0<t<1$. Particularly, (H1) implies that $g(t,u)$ may be super-strongly singular at $u=0$, see Section 4.

This paper is organized as follows. In Section 2, we transfer the problem (P) into a Hammerstein integral equation by using the Green function $G(t,s)$. (H1)–(H5) will ensure the compactness of the associated integral operator (see Lemma 2.1). In Section 3, we construct two control functions for estimating the a priori bound of solution. By applying the Guo-Krasnosel'skii fixed point theorem of norm expansion-compression type, we establish three existence theorems concerned with one, two and three positive solutions. Finally, we give an example to demonstrate the main result.

If $bd=0$, then the Green function $G(t,s)$ is nonnegative. In such a case, $g(t,u)$ cannot be super-strongly singular at $u=0$, otherwise the associated integral operator may be noncompact. For the other singular boundary value problems with nonnegative Green function, we refer to [1, 2, 7, 14].

2. Preliminaries

Let $C[0,1]$ be the Banach space of all continuous functions on $[0,1]$ equipped with the norm $\|u\|=\max_{0\le t\le1}|u(t)|$.

Let $\rho=da+ac\int_0^1\frac{\mathrm{d}t}{p(t)}+cb$. Since $da+ac+cb>0$, one has $\rho>0$. Let

$$q(t)=\min\Big\{\frac{b+a\int_0^t\frac{\mathrm{d}s}{p(s)}}{b+a\int_0^1\frac{\mathrm{d}s}{p(s)}},\frac{d+c\int_t^1\frac{\mathrm{d}s}{p(s)}}{d+c\int_0^1\frac{\mathrm{d}s}{p(s)}}\Big\},\quad 0\le t\le1.$$

Then $q(t) > 0$, $\forall 0 \le t \le 1$ and $\sigma = \min_{0\le t\le 1} q(t)$. By (H1), $0 < \sigma < 1$. Let

$$K = \{u \in C[0,1] : u(t) \ge \sigma\|u\|,\ 0 \le t \le 1\}.$$

Then K is a cone of nonnegative functions in $C[0,1]$. Write

$$\Omega(r) = \{u \in K : \|u\| < r\}, \quad \partial\Omega(r) = \{u \in K : \|u\| = r\}.$$

Let $G(t,s)$ be the Green function of the homogeneous linear problem

$$\begin{cases} -(p(t)u'(t))' = 0, \quad 0 < t < 1, \\ au(0) - bp(0)u'(0) = 0, \quad cu(1) + dp(1)u'(1) = 0. \end{cases}$$

Then $G(t,s)$ has the precise expression

$$G(t,s) = \begin{cases} \frac{1}{\rho}(b + a\int_0^s \frac{\mathrm{d}\tau}{p(\tau)})(d + c\int_t^1 \frac{\mathrm{d}\tau}{p(\tau)}), \quad 0 \le s \le t \le 1, \\ \frac{1}{\rho}(b + a\int_0^t \frac{\mathrm{d}\tau}{p(\tau)})(d + c\int_s^1 \frac{\mathrm{d}\tau}{p(\tau)}), \quad 0 \le t \le s \le 1. \end{cases}$$

Clearly, $G : [0,1] \times [0,1] \to [0,1]$ is continuous and

$$\min_{0\le t,s\le 1} G(t,s) = G(1,0) = G(0,1) = \frac{bd}{\rho} > 0.$$

For $u \in K\backslash\{0\}$, define the operator T as follows

$$(Tu)(t) = \int_0^1 G(t,s)[h(s)f(s,u(s)) + g(s,u(s))]\mathrm{d}s, \quad 0 \le t \le 1.$$

It is not difficult to see that the operator $T : K\backslash\{0\} \to C[0,1]$ is well-defined if the assumptions (H1)–(H5) hold.

Lemma 2.1 *Suppose that (H1)–(H5) hold. Then* $T : \overline{\Omega(r_2)}\backslash\Omega(r_1) \to K$ *is compact for any* $r_2 > r_1 > 0$.

