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Classification of 8-Dimensional Nilpotent Lie Algebras with 4-Dimensional Center

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Abstract In this paper, we give a complete classification of eight dimensional nilpotent Lie algebras with four-dimensional center by using the method of Skjelbred and Sund.

Keywords nilpotent Lie algebra; automorphism group; central extension.

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1. Introduction

It is well known that the classification of Lie algebras is a classical problem. Because of the Levi's Theorem, the question can be reduced to the classification of semi-simple and solvable Lie algebras. The classification of semi-simple Lie algebras has already been solved by Cartan in [1] over complex field and Gantmacher in [2] over the field of real numbers. As to the classification of solvable Lie algebras, it has relationship with that of the classification of nilpotent Lie algebras. However the classification of nilpotent Lie algebras is still a difficult problem. Many mathematic scholars are devoted to the study of the classification. In 1982, Santharoubane established a link between the nilpotent Lie algebras and Kac-Moody Lie algebras, and solved the classification of nilpotent Lie algebras of maximal rank in [3]. In 1998, Favre and the author of [3] determined the number of isomorphism classes of nilpotent Lie algebras of maximal rank and of type A_l, B_l, C_l, D_l in [4]. These works have contributed a lot to the classification of nilpotent Lie algebras. Particularly, some people are interested in investigating the classifications of nilpotent Lie algebras of small dimension. Csaba Schneider gave the number of some small-dimensional nilpotent Lie algebra over field F with small prime characters by using the computer in [5]. The classifications of nilpotent Lie algebras of dimension less than 6 have been solved by Morozov in [6]. Recently the classification of 6-dimensional nilpotent Lie algebras over all fields has been got by Cicalò, de Graaf and Schneider in [7]. The classification of 7-dimensional nilpotent Lie algebras over complex and real field has been worked out by Seeley and Gong in [8, 9]. Tsagas, Kobotis and Koukouvinos gave the classification of 9-dimensional nilpotent Lie algebras whose maximum abelian ideal is of dimension seven by using the computer in [10]. So far, the classification of 8-dimensional nilpotent Lie algebras has no complete result. In [11], the

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authors got the classification of 2-step nilpotent Lie algebras of dimension 8 with 2-dimensional center over the complex field. In [12], authors finished the classification of 2-step indecomposable nilpotent Lie algebras of dimension 8 with 4-dimensional center over the complex field. In this paper we not only give the complete classification of 2-step 8-dimensional nilpotent Lie algebras of 4-dimensional center, but also settle classifications of 3, 4-step nilpotent Lie algebras of eight dimension with 4-dimensional center over the complex field $\mathbb{C}$. In short, we solve the classification of eight dimensional nilpotent Lie algebras with 4-dimensional center.

In this paper all algebras and vector spaces are over the complex field $\mathbb{C}$.

2. Preliminaries

In this section, we recall some elementary facts about nilpotent Lie algebras mainly from the references [7] and [13].

Definition 2.1 ([14]) *Let L be a Lie algebra and $L^1 = L, L^{k+1} = [L^k, L], k \in \mathbb{Z}^+$. If there exists k such that $L^k = 0$, then L is called nilpotent. The largest value of k such that $L^k \neq 0, L^{k+1} = 0$ is called the degree of nilpotency and now we call L k-step nilpotent Lie algebra.*

Definition 2.2 ([15]) *A Lie algebra N is called a descendant of the Lie algebra L if $N/C(N) \cong L$ and $C(N) \leq N^2$ where $C(N)$ is the center of N. If $\dim C(N) = t$, then N is said to be step-t descendant.*

A descendant of nilpotent Lie algebra is nilpotent. On the contrary, if N is a nilpotent Lie algebra over a complex field $\mathbb{C}$, then N is either a descendant of a smaller-dimensional nilpotent Lie algebra or $N = N_1 \oplus \mathbb{C}$ where N_1 is an ideal of N and $\mathbb{C}$ is viewed as a one-dimensional Lie algebra. In order to make the classification of the 8-dimensional nilpotent Lie algebras we need find the isomorphism types of the descendants of the nilpotent Lie algebras with dimension at most 7. So in this paper, we need calculate step-4 descendants of the 4-dimensional nilpotent Lie algebras L and the classification of 7-dimensional nilpotent Lie algebras with 3-dimensional center.

