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Toeplitz Operators with Quasihomogeneous Symbols on the Dirichlet Space of $\mathbb{B}_n$

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Abstract In this paper, we study some algebraic properties of Toeplitz operators with quasihomogeneous symbols on the Dirichlet space of the unit ball $\mathbb{B}_n$. First, we describe commutators of a radial Toeplitz operator and characterize commuting Toeplitz operators with quasihomogeneous symbols. Then we show that finite rank product of such operators only happens in the trivial case. Finally, some necessary and sufficient conditions are given for the product of two quasihomogeneous Toeplitz operators to be a quasihomogeneous Toeplitz operator.

Keywords Dirichlet space; unit ball; Toeplitz operators; quasihomogeneous symbols.

MR(2010) Subject Classification 47B35

1. Introduction

For any integer $n \geq 1$, let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball of $\mathbb{C}^n$ and dm be the normalized Lebesgue measure on $\mathbb{B}_n$. The Sobolev space $w^{1,2}$ is defined to be the completion of smooth functions on $\mathbb{B}_n$ which satisfy

$$\|f\|^2 = |\int_{\mathbb{B}_n} f\mathrm{d}m|^2 + \sum_{i=1}^{n} \int_{\mathbb{B}_n} (|\frac{\partial f}{\partial z_i}|^2 + |\frac{\partial f}{\partial \bar{z}_i}|^2)\mathrm{d}m < \infty.$$

The inner product $\langle \cdot, \cdot \rangle$ on $w^{1,2}$ is defined by

$$\langle f, g\rangle = \int_{\mathbb{B}_n} f\mathrm{d}m \int_{\mathbb{B}_n} \bar{g}\mathrm{d}m + \sum_{i=1}^{n} \int_{\mathbb{B}_n} (\frac{\partial f}{\partial z_i}\overline{\frac{\partial g}{\partial z_i}} + \frac{\partial f}{\partial \bar{z}_i}\overline{\frac{\partial g}{\partial \bar{z}_i}})\mathrm{d}m, \quad \forall f, g \in w^{1,2}.$$

The Dirichlet space $\mathcal{D}$ of $\mathbb{B}_n$ is the closed subspace consisting of all holomorphic functions $f \in w^{1,2}$. It is easily verified that each point evaluation is a bounded linear functional on $\mathcal{D}$. Hence, for each $z \in \mathbb{B}_n$, there exists a unique reproducing kernel $K_z(w) \in \mathcal{D}$ such that

$$f(z) = \langle f, K_z\rangle, \quad \forall f \in \mathcal{D}.$$

Actually, it can be calculated that $K_z(w) = 1 + \sum_{\alpha \in \mathbb{Z}_+^n} \frac{(|\alpha|+n-1)!}{|\alpha|n!\alpha!} w^\alpha \bar{z}^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $\alpha_i \in \mathbb{Z}_+$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. For multi-indexes α and β, the

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notation $\alpha \succeq \beta$ means that

$$\alpha_i \geq \beta_i, \quad i = 1, \ldots, n,$$

$\alpha \succ \beta$ means that $\alpha \succeq \beta$ and $\alpha \neq \beta$, and $\alpha \not\succeq q$ means that there exists i_0 such that $\alpha_{i_0} < q_{i_0}$.

Let P be the orthogonal projection from $w^{1,2}$ onto $\mathcal{D}$. By the explicit formula for $K_z(w)$, we have

$$P\psi(z) = \langle P\psi, K_z \rangle = \langle \psi, K_z \rangle = \int_{\mathbb{B}_n} \psi \mathrm{d}m \int_{\mathbb{B}_n} \overline{K}_z \mathrm{d}m + \sum_{i=1}^{n} \int_{\mathbb{B}_n} \frac{\partial \psi}{\partial w_i} \overline{\frac{\partial K_z}{\partial w_i}} \mathrm{d}m(w), \quad \psi \in w^{1,2}.$$