Proof Let $j_{r_1}^{r_2}(t)$ be as in (H5). For $n = 3, 4, \dots$, let

$$\xi_n(t) = \begin{cases} \min\{j_{r_1}^{r_2}(t), ntj_{r_1}^{r_2}(\frac{1}{n})\}, & 0 \le t \le \frac{1}{n}, \\ j_{r_1}^{r_2}(t), & \frac{1}{n} \le t \le \frac{n-1}{n}, \\ \min\{j_{r_1}^{r_2}(t), n(1-t)j_{r_1}^{r_2}(\frac{n-1}{n})\}, & \frac{n-1}{n} \le t \le 1. \end{cases}$$

Then $\xi_n \in C[0,1]$, $\xi_n(0) = \xi_n(1) = 0$ and

$$\int_0^1 [j_{r_1}^{r_2}(t) - \xi_n(t)]\mathrm{d}t \to 0, \quad n \to \infty.$$

Further, let

$$g_n(t,u) = \begin{cases} \min\{g(t,u), \xi_n(t)\}, & \sigma r_1 \le u < +\infty, \\ \min\{g(t,\sigma r_1), \xi_n(t)\}, & 0 \le u \le \sigma r_1. \end{cases}$$

Then $g_n : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

For $u \in K$, define the operator T_n as follows

$$(T_n u)(t) = \int_0^1 G(t,s)[h(s)f(s,u(s)) + g_n(s,u(s))]\mathrm{d}s, \quad 0 \le t \le 1.$$

Then $T_n : \overline{\Omega(r_2)}\backslash\Omega(r_1) \to C[0,1]$ is compact by the Arzela-Ascoli theorem [3, 13]. Moreover, by [3, Lemma 2.1], one has

$$q(t)\max_{0\le t\le 1} G(t,s) \le G(t,s) \le \max_{0\le t\le 1} G(t,s), \quad \forall 0 \le t, s \le 1.$$

So, for $0 \le t \le 1$ and $u \in \overline{\Omega(r_2)}\backslash\Omega(r_1)$,

$$\begin{aligned}(T_n u)(t) &\ge q(t)\int_0^1 \max_{0\le t\le 1} G(t,s)[h(s)f(s,u(s)) + g_n(s,u(s))]\mathrm{d}s\\ &\ge q(t)\max_{0\le t\le 1}\int_0^1 G(t,s)[h(s)f(s,u(s)) + g_n(s,u(s))]\mathrm{d}s\\ &= \|T_n u\| q(t).\end{aligned}$$

It follows that $T_n : \overline{\Omega(r_2)}\backslash\Omega(r_1) \to K$. Direct computations give that

$$\begin{aligned}\sup_{u\in\overline{\Omega(r_2)}\backslash\Omega(r_1)} \|Tu - T_n u\| &= \sup_{u\in\overline{\Omega(r_2)}\backslash\Omega(r_1)} \max_{0\le t\le 1}\int_0^1 G(t,s)[g(s,u(s)) - g_n(s,u(s))]\mathrm{d}s\\ &\le \max_{0\le t,s\le 1} G(t,s)\int_0^1 [j_{r_1}^{r_2}(s) - \xi_n(s)]\mathrm{d}s \to 0.\end{aligned}$$

This shows that the compact operators T_n uniformly converge to the operator T on $\overline{\Omega(r_2)}\backslash\Omega(r_1)$. Therefore, $T : \overline{\Omega(r_2)}\backslash\Omega(r_1) \to K$ is compact. $\Box$

In order that the paper is self-contained, we state the Guo-Krasnosel'skii fixed point theorem of norm expansion-compression type.

Lemma 2.2 *Let X be a Banach space, K be a cone in X, Ω_1, Ω_2 be two bounded open subsets of K satisfying $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $T : \overline{\Omega_2}\backslash\Omega_1 \to K$ is a compact operator such that either*

(1) $\|Tu\| \le \|u\|$, $u \in \partial\Omega_1$ and $\|Tu\| \ge \|u\|$, $u \in \partial\Omega_2$, or

(2) $\|Tu\| \ge \|u\|$, $u \in \partial\Omega_1$ and $\|Tu\| \le \|u\|$, $u \in \partial\Omega_2$.

Then T has a fixed point in $\overline{\Omega_2}\backslash\Omega_1$.

