Journal of Mathematical Research with Applications
Nov., 2013, Vol. 33, No. 6, pp. 666–672
DOI:10.3770/j.issn:2095-2651.2013.06.003
Http://jmre.dlut.edu.cn

On Sums of Powers of Odd Integers

Songbai GUO[1,2], **Youjian SHEN**[1,*]

1. *School of Mathematics and Statistics, Hainan Normal University, Hainan* 571158*, P. R. China;*
2. *School of Mathematics and Physics, University of Science and Technology Beijing, Beijing* 100083*, P. R. China*

Abstract In this paper, by using superposition method, we aim to show that $\sum_{i=1}^{n}(2i-1)^{2k-1}$ is the product of n^2 and a rational polynomial in n^2 with degree $k-1$, and that $\sum_{i=1}^{n}(2i-1)^{2k}$ is the product of $n(2n-1)(2n+1)$ and a rational polynomial in $(2n-1)(2n+1)$ with degree $k-1$. Moreover, recurrence formulas to compute the coefficients of the corresponding rational polynomials are also obtained.

Keywords odd number; sums of powers; binomial theorem; superposition method.

MR(2010) Subject Classification 11A25; 11B57; 11B75

1. Introduction

The study on sums of powers of integers $\sum_{i=1}^{n} i^m$ has a long history. In recent years, it has been extensively studied and many good results have been obtained [2–6, 8, 9, 11–13, 16], some of which were improved in our former paper [10] along with [3–6, 16]. It is well-known that one of the unusual and effective methods used to express the sums of powers of integers used the so-called Bernoulli numbers, while Edwards [8] traced the knowledge of the formula to Johann Faulhaber who showed us how to obtain the coefficients by matrix inversion. By submitting problem E3204 to the Math Monthly [9], Gessel responded to the same stimulus on a bivariate generating function for Faulhaber's coefficients. Faulhaber's work, including more generally r-fold sums of powers, was nicely exposed by Knuth [11]. For the study of polynomial relations between sums of powers functions, the readers can refer to Beardon [2]. Recall that in [10] we have proved that

$$\sum_{i=1}^{n} i^{2k+1} = n^2(n+1)^2 \sum_{i=1}^{k} a_i n^{k-i}(n+1)^{k-i}, \quad k = 1, 2, \ldots, \tag{1.1}$$

where

$$\begin{cases} a_1 = \dfrac{1}{2(k+1)}, \\ a_i = -\dfrac{\sum_{r=1}^{i-1} a_r \binom{k-r+2}{2i-2r+1}}{k-i+2}, \quad i = 2, 3, \ldots, k. \end{cases} \tag{1.2}$$

Received September 28, 2012; Accepted April 18, 2013
Supported by the Natural Science Foundation of Hainan Province (Grant No. 111004).
* Corresponding author
E-mail address: gsb8369@163.com (Songbai GUO); syjian@hainnu.edu.cn (Youjian SHEN)

In fact, a_i $(i = 2, 3, \ldots, k)$ can also be rewritten as

$$\begin{cases} a_i = -\dfrac{\sum_{r=1}^{i-1} a_r \binom{k-r+2}{2i-2r+1}}{k-i+2}, & i \leqslant [\frac{k+2}{2}], \\ a_i = -\dfrac{\sum_{r=2i-k-1}^{i-1} a_r \binom{k-r+2}{2i-2r+1}}{k-i+2}, & i > [\frac{k+2}{2}]. \end{cases}$$

Therefore, we have

$$\begin{cases} a_2 = -\dfrac{k-1}{2 \cdot 3!}, \\ a_3 = \dfrac{k(k-2)(7k-1)}{3!5!}, \\ a_4 = -\dfrac{k(k-1)(k-3)}{3!7!}(31k^2 - 27k - 10), \\ a_5 = \dfrac{3k(k-1)(k-2)(k-4)}{10!}(127k^3 - 310k^2 + 37k + 90), \\ a_6 = -\dfrac{k(k-1)(k-2)(k-3)(k-5)}{3!11!}(2555k^4 - 12674k^3 + 14161k^2 \\ \qquad + 2486k - 3864), \\ a_7 = \dfrac{k(k-1)(k-2)(k-3)(k-4)(k-6)}{15!}(1414477k^5 - 11974437k^4 + \\ \qquad 31092673k^3 - 22432587k^2 - 7706534k + 6399960), \\ \cdots \\ a_{k-1} = -\dfrac{4a_{k-2} + a_{k-3}}{3}, \\ a_k = -\dfrac{a_{k-1}}{2}. \end{cases}$$