Let $\Omega = \{\varphi \in w^{1,2} : \varphi, \frac{\partial \varphi}{\partial z_i}, \frac{\partial \varphi}{\partial \bar{z}_i} \in L^\infty(\mathbb{B}_n)\}$ and $\|\varphi\|_\infty^{1,2} = \max\{\|\varphi\|_\infty, \|\frac{\partial \varphi}{\partial z_i}\|_\infty, \|\frac{\partial \varphi}{\partial \bar{z}_i}\|_\infty\}$. Given a function $\varphi \in \Omega$, the Toeplitz operator T_φ with symbol φ is defined by

$$T_\varphi f = P(\varphi f), \quad f \in \mathcal{D}.$$

It is easy to verify that the Toeplitz operator $T_\varphi : \mathcal{D} \to \mathcal{D}$ is a bounded linear operator and $\|T_\varphi\| \leq \|\varphi\|_\infty^{1,2}$, whenever $\varphi \in \Omega$.

A function f on unit disc $\mathbb{D}$ is said to be quasihomogeneous of degree p if it is of the form $f(re^{i\theta}) = e^{ip\theta}\phi(r)$, where ϕ is a radial function, i.e., $\phi(z) = \phi(|z|)$, $z \in \mathbb{D}$. In this case the associated Toeplitz operator $T_{e^{ip\theta}\phi(r)}$ is also called quasihomogeneous Toeplitz operator of degree p. If $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, we can also get the definition above on the unit ball $\mathbb{B}_n$.

The algebraic properties of Toeplitz operators on the classical Hardy spaces and Bergman spaces have been well studied, for example, as in [1–5]. It is known that on the Bergman space the commuting problem still remains open except for bounded harmonic symbols. As the quasihomogeneous functions are the nature generalization of radial functions which are nonharmonic, quasihomogeneous symbols operators excited many researchers' interest. Toeplitz operators with those symbols were intensively studied in [6–20].

On the Bergman space of the unit disk, Čučkovic and Rao [6] first studied quasihomogeneous Toeplitz operators by using the Mellin transform. Later, Louhichi and Zakariasy [7] gave some basic results and a partial characterization of commuting quasihomogeneous Toeplitz operators. In terms of T-functions, Louhichi, Strouse and Zakariasy [8] found some necessary and sufficient conditions for the product of two quasihomogeneous Toeplitz operators to be a Toeplitz operator. Louhichi and Rao [10] pointed out an unusual phenomenon that the commutant of a quasihomogeneous Toeplitz operator is equal to its bicommutant. That is, if two Toeplitz operators commute with a quasihomogeneous Toeplitz operator, then they commute with each other. At the same time, Čučkovic and Louhichi [11] studied the zero product and (semi)commutators of quasihomogeneous Toeplitz operators.

On the Bergman space of the unit ball, Zhou and Dong [12] investigated algebraic properties of Toeplitz operators with radial symbols and quasihomogeneous symbols. They also discussed algebraic properties of Toeplitz operators with separately quasihomogeneous symbols in [14] (i.e., symbols being of the form $\xi^k \phi(|z_1|, \ldots, |z_n|)$).

On the Bergman space of the polydisc, Dong [16] and Zhang [17] respectively studied the commuting problem and the finite rank product problem for the separately quasihomogeneous

Toeplitz operators.

In recent years, Toeplitz operators on Dirichlet spaces attracted more and more attention of mathematicians. Chen [18] showed that every continuous function f in $L^\infty(\mathbb{D}, \mathrm{d}A)$ has the following polar decomposition

$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r)$$

if and only if $f(re^{i\theta})$ is absolutely continuous ($\|\cdot\|_D$) on $\theta \in [0, 2\pi]$ for almost every $r \in [0, 1)$. Then, (semi-)commuting Toeplitz operators whose symbols have the decomposition above were studied. Moreover, since $T_\phi = 0$ may not imply $\phi = 0$ in this case, Chen also showed that radial Toeplitz operators not only commute with another such operator. Deng, Pan and Chen studied the boundedness, compactness and product of quasihomogeneous Toeplitz operators in [19, 20].