3. Main results

In this section, we use the following constants:

$$A = \max_{0\le t\le 1}\int_0^1 G(t,s)h(s)\mathrm{d}s, \quad B = \min_{0\le t\le 1}\int_0^1 G(t,s)h(s)\mathrm{d}s,$$
$$C = \max_{0\le t,s\le 1} G(t,s), \quad D = \min_{0\le t,s\le 1} G(t,s).$$

If $p(t), h(t), a, b, c, d$ are known, then A, B, C, D are computable. Moreover, for $r > 0$, we use the following two control functions:

$$A\varphi(r) + C\mu(r), \quad B\psi(r) + D\nu(r),$$

where

$$\varphi(r) = \max\{f(t,u) : (t,u) \in [0,1]\times[\sigma r, r]\},$$
$$\psi(r) = \min\{f(t,u) : (t,u) \in [0,1]\times[\sigma r, r]\},$$

$$\mu(r) = \int_0^1 \max\{g(t,u) : u \in [\sigma r, r]\}\mathrm{d}t,$$
$$\nu(r) = \int_0^1 \min\{g(t,u) : u \in [\sigma r, r]\}\mathrm{d}t.$$

If (H1)–(H5) hold, then $\varphi(r)$, $\psi(r)$, $\mu(r)$, $\nu(r)$ are nonnegative real numbers.

We obtain the following existence results.

Theorem 3.1 *Suppose that (H1)–(H5) hold and there exist two positive numbers $r_1 < r_2$ such that one of the following conditions is satisfied:*

(a1) $A\varphi(r_1) + C\mu(r_1) \le r_1$, $B\psi(r_2) + D\nu(r_2) \ge r_2$.

(a2) $B\psi(r_1) + D\nu(r_1) \ge r_1$, $A\varphi(r_2) + C\mu(r_2) \le r_2$.

Then the problem (P) has at least one positive solution $u^* \in K$ and $r_1 \le \|u^*\| \le r_2$.

Proof Without loss of generality, we only prove the case (a1).

If $u \in \partial\Omega(r_1)$, then $\|u\| = r_1$ and $\sigma r_1 \le u(t) \le r_1, \forall 0 \le t \le 1$. Thus, $\max_{0\le t\le 1} f(t,u(t)) \le \varphi(r_1)$ and $\int_0^1 g(t,u(t))\mathrm{d}t \le \mu(r_1)$. It follows that

$$\begin{aligned}
\|Tu\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)[h(s)f(s,u(s)) + g(s,u(s))]\mathrm{d}s \\
&\le \max_{0\le t\le 1}\int_0^1 G(t,s)h(s)f(s,u(s))\mathrm{d}s + \max_{0\le t\le 1}\int_0^1 G(t,s)g(s,u(s))\mathrm{d}s \\
&\le \varphi(r_1)\max_{0\le t\le 1}\int_0^1 G(t,s)h(s)\mathrm{d}s + \max_{0\le t,s\le 1} G(t,s)\int_0^1 g(s,u(s))\mathrm{d}s \\
&\le A\varphi(r_1) + C\mu(r_1) \le r_1 = \|u\|.
\end{aligned}$$

If $u \in \partial\Omega(r_2)$, then $\|u\| = r_2$ and $\sigma r_2 \le u(t) \le r_2, \forall 0 \le t \le 1$. Thus, $\min_{0\le t\le 1} f(t,u(t)) \ge \psi(r_2)$ and $\int_0^1 g(t,u(t))\mathrm{d}t \ge \nu(r_2)$. It follows that

$$\begin{aligned}
\|Tu\| &\ge \min_{0\le t\le 1}\int_0^1 G(t,s)[h(s)f(s,u(s)) + g(s,u(s))]\mathrm{d}s \\
&\ge \min_{0\le t\le 1}\int_0^1 G(t,s)h(s)f(s,u(s))\mathrm{d}s + \min_{0\le t\le 1}\int_0^1 G(t,s)g(s,u(s))\mathrm{d}s \\
&\ge \psi(r_2)\min_{0\le t\le 1}\int_0^1 G(t,s)h(s)\mathrm{d}s + \min_{0\le t,s\le 1} G(t,s)\int_0^1 g(s,u(s))\mathrm{d}s \\
&\ge B\psi(r_2) + D\nu(r_2) \ge r_2 = \|u\|.
\end{aligned}$$

By Lemmas 2.1 and 2.2, T has at least one fixed point $u^* \in \overline{\Omega(r_2)}\backslash\Omega(r_1)$. So, $r_1 \le \|u^*\| \le r_2$ and $u^*(t) \ge \sigma r_1 > 0$, $\forall 0 \le t \le 1$.

Direct verifications show that $u^*(t)$ satisfies (P). Therefore, $u^*(t)$ is a positive solution of the problem (P). □

Theorem 3.2 *Suppose that (H1)–(H5) hold and there exist three positive numbers $r_1 < r_2 < r_3$ such that one of the following conditions is satisfied:*

(b1) $A\varphi(r_1) + C\mu(r_1) \le r_1$, $B\psi(r_2) + D\nu(r_2) > r_2$, $A\varphi(r_3) + C\mu(r_3) \le r_3$.