In [10], we have also proved that

$$\sum_{i=1}^{n} i^{2k} = n(n+1)(2n+1) \sum_{i=1}^{k} b_i n^{k-i} (n+1)^{k-i}, \quad k = 1, 2, \ldots, \tag{1.3}$$

where

$$\begin{cases} b_1 = \dfrac{1}{(4k+2)}, \\ b_i = -\displaystyle\sum_{r=1}^{i-1} \frac{b_r \binom{k-r+2}{2i-2r+1}}{k-r+2}, \quad i = 2, 3, \ldots, k, \end{cases} \tag{1.4}$$

and b_i $(i = 2, 3, \ldots, k)$ can be rewritten as

$$\begin{cases} b_i = -\displaystyle\sum_{r=1}^{i-1} \frac{b_r \binom{k-r+2}{2i-2r+1}}{k-r+2}, & i \leqslant [\frac{k+2}{2}], \\ b_i = -\displaystyle\sum_{r=2i-k-1}^{i-1} \frac{b_r \binom{k-r+2}{2i-2r+1}}{k-r+2}, & i > [\frac{k+2}{2}]. \end{cases}$$

However, to the best of our knowledge, the results concerning sums of powers of odd integers

are very few [1, 14, 15]. In this case, it follows straightforward that

$$\sum_{i=1}^{n}(2i-1)^m=\sum_{i=1}^{2n}i^m-2^m\sum_{i=1}^{n}i^m. \tag{1.5}$$

Thus, using (1.1), (1.2) and (1.5), we obtain that,

$$\begin{cases}\displaystyle\sum_{i=1}^{n}(2i-1)=n^2,\\ \displaystyle\sum_{i=1}^{n}(2i-1)^3=n^2(2n^2-1),\\ \displaystyle\sum_{i=1}^{n}(2i-1)^5=\frac{n^2(16n^4-20n^2+7)}{3}.\end{cases} \tag{1.6}$$

In an analogous way, (1.3) together with (1.4) and (1.5) yields that,

$$\begin{cases}\displaystyle\sum_{i=1}^{n}(2i-1)^2=\frac{n(2n-1)(2n+1)}{3},\\ \displaystyle\sum_{i=1}^{n}(2i-1)^4=n(2n-1)(2n+1)[\frac{(2n-1)(2n+1)}{5}-\frac{4}{15}],\\ \displaystyle\sum_{i=1}^{n}(2i-1)^6=n(2n-1)(2n+1)[\frac{(2n-1)^2(2n+1)^2}{7}-\frac{4(2n-1)(2n+1)}{7}+\frac{16}{21}].\end{cases} \tag{1.7}$$

The relations in (1.6) and (1.7) motive us to conjecture that for $k=1,2,\ldots,$

$$\sum_{i=1}^{n}(2i-1)^{2k-1}=n^2\sum_{i=1}^{k}c_in^{2k-2i}, \tag{1.8}$$

$$\sum_{i=1}^{n}(2i-1)^{2k}=n(2n-1)(2n+1)\sum_{i=1}^{k}d_i(2n-1)^{k-i}(2n+1)^{k-i}, \tag{1.9}$$

where c_i and d_i $(i=1,2,\ldots,k)$ are undetermined constants.

In fact, $\sum_{i=1}^{n}(2i-1)^{2k-1}$ is a rational polynomial in n^2 from Faulhaber's theorem on sums of odd powers [1, 7, 11]. However, no explicit calculation formula on $\sum_{i=1}^{n}(2i-1)^{2k-1}$ has been given. In this paper, we aim to establish two concise calculation formulae on sums of powers of odd integers by using superposition method. In other words, we will show that our conjectures (1.8) and (1.9) are indeed valid. In addition, we obtain recurrence formulas to compute the coefficients of rational polynomials in formulas above. Furthermore, we also give the relations among a_i, b_i, c_i and d_i $(i=1,2,\ldots,k)$. Finally, we put forward a conjecture on signs of the coefficients in formulas (1.8) and (1.9).

2. The summation theorems

Theorem 2.1 *For any $n \in N^+$, (1.8) holds if and only if*

$$\begin{cases} c_1 = \dfrac{4^{k-1}}{k}, \\ c_i = -\dfrac{\sum\limits_{r=1}^{i-1} 4^r c_r \binom{2k-2r+2}{2i-2r+1}}{2^{2i+1}(k-i+1)}, \quad i = 2, 3, \ldots, k. \end{cases} \tag{2.1}$$

Theorem 2.2 *Let $P(n, r) := n(n-1)\cdots(n-r+1)(n, r \in N^+, r \leqslant n)$ and $\alpha_1 = 2$. Then*

$$c_i = -4^{k-i} P(2k-1, 2i-3)\alpha_i, \quad i = 2, 3, \ldots, k, \tag{2.2}$$

where $\alpha_2 = \frac{1}{3}$, $\alpha_i = \frac{\alpha_1}{(2i-1)!} - \frac{\alpha_2}{(2i-3)!} - \cdots - \frac{\alpha_{i-1}}{3!}$ $(i = 3, 4, \ldots, k)$.