Unlike the case of Bergman space, little has been known about quasihomogeneous Toeplitz operators on the Dirichlet space of unit ball. The authors [21] gave some basic properties of Toeplitz operators with pluriharmonic symbols and discuss the commuting problem of Toeplitz operators with $z^p\bar{z}^q\phi(|z|)$ symbols. Motivated by the work in [12, 14, 18] and [21], in this paper we will investigate some properties of quasihomogeneous Toeplitz operators. In Section 2, we give some basic properties of quasihomogeneous Toeplitz operators and characterize the commuting quasihomogeneous Toeplitz operators. In Section 3, we discuss the problems of finite rank product and zero product of those operators. At last, we obtain the necessary and sufficient conditions for the product of two quasihomogeneous Toeplitz operators to be a Toeplitz operator.

2. Commuting Toeplitz operators with quasihomogeneous symbols

In this section, we will characterize commuting Toeplitz operators with bounded quasihomogeneous symbols on the Dirichlet space of the unit ball. The definition of quasihomogeneous function on the unit disk has been given in many papers and a similar definition on the unit ball has also been given in [22].

Definition 2.1 *Let p, $s \in \mathbb{Z}_+^n$ and $f \in L^1(\mathbb{B}_n, \mathrm{d}m)$. f is called a quasihomogeneous function of degree (p, s) if*

$$f(r\xi) = \xi^p\overline{\xi}^s\phi(r)$$

for any ξ in the unit sphere $\mathbb{S}_n$ and $r = \sqrt{|z_1|^2 + \cdots + |z_n|^2} \in [0, 1)$.

First, we make some notations. Let $\phi = \phi(r)$ be a radial function, $\Sigma = \{\phi : \phi, \phi' \in L^1([0, 1])\}$ and $\Sigma' = \{\phi : \phi \in \Sigma$ and ϕ is absolutely continuous on $[0, 1)\}$. In the remaining part of this paper, we will always assume $\phi \in \Sigma$. For $\phi \in \Sigma$ and $k \in \mathbb{Z}_+$, let $\tilde{\phi}(k) = \int_0^1 r^{k-1}\phi(r)\mathrm{d}r$ and $\hat{\phi}(k) = \int_0^1 r^{k-1}[\phi + \int_r^1 \phi'(t)\mathrm{d}t]\mathrm{d}r$. Before discussing the commutivity of Toeplitz operators with quasihomogeneous symbols, we need the following lemma which can be obtained by direct computation.

Lemma 2.2 *Let p, $s \in \mathbb{Z}_+^n$ and $\phi \in \Sigma$. Then for any $\alpha \in \mathbb{Z}_+^n$,*

$$T_{\xi^p\phi} z^\alpha = \begin{cases} (\tau_\alpha + |p|)\hat{\phi}(\tau_\alpha + |p|) z^{\alpha+p}, & p+\alpha \succ 0; \\ 2n\int_0^1 r^{2n-1}\phi(r)\mathrm{d}r, & p = \alpha = 0, \end{cases}$$

$$T_{\bar{\xi}^s\phi} z^\alpha = \begin{cases} d(\alpha, \alpha - s)(\tau_\alpha - |s|)\hat{\phi}(\tau_\alpha - |s|) z^{\alpha - s}, & \alpha \succ s; \\ \frac{2n!s!}{(n+|s|-1)!}\int_0^1 r^{2n+|s|-1}\phi(r)\mathrm{d}r, & \alpha = s; \\ 0, & \alpha \not\succeq s, \end{cases}$$

$$T_{\xi^p\bar{\xi}^s\phi} z^\alpha = \begin{cases} d(\alpha+p, \alpha+p-s)(\tau_\alpha + |p| - |s|)\hat{\phi}(\tau_\alpha + |p| - |s|) z^{\alpha+p-s}, & \alpha+p-s \succ 0; \\ \frac{2n!(p+\alpha)!}{(n+|p+\alpha|-1)!}\int_0^1 r^{2n+|\alpha|-1}\phi(r)\mathrm{d}r, & \alpha+p-s = 0; \\ 0, & \alpha+p-s \not\succeq 0, \end{cases}$$

where $\tau_\alpha = 2n + 2|\alpha| - 2, d(\alpha, \alpha - q) = \frac{\alpha!}{(n+|\alpha|-1)!} / \frac{(\alpha-q)!}{(n+|\alpha-q|-1)!}$.