Definition 2.3 *Let L be a Lie algebra and V a vector space over the complex field $\mathbb{C}$. The skew- symmetric bilinear map $\tau : L \times L \longrightarrow V$ with*

$$\tau([x_1, x_2], x_3) + \tau([x_3, x_2], x_1) + \tau([x_2, x_3], x_1) = 0 \text{ for all } x_1, x_2, x_3 \in L$$

is called cocycle. Let $Z^2(L, V)$ denote the set of all above τ. We define, for a linear map $\nu : L \to V$, a skew-symmetric map $\eta_\nu : L \times L \to V$ as $\eta_\nu(x, y) = \nu([x, y])$. We write the set $\{\eta_\nu | \nu : L \to V\} = B^2(L, V)$ and call the elements of $B^2(L, V)$ coboundaries. It is easy to check that $B^2(L, V)$ is a subspace of $Z^2(L, V)$. Let $H^2(L, V) = Z^2(L, V)/B^2(L, V)$. We call $H^2(L, V)$ the second cohomology.

Let L be a Lie algebra and V a vector space over the complex field $\mathbb{C}$. For $\tau \in Z^2(L, V)$, let $L_\tau = L \oplus V$. The Lie bracket of L_τ is $[x + v, y + w]_\tau = [x, y]_L + \tau(x, y)$, where $x, y \in L, v, w \in V$, $[x, y]_L$ is the Lie bracket on L. Then L_τ is a Lie algebra and is the central extension of L by

τ. We know that $L_{\tau_1} \cong L_{\tau_2}$ if and only if $\tau_1 - \tau_2 \in B^2(L, V)$. So the isomorphism type of L_τ depends on the element $\tau + B^2(L, V)$ of $H^2(L, V)$.

Let N be a Lie algebra with $C(N) \neq 0$. Set $V = C(N)$ and $L = N/C(N)$. Let $\pi : N \to L$ be the projection map. Choose an injective linear map $\rho : L \to N$ such that $\pi(\rho(x)) = x$ for all $x \in L$. Define $\tau : L \times L \longrightarrow V$ by $\tau(x, y) = [\rho(x), \rho(y)] - \rho([x, y])$. Then τ is a cocycle such that $N \cong L_\tau$. Though τ depends on the choice of ρ, the coset $\tau + B^2(L, B)$ is independent of ρ. So the central extension N of L determines elements of $H^2(L, V)$.

Definition 2.4 *For $\tau \in Z^2(L, V)$, let $\tau^\perp$ denote the radical of τ, i.e.,*

$$\tau^\perp = \{x \in L | \tau(x, y) = 0, \ \forall y \in L\}.$$

Then $C(L_\tau) = (\tau^\perp \cap C(L)) + V$.

Let $v_1, v_2, \ldots, v_s$ be a basis of V. τ is an element of $Z^2(L, V)$. Then $\tau(x, y) = \sum_{i=1}^{s} \tau_i(x, y)v_i$, where $\tau_i \in Z^2(L, \mathbb{C})$. And if $\tau \in B^2(L, V)$, then all $\tau_i \in B^2(L, \mathbb{C})$. At the moment we know $C(L_\tau) = (\tau^\perp \cap C(L)) + V$ and $\tau^\perp = \cap_{\vartheta \in (\tau_1, \ldots, \tau_s)} \vartheta^\perp = \tau_1^\perp \cap \cdots \cap \tau_s^\perp$. Therefore, $C(L_\tau) = (\tau_1^\perp \cap \cdots \cap \tau_s^\perp \cap C(L)) + V$.

The automorphism group $\mathrm{Aut}(L)$ acts on $Z^2(L, V)$ by $\psi\tau(x, y) = \tau(\psi(x), \psi(y)), \psi \in \mathrm{Aut}(L)$. It is easy to know that $\vartheta \in B^2(L, V)$ implies $\psi\vartheta \in B^2(L, V)$. So $\mathrm{Aut}(L)$ acts on $H^2(L, V)$.

Theorem 2.5 *Let L be a Lie algebra, V be a vector space with fixed basis $v_1, \ldots, v_s$ over a complex field $\mathbb{C}$ and $\tau, \eta \in Z^2(L, V)$.*

(i) The Lie algebras L_τ is a step-t descendent of L if and only if $\tau^\perp \cap C(L) = 0$ and the image of the subspace $(\tau_1, \ldots, \tau_s)$ in $H^2(L, \mathbb{C})$ is t-dimensional.