Theorem 2.3 *For any $n \in N^+$, (1.9) holds if and only if*

$$\begin{cases} d_1 = \dfrac{1}{2k+1}, \\ d_i = -4^i \sum\limits_{r=1}^{i-1} \dfrac{d_r \binom{k-r+2}{2i-2r+1}}{4^r(k-r+2)}, \quad i = 2, 3, \ldots, k, \end{cases} \tag{2.3}$$

and d_i $(i = 2, 3, \ldots, k)$ can also be rewritten as

$$\begin{cases} d_i = -4^i \sum\limits_{r=1}^{i-1} \frac{d_r \binom{k-r+2}{2i-2r+1}}{4^r(k-r+2)}, & i \leqslant [\frac{k+2}{2}], \\ d_i = -4^i \sum\limits_{r=2i-k-1}^{i-1} \frac{d_r \binom{k-r+2}{2i-2r+1}}{4^r(k-r+2)}, & i > [\frac{k+2}{2}]. \end{cases}$$

From (1.2) and (1.4) we get immediately the following corollary.

Corollary 2.1 *It holds that $b_i = \frac{k-i+2}{2k+1} a_i$ $(i = 1, 2, \ldots, k)$, where a_i and b_i are given by (1.2) and (1.4), respectively.*

From which and (1.4), (2.3) we have the following corollary.

Corollary 2.2 *It holds that $d_i = 2^{2i-1} b_i = 2^{2i-1} \frac{k-i+2}{2k+1} a_i$ $(i = 1, 2, \ldots, k)$, where a_i, b_i and d_i are as presented in (1.2), (1.4) and (2.3), respectively.*

To end this section, we put forward the following conjecture on the signs of the coefficients c_i and d_i in Theorems 2.1 and 2.3.

Conjecture It holds that $\operatorname{sgn} c_i = (-1)^{i-1}$, $\operatorname{sgn} d_i = (-1)^{i-1}$, $i = 1, 2, \ldots, k$.

3. Proofs of the summation theorems

In this section, we present the proofs of our results stated in the above section, by using superposition method.

Proof of Theorem 2.1 Let $b_m = (2m-1)^{2k-1}$ and $s_0 = 0$, $s_m = \sum_{i=1}^{k} c_i m^{2k-2i+2}$ $(m = 1, 2, \ldots, n)$. Clearly, (1.8) holds if and only if $b_m = s_m - s_{m-1}$. On the other hand, since

$$c_i m^{2k-2i+2} - c_i (m-1)^{2k-2i+2} = \frac{c_i}{2^{2k-2i+2}} [(2m)^{2k-2i+2} - (2m-2)^{2k-2i+2}]$$

$$= \frac{c_i}{2^{2k-2i+2}}[(2m-1+1)^{2k-2i+2} - (2m-1-1)^{2k-2i+2}]$$
$$= \frac{c_i}{2^{2k-2i+1}} \sum_{r=1}^{k-i+1} \binom{2k-2i+2}{2r-1}(2m-1)^{2k-2i-2r+3},$$

we have

$$s_m - s_{m-1} = \sum_{i=1}^{k} \frac{c_i}{2^{2k-2i+1}} \sum_{r=1}^{k-i+1} \binom{2k-2i+2}{2r-1}(2m-1)^{2k-2i-2r+3}$$
$$= \sum_{i=1}^{k}\sum_{r=1}^{i} \frac{c_r}{2^{2k-2r+1}} \binom{2k-2r+2}{2i-2r+1}(2m-1)^{2k-2i+1}.$$

Hence, $s_m - s_{m-1} = (2m-1)^{2k-1}$ holds if and only if c_i $(i = 1, 2, \ldots, k)$ satisfy the following system of linear equations.

$$\begin{cases} c_1\binom{2k}{1} = 2^{2k-1}, \\ c_1\binom{2k}{3} + c_2 2^2 \binom{2k-2}{1} = 0, \\ \cdots \\ c_1\binom{2k}{2i-1} + c_2 2^2 \binom{2k-2}{2i-3} + \cdots + c_i 2^{2i-2}\binom{2k-2i+2}{1} = 0, \\ \cdots \\ c_1\binom{2k}{2k-1} + c_2 2^2 \binom{2k-2}{2k-3} + \cdots + c_k 2^{2k-2}\binom{2}{1} = 0. \end{cases} \tag{3.1}$$

The equivalence between (2.1) and (3.1) follows from direct calculation. The proof is completed. □

Proof of Theorem 2.2 We derive the result by using the second induction method.