Proof By taking $p = 0$ or $s = 0$ in the third equation, we can get the other two equations. So, we only need to prove the third equation. For multi-index $\alpha, \beta \in \mathbb{Z}_+^n$, we calculate

$$\langle T_{\xi^p\bar{\xi}^s\phi} z^\alpha, z^\beta\rangle = \langle \xi^p\bar{\xi}^s\phi z^\alpha, z^\beta\rangle = \langle \phi r^{-|p+s|} z^{p+\alpha}\bar{z}^s, z^\beta\rangle.$$

Denote $F(r) = \phi \cdot r^{-|p+s|}$, then

$$\frac{\partial F}{\partial z_i} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial z_i} = F'\frac{\overline{z_i}}{2r}. \tag{2.1}$$

For any positive integer k such that $2n + 2k - |p+s| - 3 > 0$, we have

$$\begin{aligned} &\widetilde{F'}(2n+2k-1) + 2(n+k-1)\widetilde{F}(2n+2k-2) \\ &= \int_0^1 F'(r) r^{2n+2k-2}\mathrm{d}r + 2(n+k-1)\int_0^1 F(r) r^{2n+2k-3}\mathrm{d}r \\ &= \int_0^1 \phi' r^{2n+2k-|p+s|-2}\mathrm{d}r + (2n+2k-|p+s|-2)\int_0^1 \phi r^{2n+2k-|p+s|-3}\mathrm{d}r \\ &= (2n+2k-|p+s|-2)\hat{\phi}(2n+2k-|p+s|-2). \end{aligned} \tag{2.2}$$

The last equation holds with integration by part since ϕ' belongs to $L^1([0,1])$ and $r^{2n+2k-|p+s|-2}$ is absolutely continuous on $[0,1]$.

By (2.1) and the well known equation $\int_{\mathbb{B}_n} f\mathrm{d}m = 2n\int_0^1 r^{2n-1}\mathrm{d}r\int_{\mathbb{S}_n} f(r\xi)\mathrm{d}\sigma$, we have

$$\begin{aligned} \langle T_{\xi^p\bar{\xi}^s\phi} z^\alpha, z^\beta\rangle &= \int_{\mathbb{B}_n} \xi^p\bar{\xi}^s\phi z^\alpha\mathrm{d}m\int_{\mathbb{B}_n}\overline{z^\beta}\mathrm{d}m + \sum_{i=1}^n\int_{\mathbb{B}_n}\frac{\partial}{\partial z_i}(Fz^{p+\alpha}\bar{z}^s)\overline{\frac{\partial z^\beta}{\partial z_i}}\mathrm{d}m \\ = &\int_{\mathbb{B}_n} \xi^p\bar{\xi}^s\phi z^\alpha\mathrm{d}m\int_{\mathbb{B}_n}\overline{z^\beta}\mathrm{d}m + \sum_{\beta_i>0}\frac{\beta_i}{2}\int_{\mathbb{B}_n} F' r^{-1} z^{p+\alpha}\bar{z}^{s+\beta}\mathrm{d}m + \\ &\sum_{\beta_i>0, p_i+\alpha_i>0}\beta_i(p_i+\alpha_i)\int_{\mathbb{B}_n} F z^{p+\alpha-e_i}\bar{z}^{s+\beta-e_i}\mathrm{d}m \\ = &\int_{\mathbb{B}_n} \xi^p\bar{\xi}^s\phi z^\alpha\mathrm{d}m\int_{\mathbb{B}_n}\overline{z^\beta}\mathrm{d}m + \sum_{\beta_i>0}\frac{\beta_i}{2}\int_0^1 2nF'(r) r^{2n-2+|p+\alpha|+|s+\beta|}\mathrm{d}r\int_{\mathbb{S}_n}\xi^{p+\alpha}\bar{\xi}^{s+\beta}\mathrm{d}\sigma + \\ &\sum_{\beta_i>0, p_i+\alpha_i>0}\beta_i(p_i+\alpha_i)\int_0^1 2nF(r) r^{2n-3+|p+\alpha|+|s+\beta|}\mathrm{d}r\int_{\mathbb{S}_n}\xi^{p+\alpha-e_i}\bar{\xi}^{s+\beta-e_i}\mathrm{d}\sigma \end{aligned}$$