(ii) Suppose that η is an other element of $Z^2(L, V)$ and that L_τ, L_η are descendants of L. Then $L_\tau \cong L_\eta$ if and only if images of the subspaces $\langle \tau_1, \ldots, \tau_s \rangle$ and $\langle \eta_1, \ldots, \eta_s \rangle$ in $H^2(L, \mathbb{C})$ are in the same orbit under the action of $\mathrm{Aut}(L)$.

Definition 2.6 *A subspace W of $H^2(L, \mathbb{C})$ is said to be allowable if $(\cap_{\tau \in W} \tau^\perp) \cap C(L) = 0$.*

At the moment, we can deduce that there is a one-to-one correspondence between the $\mathrm{Aut}(L)$-orbits on the t-dimensional allowable subspaces of $H^2(L, \mathbb{C})$ and the set of isomorphism types of step-t descendants of L by Theorem 2.5. So we need determine these orbits. The process of conclusion is in Section 4.

In order to solve the classification of 8-dimensional nilpotent Lie algebras with 4-dimensional center, our steps are as follows:

(1) Calculate the second cohomology $H^2(L, \mathbb{C})$ and $\mathrm{Aut}(L)$;

(2) Decide all allowable spaces;

(3) Testing for isomorphisms,

where L is a 4-dimensional nilpotent Lie algebra.

3. Our conclusion

Every 8-dimensional nilpotent Lie algebra is marked $L_{a,b,c,d}$, where a represents the dimension of nilpotent Lie algebra, b represents the dimension of center of nilpotent Lie algebra, c represents the degree of nilpotent Lie algebra in the section. And $x_1, \ldots, x_8$ is a basis of 8-dimensional nilpotent Lie algebra. The complete classification of 8-dimensional nilpotent Lie algebras with 4-dimensional center are as follows:

$$
\begin{aligned}
&L_{8,4,2,1} = \langle [x_1,x_2]=x_5, [x_3,x_4]=x_5 \rangle,\\
&L_{8,4,2,2} = \langle [x_1,x_2]=x_5, [x_1,x_3]=x_6, [x_3,x_4]=x_5 \rangle,\\
&L_{8,4,2,3} = \langle [x_1,x_2]=x_5, [x_1,x_3]=x_6, [x_2,x_4]=x_6, [x_3,x_4]=x_5 \rangle,\\
&L_{8,4,2,4} = \langle [x_1,x_2]=x_5, [x_2,x_3]=x_6, [x_2,x_4]=x_7 \rangle,\\
&L_{8,4,2,5} = \langle [x_1,x_2]=x_5, [x_2,x_3]=x_6, [x_3,x_4]=x_7 \rangle,\\
&L_{8,4,2,6} = \langle [x_1,x_2]=x_5, [x_2,x_3]=x_6, [x_2,x_4]=x_7, [x_3,x_4]=x_5 \rangle,\\
&L_{8,4,2,7} = \langle [x_1,x_2]=x_5, [x_1,x_3]=x_6, [x_2,x_4]=x_7, [x_3,x_4]=x_5 \rangle,\\
&L_{8,4,2,8} = \langle [x_1,x_2]=x_5, [x_2,x_3]=x_6, [x_3,x_4]=x_7, [x_2,x_4]=x_8 \rangle,\\
&L_{8,4,2,9} = \langle [x_1,x_2]=x_5, [x_2,x_3]=x_6, [x_3,x_4]=x_7, [x_1,x_4]=x_8 \rangle,\\
&L_{8,4,2,10} = \langle [x_1,x_2]=x_5, [x_2,x_3]=x_6, [x_3,x_4]=x_7, [x_1,x_3]=x_8, [x_2,x_4]=x_8 \rangle,\\
&L_{8,4,3,1} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_2,x_4]=x_5 \rangle,\\
&L_{8,4,3,2} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_2,x_4]=x_6 y \rangle,\\
&L_{8,4,3,3} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_1,x_4]=x_6, [x_2,x_4]=x_5 \rangle,\\
&L_{8,4,3,4} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_1,x_4]=x_6 \rangle,\\
&L_{8,4,3,5} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_2,x_3]=x_6, [x_2,x_4]=x_5 y \rangle,\\
&L_{8,4,3,6} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_1,x_4]=x_6, [x_2,x_3]=x_6, [x_2,x_4]=x_5 \rangle,\\
&L_{8,4,3,7} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_1,x_4]=x_7, [x_2,x_4]=x_6 \rangle,\\
&L_{8,4,3,8} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_1,x_4]=x_7, [x_2,x_3]=x_6 y \rangle,\\
&L_{8,4,3,9} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_1,x_4]=x_7, [x_2,x_3]=x_6, [x_2,x_4]=x_5 \rangle,\\
&L_{8,4,3,10} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_5, [x_1,x_4]=x_6, [x_2,x_3]=x_7, [x_2,x_4]=x_8 \rangle,\\
&L_{8,4,4,1} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_4, [x_1,x_4]=x_5 \rangle,\\
&L_{8,4,4,2} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_4, [x_1,x_4]=x_5, [x_2,x_3]=x_5 \rangle,\\
&L_{8,4,4,3} = \langle [x_1,x_2]=x_3, [x_1,x_3]=x_4, [x_1,x_4]=x_5, [x_2,x_3]=x_6 \rangle.
\end{aligned}
$$