From (2.1), $c_2 = \frac{-4^{k-2}(2k-1)}{3}$, so that for $i = 2$, (2.2) is true.

Assume that (2.2) is true for $i \leqslant m$ $(2 \leqslant m < k)$. Then from (2.1) we have

$$c_{m+1} = -\frac{\sum_{r=1}^{m} 4^r c_r \binom{2k-2r+2}{2m-2r+3}}{2^{2m+3}(k-m)}$$
$$= -\frac{4c_1\binom{2k}{2m+1}}{2^{2m+3}(k-m)} + \frac{\sum_{r=2}^{m} 4^k P(2k-1, 2r-3)\alpha_r \binom{2k-2r+2}{2m-2r+3}}{2^{2m+3}(k-m)}$$
$$= -\frac{4^{k-m-1}P(2k-1, 2m-1)\alpha_1}{(2m+1)!} + \sum_{r=2}^{m} \frac{4^{k-m-1}P(2k-1, 2m-1)\alpha_r}{(2m-2r+3)!}$$
$$= -4^{k-m-1}P(2k-1, 2m-1)\alpha_{m+1},$$

hence (2.2) is true for $i = m+1$. The proof is completed. □

Proof of Theorem 2.3 Let $b_m = (2m-1)^{2k}$ and $s_0 = 0, s_m = \sum_{i=1}^{k} d_i m(2m-1)^{k-i+1}(2m+1)^{k-i+1}$ $(m = 1, 2, \ldots, n)$. Clearly, (1.9) holds if and only if $b_m = s_m - s_{m-1}$. For simplicity, we let $p = k - i + 1$, then it holds that

$$d_i m(2m-1)^p(2m+1)^p - d_i(m-1)(2m-3)^p(2m-1)^p$$
$$= \frac{d_i}{2}(2m-1)^p[2m(2m+1)^p - (2m-2)(2m-3)^p]$$
$$= \frac{d_i}{2}(2m-1)^p[(2m-1+1)(2m-1+2)^p - (2m-1-1)(2m-1-2)^p]$$

$$
\begin{aligned}
&= \frac{d_i}{2}(2m-1)^p\{(2m-1)[(2m-1+2)^p-(2m-1-2)^p]+\\
&\quad [(2m-1+2)^p+(2m-1-2)^p]\}\\
&= d_i\sum_{r=1}^{[\frac{p}{2}]+1} 2^{2r-2}(2\binom{p}{2r-1}+\binom{p}{2r-2})(2m-1)^{2p-2r+2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
s_m - s_{m-1} &= \sum_{i=1}^{k} d_i \sum_{r=1}^{[\frac{p}{2}]+1} 2^{2r-2}(2\binom{p}{2r-1}+\binom{p}{2r-2})(2m-1)^{2p-2r+2}\\
&= \sum_{i=1}^{k}\sum_{r=1}^{i} d_r 2^{2i-2r}(2\binom{k-r+1}{2i-2r+1}+\binom{k-r+1}{2i-2r})(2m-1)^{2k-2i+2}\\
&= \sum_{i=1}^{[\frac{k+1}{2}]}\sum_{r=1}^{i} d_r 2^{2i-2r}(2\binom{k-r+1}{2i-2r+1}+\binom{k-r+1}{2i-2r})(2m-1)^{2k-2i+2}+\\
&\quad \sum_{i=[\frac{k+1}{2}]+1}^{k}[\sum_{r=2i-k}^{i} d_r 2^{2i-2r}(2\binom{k-r+1}{2i-2r+1}+\binom{k-r+1}{2i-2r})+\\
&\quad d_{2i-k-1}2^{2k-2i+2}](2m-1)^{2k-2i+2}\\
&= \sum_{p=[\frac{k}{2}]+1}^{k}\sum_{r=1}^{k-p+1} d_r 2^{2k-2p-2r+2}(2\binom{k-r+1}{2k-2p-2r+3}+\binom{k-r+1}{2k-2p-2r+2})(2m-1)^{2p}+\\
&\quad \sum_{p=1}^{[\frac{k}{2}]}[\sum_{r=k-2p+2}^{k-p+1} d_r 2^{2k-2p-2r+2}(2\binom{k-r+1}{2k-2p-2r+3}+\binom{k-r+1}{2k-2p-2r+2})+\\
&\quad d_{k-2p+1}2^{2p}](2m-1)^{2p}\\
&= \sum_{p=[\frac{k}{2}]+1}^{k}\sum_{l=p}^{k} d_{k-l+1}2^{2l-2p}(2\binom{l}{2l-2p+1}+\binom{l}{2l-2p})(2m-1)^{2p}\ (l=k-r+1)+\\
&\quad \sum_{p=1}^{[\frac{k}{2}]}[\sum_{l=p}^{2p-1} d_{k-l+1}2^{2l-2p}(2\binom{l}{2l-2p+1}+\binom{l}{2l-2p})+d_{k-2p+1}2^{2p}](2m-1)^{2p}.
\end{aligned}
$$