(b2) $B\psi(r_1)+D\nu(r_1)\geq r_1$, $A\varphi(r_2)+C\mu(r_2)<r_2$, $B\psi(r_3)+D\nu(r_3)\geq r_3$.

Then the problem (P) has at least two positive solutions $u_1^*, u_2^*\in K$ *and* $r_1\leq \|u_1^*\|<r_2<\|u_2^*\|\leq r_3$.

Proof Let $\Phi(r)=A\varphi(r)+C\mu(r),\ \Psi(r)=B\psi(r)+D\nu(r)$. Then $\Phi,\Psi:(0,+\infty)\to[0,+\infty)$ are continuous by (H2)–(H5).

If (b1) holds, then there exist $\bar r_2\in(r_1,r_2), \tilde r_2\in(r_2,r_3)$ such that $\Psi(\bar r_2)\geq \bar r_2$, $\Psi(\tilde r_2)\geq\tilde r_2$. It follows that

$$A\varphi(r_1)+C\mu(r_1)\leq r_1,\quad B\psi(\bar r_2)+D\nu(\bar r_2)\geq \bar r_2;$$
$$B\psi(\tilde r_2)+D\nu(\tilde r_2)\geq\tilde r_2,\quad A\varphi(r_3)+C\mu(r_3)\leq r_3.$$

By Theorem 3.1, (P) has two positive solutions $u_1^*,u_2^*\in K$ and $r_1\leq\|u_1^*\|\leq\bar r_2<r_2<\tilde r_2\leq\|u_2^*\|\leq r_3$.

If (b2) holds, the proof is similar. $\square$

Theorem 3.3 *Suppose that (H1)–(H5) hold and there exist four positive numbers* $r_1<r_2<r_3<r_4$ *such that one of the following conditions is satisfied:*

(c1) $A\varphi(r_1)+C\mu(r_1)\leq r_1$, $B\psi(r_2)+D\nu(r_2)>r_2$, $A\varphi(r_3)+C\mu(r_3)<r_3$ *and* $B\psi(r_4)+D\nu(r_4)\geq r_4$.

(c2) $B\psi(r_1)+D\nu(r_1)\geq r_1$, $A\varphi(r_2)+C\mu(r_2)<r_2$, $B\psi(r_3)+D\nu(r_3)>r_3$ *and* $A\varphi(r_4)+C\mu(r_4)\leq r_4$.

Then the problem (P) has at least three positive solutions $u_1^*,u_2^*,u_3^*\in K$ *and* $r_1\leq\|u_1^*\|<r_2<\|u_2^*\|<r_3<\|u_3^*\|\leq r_4$.

Obviously, we can prove similar results for any positive integer k.

If $\lim_{u\to+0}\min_{0\leq t\leq1}g(t,u)=+\infty$, then Corollary 3.4 is very convenient.

Corollary 3.4 *Suppose that (H1)–(H5) hold and the following conditions are satisfied:*

(d1) There exist $\hat r>0$, $0\leq\theta<1$ *and a nonnegative function* $\gamma\in L^1[0,1]$ *such that* $g(t,u)\leq\gamma(t)u^\theta$, $\forall(t,u)\in[0,1]\times[\sigma\hat r,+\infty)$.

(d2) There exist $0\leq\alpha<\beta\leq1$ *such that* $\lim_{u\to+0}\min_{\alpha\leq t\leq\beta}g(t,u)>0$.

(d3) $\lim_{u\to+\infty}\max_{0\leq t\leq1}\frac{f(t,u)}{u}<A^{-1}$.

Then the problem (P) has at least one positive solution $u^*\in K$.

Proof By (d2), there exist $L>0$ and $\bar r>0$ such that

$$\max\{g(t,u):(t,u)\in[\alpha,\beta]\times(0,\bar r]\}\geq L.$$

Let $r_1=\min\{DL(\beta-\alpha),\bar r\}$. Then $r_1>0$ and

$$\max\{g(t,u):(t,u)\in[\alpha,\beta]\times[\sigma r_1,r_1]\}\geq L.$$

If $u\in\partial\Omega(r_1)$, then $\sigma r_1\leq u(t)\leq r_1$, $0\leq t\leq1$. Thus,

$$\nu(r_1)\geq\int_\alpha^\beta\min\{g(t,u):\sigma r_1\leq u\leq r_1\}\mathrm{d}t\geq L(\beta-\alpha).$$