If $[x_i, x_j]$ does not appear in the multiplication table of the $L_{a,b,c,d}$, it means that $[x_i, x_j]=0$.

4. The calculation of the results

In this section, we calculate the results stated in the above section. Let L be a 4-dimensional nilpotent Lie algebra. In [2], we know that there exist three 4-dimensional nilpotent Lie algebras up to isomorphism: $L_{4,1}, L_{4,2}, L_{4,3}$. Let $\Phi_{i,j}$ with $i \le j$ denote the basis of the space $Z^2(L,\mathbb{C})$, $\phi_{i,j}$ $(i \le j)$ denote the basis of the space $H^2(L,\mathbb{C})$. By the known Lie bracket of L, we will get the basis of $Z^2(L,\mathbb{C})$, $B^2(L,\mathbb{C})$ and the automorphism group of L. It is easy to know that the automorphism group of L is a group of 4×4-matrices with respect to the given basis of L. In the following, we ignore the proof of a basis of $Z^2(L,\mathbb{C})$, $B^2(L,\mathbb{C})$ and the counting process of the automorphism group.

Now we divide the conclusion into three cases.

Case 1 The classification of 8-dimensional 2-step nilpotent Lie algebras with 4-dimensional center

(1) There exists a nilpotent Lie algebra N with $\dim C(N)=4$ and $N^2\neq 0, N^3=0$ in [7, 8].

$$\begin{aligned}
&L_{8,4,2,1}=\langle [x_1,x_2]=x_5,[x_3,x_4]=x_5\rangle,\\
&L_{8,4,2,2}=\langle [x_1,x_2]=x_5,[x_1,x_3]=x_6,[x_3,x_4]=x_5\rangle,\\
&L_{8,4,2,3}=\langle [x_1,x_2]=x_5,[x_1,x_3]=x_6,[x_2,x_4]=x_6,[x_3,x_4]=x_5\rangle,\\
&L_{8,4,2,4}=\langle [x_1,x_2]=x_5,[x_2,x_3]=x_6,[x_2,x_4]=x_7\rangle,\\
&L_{8,4,2,5}=\langle [x_1,x_2]=x_5,[x_2,x_3]=x_6,[x_3,x_4]=x_7\rangle,\\
&L_{8,4,2,6}=\langle [x_1,x_2]=x_5,[x_2,x_3]=x_6,[x_2,x_4]=x_7,[x_3,x_4]=x_5\rangle,\\
&L_{8,4,2,7}=\langle [x_1,x_2]=x_5,[x_1,x_3]=x_6,[x_2,x_4]=x_7,[x_3,x_4]=x_5\rangle.
\end{aligned}$$

(2) Calculate step-4 descendants of the nilpotent Lie algebra $L=L_{4,1}$.