Hence, $s_m - s_{m-1} = (2m-1)^{2k}$ holds if and only if d_i $(i = 1, 2, \ldots, k)$ satisfy the following system of linear equations.

$$
\begin{cases}
d_1(2\binom{k}{1}+1) = 1,\\
d_1 2^2(2\binom{k}{3}+\binom{k}{2}) + d_2(2\binom{k-1}{1}+1) = 0,\\
\cdots\\
d_1 2^{2i-2}(2\binom{k}{2i-1}+\binom{k}{2i-2}) + d_2 2^{2i-4}(2\binom{k-1}{2i-3}+\binom{k-1}{2i-4}) + \cdots + d_i(2\binom{k-i+1}{1}+1) = 0,\\
\cdots\\
d_{k-1}2^2\binom{2}{2} + d_k(2\binom{1}{1}+1) = 0.
\end{cases}
\tag{3.2}
$$

Along the same line as in Theorem 2.1, we can show the equivalence between (2.3) and (3.2). The proof is completed. □

Acknowledgements The authors would like to thank Prof. Zuohuan ZHENG and Dr. Yang GAO of Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China, for their helpful suggestions during the preparation of the paper.

References

[1] A. BAZSÓ, Á. PINTÉR, H. M. SRIVASTAVA. *A refinement of Faulhaber's theorem concerning sums of powers of natural numbers*. Appl. Math. Lett., 2012, **25**(3): 486–489.
[2] A. F. BEARDON. *Sums of powers of integers*. Amer. Math. Monthly, 1996, **103**(3): 201–213.
[3] Jingrun CHEN, Jianyu LI. *A further study on sums of powers of natural numbers*. Chinese Sci. Bull., 1985, **30**(17): 1281–1284. (in Chinese)
[4] Jingrun CHEN, Jianyu LI. *The generic property on sums of powers of natural numbers*. J. Math. Res. Exposition, 1986, **6**(1): 43–50. (in Chinese)
[5] Jingrun CHEN, Jianyu LI. *On the sums of powers of natural numbers*. Chinese Quart. J. Math., 1987, **2**(1): 1–17.
[6] Ruiqing CHEN. *Some new results on problem of sums of powers*. Math. Practice Theory, 1994, **26**(1): 66–69. (in Chinese)
[7] W. Y. C. CHEN, A. M. FU, I. F. ZHANG. *Faulhaber's theorem on power sums*. Discrete Math., 2009, **309**(10): 2974–2981.
[8] A. W. F. EDWARDS. *A quick route to sums of powers*. Amer. Math. Monthly, 1986, **93**(6): 451–455.
[9] I. GESSEL, University of South Alabama Problem Group. *A formula for power sums*. Amer. Math. Monthly, 1988, **95**(10): 961–962.
[10] Songbai GUO, Youjian SHEN. *The formula on sums of powers of integers*. Studies in College Mathematics, 2010, **13**(1): 61–63. (in Chinese)
[11] D. E. KNUTH. *Johann Faulhaber and sums of powers*. Math. Comput., 1993, **61**(203): 277–294.
[12] Qinying LIU, Jianjin LUO. *Power sums of integer: a combinatorial model*. J. Zhengzhou Univ. Nat. Sci. Ed., 2003, **35**(2): 4–8.
[13] G. MACKIW. *A combinatorial approach to sums of integer powers*. Math. Mag., 2000, **73**(1): 44–46.
[14] C. S. RYOO. *On the sums of powers of consecutive odd integers*. Far East J. Math. Sci. (FJMS), 2011, **48**(1): 55–61.
[15] C. S. RYOO. *On the alternating sums of powers of consecutive odd integers*. J. Comput. Anal. Appl., 2011, **13**(6): 1019–1024.
[16] S. A. SHIRALI. *On sums of powers of integers*. Resonance, 2007, **12**(7): 27–43.