It follows that

$$B\psi(r_1)+D\nu(r_1)\geq D\nu(r_1)\geq DL(\beta-\alpha)\geq r_1.$$

Let $\varepsilon=\frac{1}{3}[A^{-1}-\lim_{u\to+\infty}\max_{0\leq t\leq 1}\frac{f(t,u)}{u}]$. By (d3), then $\varepsilon>0$. So, there exists $r_2>0$ such that

$$\max\{\frac{f(t,u)}{u}:(t,u)\in[0,1]\times[r_2,+\infty)\}\leq A^{-1}-2\varepsilon.$$

Since $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous, one has

$$W=\max\{f(t,u):(t,u)\in[0,1]\times[0,r_2]\}<+\infty.$$

By (d1), then for any $r\geq\hat{r}$,

$$\max\{g(t,u):\sigma r\leq u\leq r\}\leq\gamma(t)r^{\theta},\quad\forall 0\leq t\leq 1.$$

It follows that

$$\lim_{r\to+\infty}\frac{\mu(r)}{r}\leq\lim_{r\to+\infty}\frac{1}{r^{1-\theta}}\int_0^1\gamma(t)\mathrm{d}t=0.$$

So, there exists $r_3>0$ such that $C\mu(r)<A\varepsilon r$, $\forall r\geq r_3$.

Choose $r_4=\max\{r_1+\hat{r},r_2,r_3,W\varepsilon^{-1}\}$. Then

$$\begin{aligned}\varphi(r_4)=&\max\{f(t,u):(t,u)\in[0,1]\times[0,r_4]\}\\ \leq&\max\{f(t,u):(t,u)\in[0,1]\times[0,r_2]\}+\max\{f(t,u):(t,u)\in[0,1]\times[r_2,r_4]\}\\ \leq& W+(A^{-1}-2\varepsilon)r_4<(A^{-1}-\varepsilon)r_4.\end{aligned}$$

It follows that

$$A\varphi(r_4)+C\mu(r_4)<A(A^{-1}-\varepsilon)r_4+A\varepsilon r_4=r_4.$$

By Theorem 3.1 (a2), (P) has at least one positive solution $u^*\in K$. $\square$

4. An example

Consider the following nonlinear Sturm-Liouville boundary value problem

$$\begin{cases}(e^{-t}u'(t))'+\frac{(1+\sin(u(t)))\sqrt{u(t)}}{\sqrt{t(1-t)}}+[2+t\arctan u(t)]^{\frac{1}{u(t)}}=0,\quad 0<t<1,\\ u(0)-u'(0)=0,\quad u(1)+\frac{1}{e}u'(1)=0.\end{cases}$$

Here, $a=b=c=d=1$, $p(t)=e^{-t}$, $h(t)=\frac{1}{\sqrt{t(1-t)}}$,

$$f(t,u)=f(u)=(1+\sin u)\sqrt{u},\quad g(t,u)=[2+t\arctan u]^{\frac{1}{u}}.$$

So, $h(t)$ is singular at $t=0$, $t=1$, and $g(t,u)$ is singular at $u=0$.

Obviously, the assumptions (H1)–(H5) are satisfied. Moreover,

$$\lim_{u\to+0}\min_{0\leq t\leq 1}g(t,u)\geq\lim_{u\to+0}2^{\frac{1}{u}}=+\infty,$$

$$\lim_{u\to+\infty}\max_{0\leq t\leq 1}\frac{f(t,u)}{u}=\lim_{u\to+\infty}\frac{f(u)}{u}\leq\lim_{u\to+\infty}\frac{2\sqrt{u}}{u}=0.$$

For $u \geq 2$ and $0 < t < 1$, one has

$$g(t,u) \leq [2+tu]^{\frac{1}{u}} \leq [2+tu]^{\frac{1}{2}} \leq [u+tu]^{\frac{1}{2}} = \sqrt{1+t}u^{\frac{1}{2}}.$$

By Corollary 3.4, the problem has a positive solution $u^* \in K$. Since for any $0 \leq t \leq 1$ and any k,

$$\lim_{u\to+0} u^k g(t,u) \geq \lim_{u\to+0} u^k 2^{\frac{1}{u}} = +\infty,$$

the function $g(t,u)$ is super-strongly singular at $u=0$.

The conclusion cannot be derived from the existing literature, for example, from [5, 7, 9, 10], because of the super-strong singularities of $g(t,u)$ at $u=0$.

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