It is routine to check that $\mathrm{Aut}(L)$ is the group of invertible matrices of the form

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}.$$

By calculation, we get $Z^2(L,\mathbb{C})=\langle \Phi_{1,2},\Phi_{1,3},\Phi_{1,4},\Phi_{2,3},\Phi_{2,4}\rangle$, $B^2(L,\mathbb{C})=0$, and so $H^2(L,\mathbb{C})=\langle \phi_{1,2},\phi_{1,3},\phi_{1,4},\phi_{2,3},\phi_{2,4},\phi_{3,4}\rangle$. The dimension of allowable subspace W is four. We can suppose $W=\{\tau_1,\tau_2,\tau_3,\tau_4\},\tau_i=(a_i,b_i,c_i,d_i,e_i,f_i)\in H^2(L,\mathbb{C}), i=1,2,3,4$ such that $\tau_1^{\perp}\cap\tau_2^{\perp}\cap\tau_3^{\perp}\cap\tau_4^{\perp}\cap C(L)=0$. If $\tau=(a,b,c,d,e,f)\in H^2(L,\mathbb{C})$, then $A\tau=(\hat a,\hat b,\hat c,\hat d,\hat e,\hat f)$. Let $\sum_{ij}^{kl}=a_{ik}a_{jl}-a_{il}a_{jk}$ and $\Lambda_{kl}=a\sum_{12}^{kl}+b\sum_{13}^{kl}+c\sum_{14}^{kl}+d\sum_{23}^{kl}+e\sum_{24}^{kl}+f\sum_{34}^{kl}, i,j,k,l=1,2,3,4$. We have $\hat a=\Lambda_{12},\hat b=\Lambda_{13},\hat c=\Lambda_{14},\hat d=\Lambda_{23},\hat e=\Lambda_{24},\hat f=\Lambda_{34}$.

Without loss of generality, we can suppose $a_1\neq 0,d_2\neq 0,f_3\neq 0,e_4\neq 0$. Let

$$A=\begin{pmatrix} d_2/a_1 & 0 & 0 & 0\\ 0 & f_3/e_4 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & d_2/e_4\end{pmatrix}.$$

Then we get

$$\begin{aligned}
&A\tau_1=(d_2f_3/e_4,\ b_1d_2/a_1,\ c_1d_2^2/a_1e_4,\ d_1f_3/e_4,\ e_1f_3d_2/e_4^2,\ f_1d_2/e_4),\\
&A\tau_2=(a_2d_2f_3/a_1e_4,\ b_2d_2/a_1,\ c_2d_2^2/a_1e_4,\ d_2f_3/e_4,\ e_2f_3d_2/e_4^2,\ f_2d_2/e_4),
\end{aligned}$$

$$A\tau_3 = (a_3d_2f_3/a_1e_4,\ b_3d_2/a_1,\ c_3d_2^2/a_1e_4,\ d_3f_3/e_4,\ e_3f_3d_2/e_4^2,\ f_3d_2/e_4),$$
$$A\tau_4 = (a_4d_2f_3/a_1e_4,\ b_4d_2/a_1,\ c_4d_2^2/a_1e_4,\ d_4f_3/e_4,\ f_3d_2/e_4,\ f_4d_2/e_4).$$

Because W is a vector space, we can assume

$$\tau_1 = (1, b_1, c_1, 0, 0, 0),\ \tau_2 = (0, b_2, c_2, 1, 0, 0),\ \tau_3 = (0, b_3, c_3, 0, 0, 1,),\ \tau_4 = (0, b_4, c_4, 0, 1, 0).$$

Let $A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -b_1 & -c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -b_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c_2/b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -c_3 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{pmatrix}$.

Then we have

$$A_1\tau_1 = (1, 0, 0, 0, 0, 0),\ A_1\tau_2 = (0, b_2, c_2, 1, 0, c_1),$$
$$A_1\tau_3 = (0, b_3, c_3, 0, 0, 1,),\ A_1\tau_4 = (0, b_4, c_4, 0, 1, -b_1).$$