If $\alpha+p\succ s$ and $\beta\neq\alpha+p-s$, then $\langle T_{\xi^p\bar{\xi}^s\phi}z^\alpha,z^\beta\rangle=0$. If $\alpha+p\succ s$ and $\beta=\alpha+p-s$, by (2.2) we calculate

$$\begin{aligned}
&\langle T_{\xi^p\bar{\xi}^s\phi}z^\alpha,z^\beta\rangle\\
&=\sum_{i=1,\beta_i>0}^{n}\beta_i n\widetilde{F'}(2n+2|p+\alpha|-1)\int_{\mathbb{S}_n}|\xi^{p+\alpha}|^2\mathrm{d}\sigma+\\
&\quad\sum_{p_i+\alpha_i-s_i>0,p_i+\alpha_i>0}(p_i+\alpha_i-s_i)(p_i+\alpha_i)2n\widetilde{F}(2n+2|p+\alpha|-2)\int_{\mathbb{S}_n}|\xi^{p+\alpha-e_i}|^2\mathrm{d}\sigma\\
&=\sum_{\beta_i>0}\frac{\beta_i n!(p+\alpha)!}{(n-1+|p+\alpha|)!}\widetilde{F'}(2n+2|p+\alpha|-1)+\\
&\quad\sum_{p_i+\alpha_i-s_i>0,p_i+\alpha_i>0}(p_i+\alpha_i-s_i)(p_i+\alpha_i)\frac{2n!(p+\alpha-e_i)!}{(n-2+|p+\alpha|)!}\widetilde{F}(2n+2|p+\alpha|-2)\\
&=|p+\alpha-s|\frac{n!(p+\alpha)!}{(n-1+|p+\alpha|)!}\widetilde{F'}(2n+2|p+\alpha|-1)+\\
&\quad\sum_{p_i+\alpha_i-s_i>0}(p_i+\alpha_i-s_i)\frac{2n!(p+\alpha)!}{(n-2+|p+\alpha|)!}\widetilde{F}(2n+2|p+\alpha|-2)\\
&=\frac{|p+\alpha-s|n!(p+\alpha)!}{(n-1+|p+\alpha|)!}(\widetilde{F'}(2n+2|p+\alpha|-1)+2(n+|p+\alpha|-1)\widetilde{F}(2n+2|p+\alpha|-2))\\
&=d(p+\alpha,p+\alpha-s)||z^{p+\alpha-s}||^2(2n+2|\alpha+p|-|p+s|-2)\hat{\phi}(2n+2|\alpha+p|-|p+s|-2)\\
&=d(p+\alpha,p+\alpha-s)||z^{p+\alpha-s}||^2(\tau_\alpha+|p|-|s|)\hat{\phi}(\tau_\alpha+|p|-|s|).
\end{aligned}$$

If $\alpha+p=s$,

$$\langle T_{\xi^p\bar{\xi}^s\phi}z^\alpha,z^\beta\rangle=\begin{cases}\frac{2n!(p+\alpha)!}{(n-1+|p+\alpha|)!}\int_0^1\phi r^{2n+|\alpha|-1}\mathrm{d}r, & \beta=0;\\ 0, & \beta\succ 0.\end{cases}$$

If $\alpha+p\not\succeq s$, $\langle T_{\xi^p\bar{\xi}^s\phi}z^\alpha,z^\beta\rangle=0$ for any multi-index β.

Thus we get the third equation. The proof is completed. $\square$

Assume $\phi\in\Sigma'$, ϕ is absolutely continuous on [0,1). Integrating by parts, we have $m\hat{\phi}(m)=\lim_{r\to1^-}\phi(r)\doteq\phi(1^-)$, for any positive integer m. Consequently, we can get the following lemma immediately from Lemma 2.2.