Furthermore, we can suppose

$$\tau_1 = (1, 0, 0, 0, 0, 0),\ \tau_2 = (0, b_2, c_2, 1, 0, 0),\ \tau_3 = (0, b_3, c_3, 0, 0, 1,),\ \tau_4 = (0, b_4, c_4, 0, 1, 0),$$

where we always ensure $b_2 \neq 0$. Consider the action of B_1 on $\tau_1, \tau_2, \tau_3, \tau_4$, then

$$B_1\tau_1 = (1, 0, 0, 0, 0, 0),\quad B_1\tau_2 = (0, 0, 0, 1, 0, 0),$$
$$B_1\tau_3 = (0, b_3, -b_3c_2/b_2 + c_3, -b_3b_2, -c_3b_2, 1,),$$
$$B_1\tau_4 = (0, b_4, -b_4c_2/b_2 + c_4, -b_4b_2, 1 - c_4b_2, 0),$$

where $1 - c_4b_2 \neq 0$. Furthermore, we get

$$\tau_1 = (1, 0, 0, 0, 0, 0),\ \tau_2 = (0, 0, 0, 1, 0, 0),\ \tau_3 = (0, b_3, c_3, 0, 0, 1,),\ \tau_4 = (0, b_4, c_4, 0, 1, 0).$$

By the same method, considering the action of C_1 on $\tau_1, \tau_2, \tau_3, \tau_4$, we obtain

$$C_1\tau_1 = (1, 0, 0, 0, 0, 0),\ C_1\tau_2 = (0, 0, 0, 1, 0, 0),$$
$$C_1\tau_3 = (0, 0, 0, 0, 0, 1,),\ C_1\tau_4 = (-b_3, b_4, c_4, 0, 1, 0).$$

Then we can put

$$\tau_1 = (1, 0, 0, 0, 0, 0),\ \tau_2 = (0, 0, 0, 1, 0, 0),\ \tau_3 = (0, 0, 0, 0, 0, 1,),\ \tau_4 = (0, b_4, c_4, 0, 1, 0).$$

(i) If $b_4, c_4 = 0$, we get

$$\tau_1 = (1, 0, 0, 0, 0, 0),\ \tau_2 = (0, 0, 0, 1, 0, 0),\ \tau_3 = (0, 0, 0, 0, 0, 1,),\ \tau_4 = (0, 0, 0, 0, 1, 0),$$

and

$$L_{8,4,2,8} = \langle [x_1, x_2] = x_5, [x_2, x_3] = x_6, [x_3, x_4] = x_7, [x_2, x_4] = x_8 \rangle.$$

(ii) If $c_4 \neq 0$, let

$$B = \begin{pmatrix} 1 & -1/c_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -b_4/c_4 & 1 \end{pmatrix}.$$

Then

$$B\tau_1 = (1,0,0,0,0,0),\ B\tau_2 = (0,0,0,1,0,0),\ B\tau_3 = (0,0,0,0,0,1,),\ B\tau_4 = (0,0,c_4,0,0,0).$$

Furthermore, we obtain

$$\tau_1 = (1,0,0,0,0,0),\ \tau_2 = (0,0,0,1,0,0),\ \tau_3 = (0,0,0,0,0,1,),\ \tau_4 = (0,0,1,0,0,0),$$

and

$$L_{8,4,2,9} = \langle [x_1,x_2] = x_5, [x_2,x_3] = x_6, [x_3,x_4] = x_7, [x_1,x_4] = x_8 \rangle.$$

(iii) If $c_4 = 0, b_4 \neq 0$, then

$$\tau_1 = (1,0,0,0,0,0),\ \tau_2 = (0,0,0,1,0,0),\ \tau_3 = (0,0,0,0,0,1,),\ \tau_4 = (0,b_4,0,0,1,0).$$

Furthermore, we get

$$\tau_1 = (1,0,0,0,0,0),\ \tau_2 = (0,0,0,1,0,0),\ \tau_3 = (0,0,0,0,0,1,),\ \tau_4 = (0,1,0,0,1,0),$$

and

$$L_{8,4,2,10} = \langle [x_1,x_2] = x_5, [x_2,x_3] = x_6, [x_3,x_4] = x_7, [x_1,x_3] = x_8, [x_2,x_4] = x_8 y \rangle.$$

By comparing the orbits, all Lie algebras $L_{8,4,2,8}$, $L_{8,4,2,9}$, $L_{8,4,2,10}$ are not isomorphism. Therefore, the 8-dimensional nilpotent Lie algebra of 4-dimensional center has three isomorphism types in this case.