Lemma 2.3 *Let* p, $s\in\mathbb{Z}_+^n$ *and* $\phi\in\Sigma'$. *Then for any* $\alpha\in\mathbb{Z}_+^n$,

$$T_{\xi^p\phi}z^\alpha=\begin{cases}\phi(1-)z^{\alpha+p}, & p+\alpha\succ 0;\\ 2n\int_0^1 r^{2n-1}\phi(r)\mathrm{d}r, & p=\alpha=0,\end{cases}$$

$$T_{\bar{\xi}^s\phi}z^\alpha=\begin{cases}d(\alpha,\alpha-s)\phi(1-)z^{\alpha-s}, & \alpha\succ s;\\ \frac{2n!s!}{(n+|s|-1)!}\int_0^1 r^{2n+|s|-1}\phi(r)\mathrm{d}r, & \alpha=s;\\ 0, & \alpha\not\succeq s,\end{cases}$$

$$T_{\xi^p\bar{\xi}^s\phi}z^\alpha=\begin{cases}d(\alpha+p,\alpha+p-s)\phi(1-)z^{\alpha+p-s}, & \alpha+p-s\succ 0;\\ \frac{2n!(p+\alpha)!}{(n+|p+\alpha|-1)!}\int_0^1 r^{2n+|\alpha|-1}\phi(r)\mathrm{d}r, & \alpha+p-s=0;\\ 0, & \alpha+p-s\not\succeq 0.\end{cases}$$

Special attention should be paid to the following cases.

Corollary 2.4 *Let $\phi \in \Sigma'$ and k have the index decomposition $k = p - s$ such that $p \perp s$ with $p, s \succ 0$. Then for any multi-index $\alpha \succeq 0$,*

$$T_{\xi^k \phi} z^\alpha = T_{\xi^p \bar{\xi}^s \phi} z^\alpha = \begin{cases} d(p+\alpha, p+\alpha-s)\phi(1-) z^{\alpha+p-s}, & \alpha \succeq s; \\ 0, & \alpha \not\succeq s. \end{cases}$$

Corollary 2.5 *Let $\phi \in \Sigma'$. Then for any multi-index $\alpha \succeq 0$,*

$$T_\phi z^\alpha = \begin{cases} \phi(1-) z^\alpha, & \alpha \succ 0; \\ 2n \int_0^1 \phi r^{2n-1} \mathrm{d}r (= 2n\tilde{\phi}(2n)), & \alpha = 0. \end{cases}$$

In the section 4 of [21], we discuss the zero product and commuting problem of Toeplitz operators with $z^p \bar{z}^q \phi(r^2)$ symbols. With analogue argument, by Lemmas 2.2 and 2.3 we can get similar consequences corresponding to Toeplitz operators with $\xi^p \bar{\xi}^q \phi(r)$ symbols. We just exhibit the results directly and omit the proof.

Theorem 2.6 *Let $p, q \succ 0$, $\phi, \psi \in \Sigma$. $T_{\xi^p \phi(r)} T_{\xi^q \psi(r)} = T_{\xi^q \psi(r)} T_{\xi^p \phi(r)}$ if and only if $(\tau_\alpha + |q|)\hat{\psi}(\tau_\alpha + |q|)(\tau_\alpha + 2|q| + |p|)\hat{\phi}(\tau_\alpha + 2|q| + |p|) = (\tau_\alpha + |p|)\hat{\phi}(\tau_\alpha + |p|)(\tau_\alpha + 2|p| + |q|)\hat{\psi}(\tau_\alpha + 2|p| + |q|)$ holds for any multi-index $\alpha \succeq 0$. In particular, if $|p| = |q|$, then $T_{\xi^p \phi(r)} T_{\xi^q \phi(r)} = T_{\xi^q \phi(r)} T_{\xi^p \phi(r)}$.*

Note that by Lemma 2.3 for $\phi \in \Sigma'$ and $p \succ 0$, $T_{\xi^p \phi} = 0$ if and only if $\phi(1-) = 0$. Direct calculation leads to the following, which is somewhat analogous to Theorem 4.2 in [21].