Case 2 The classification of 8-dimensional 3-step nilpotent Lie algebras with 4-dimensional center

(1) There exists a nilpotent Lie algebra N with $\dim C(N) = 4$ and $N^3 \neq 0, N^4 = 0$ in [7, 8].

$$\begin{aligned}
L_{8,4,3,1} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_2,x_4] = x_5 \rangle,\\
L_{8,4,3,2} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_2,x_4] = x_6 \rangle,\\
L_{8,4,3,3} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_1,x_4] = x_6, [x_2,x_4] = x_5 \rangle,\\
L_{8,4,3,4} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_1,x_4] = x_6 \rangle,\\
L_{8,4,3,5} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_2,x_3] = x_6, [x_2,x_4] = x_5 \rangle,\\
L_{8,4,3,6} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_1,x_4] = x_6, [x_2,x_3] = x_6, [x_2,x_4] = x_5 \rangle,\\
L_{8,4,3,7} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_1,x_4] = x_7, [x_2,x_4] = x_6 \rangle,\\
L_{8,4,3,8} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_1,x_4] = x_7, [x_2,x_3] = x_6 \rangle,\\
L_{8,4,3,9} &= \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_1,x_4] = x_7, [x_2,x_3] = x_6, [x_2,x_4] = x_5 \rangle.
\end{aligned}$$

(2) Calculate step-4 descendants of the nilpotent Lie algebras $L = L_{4,2}$.

It is routine to check that $\mathrm{Aut}(L)$ is the group of invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & u & 0 \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix},$$

where $u = a_{11}a_{22} - a_{12}a_{21}$. By calculation, we get $Z^2(L,\mathbb{C}) = \langle \Phi_{1,2}, \Phi_{1,3}, \Phi_{1,4}, \Phi_{2,3}, \Phi_{2,4}\rangle$, $B^2(L,\mathbb{C}) = \langle \Phi_{1,2}\rangle$, and so $H^2(L,\mathbb{C}) = \langle \phi_{1,3}, \phi_{1,4}, \phi_{2,3}, \phi_{2,4}\rangle$. The dimension of allowable subspaces W is four. Because of dim $H^2(L,\mathbb{C}) = 4$, there is only one suitable allowable subspace such that

$$\tau_1 = (1,0,0,0),\ \tau_2 = (0,1,0,0),\ \tau_3 = (0,0,1,0),\ \tau_4 = (0,0,0,1) \in H^2(L,\mathbb{C}).$$

In a word, the Lie algebra $L = L_{4.2}$ of step-4 descendants has only one isomorphism class, namely

$$L_{8,4,3,10} = \langle [x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_1,x_4] = x_6, [x_2,x_3] = x_7, [x_2,x_4] = x_8\rangle.$$

Case 3 The classification of 8-dimensional 4-step nilpotent Lie algebras with 4-dimensional center

(1) There exists a nilpotent Lie algebra N which satisfies the conditions of $C(N) = 4$ and $N^4 \neq 0, N^5 = 0$ in [7].

$$L_{8,4,4,1} = \langle [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5\rangle,$$
$$L_{8,4,4,2} = \langle [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5, [x_2,x_3] = x_5\rangle,$$
$$L_{8,4,4,3} = \langle [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5, [x_2,x_3] = x_6\rangle.$$

(2) Calculate step-4 descendants of the nilpotent Lie algebras $L = L_{4,3}$.

It is routine to check that $\mathrm{Aut}(L)$ is the group of invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & u & v \\ a_{41} & a_{42} & w & x \end{pmatrix},$$

where $u = a_{11}a_{22} - a_{12}a_{21}$, $v = a_{11}a_{23} - a_{23}a_{21}$, $w = a_{11}a_{32} - a_{12}a_{31}$, $x = a_{11}a_{33} - a_{13}a_{31}$. By calculation, we get $Z^2(L,\mathbb{C}) = \langle \Phi_{1,2}, \Phi_{1,3}, \Phi_{1,4}, \Phi_{2,3}\rangle$, $B^2(L,\mathbb{C}) = \langle \Phi_{1,2}, \Phi_{1,3}\rangle$, and so $H^2(L,\mathbb{C}) = \langle \phi_{1,4}, \phi_{2,3}\rangle$. The set of allowable subspaces W is of dimension four. By virture of $\dim H^2(L,\mathbb{C}) = 2 < 4$, there is not the allowable subspace.

In short, the Lie algebra $L = L_{4.3}$ has no step-4 descendants.

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