Theorem 2.7 *Let $p_i \succ 0$ and $\phi_i \in \Sigma'$. Then the followings hold.*

(1) $T_{\xi^{p_1} \phi_1} T_{\xi^{p_2} \phi_2} = T_{\xi^{p_2} \phi_2} T_{\xi^{p_1} \phi_1} = T_{\xi^{p_1+p_2} \phi_1 \phi_2}$.

(2) $T_{\xi^{p_1} \phi_1} \times \cdots \times T_{\xi^{p_k} \phi_k} = 0$ if and only if $\phi_i(1^-) = 0$ for some i, that is $T_{\xi^{p_i} \phi_i} = 0$ for some i.

(3) Let $p_i \neq p_j$ for $i \neq j$. $T_{\xi^{p_1} \phi_1} + \cdots + T_{\xi^{p_k} \phi_k} = 0$ if and only if $T_{\xi^{p_i} \phi_i} = 0$ for each i if and only if $\phi_i(1^-) = 0$ for each i.

As for Toeplitz operators with $\bar{\xi}^s \phi(r)$ symbols, direct calculation leads to the following theorem, which is analogous to Theorem 4.3 in [21].

Theorem 2.8 *Let $s, t \succ 0$, $\phi, \psi \in \Sigma$. $T_{\bar{\xi}^s \phi(r)} T_{\bar{\xi}^t \psi(r)} = T_{\bar{\xi}^t \psi(r)} T_{\bar{\xi}^s \phi(r)}$ if and only if $(2n+2|s|+|t|-2)\hat{\psi}(2n+2|s|+|t|-2) \int_0^1 \phi r^{2n+|s|-1} dr = (2n+2|t|+|s|-2)\hat{\psi}(2n+2|t|+|s|-2) \int_0^1 \psi r^{2n+|t|-1} \mathrm{d}r$ and $(\tau_\alpha - |t|)\hat{\psi}(\tau_\alpha - |t|)(\tau_\alpha - 2|t| - |s|)\hat{\phi}(\tau_\alpha - 2|t| - |s|) = (\tau_\alpha - |s|)\hat{\phi}(\tau_\alpha - |s|)(\tau_\alpha - 2|s| - |t|)\hat{\psi}(\tau_\alpha - 2|s| - |t|)$ holds for any multi-index $\alpha \succ 0$. In particular, if $|p| = |q|$, then $T_{\bar{\xi}^p \phi(r)} T_{\bar{\xi}^q \phi(r)} = T_{\bar{\xi}^q \phi(r)} T_{\bar{\xi}^p \phi(r)}$.*

Theorem 2.9 *Let $s, t, s_i \succ 0$, $s_i \neq s_j$ for $i \neq j$ and $\phi, \psi, \phi_i \in \Sigma'$. Then the following assertions hold.*

(1) $T_{\bar{\xi}^s \phi(r)} T_{\bar{\xi}^t \psi(r)} = T_{\bar{\xi}^t \psi(r)} T_{\bar{\xi}^s \phi(r)}$ if and only if

$$\psi(1^-) \int_0^1 r^{2n+|s|-1} \phi(r) \mathrm{d}r = \phi(1^-) \times \int_0^1 r^{2n+|t|-1} \psi(r) \mathrm{d}r.$$

In this case, $T_{\bar{\xi}^s \phi(r)} T_{\bar{\xi}^t \psi(r)}$ may not equal $T_{\bar{\xi}^{s+t} \phi\psi}$.

(2) $T_{\bar{\xi}^{s_1}\phi_1} \times \cdots \times T_{\bar{\xi}^{s_k}\phi_k} = 0$ *if and only if one of the following holds:*

(i) $\phi_1(1^-) = 0$ *and* $\int_0^1 r^{2n+|s_1|-1}\phi_1(r)\mathrm{d}r = 0$;

(ii) There exists i_0 *where* $2 \le i_0 \le k$ *such that* $\phi_{i_0}(1^-) = 0$.

(3) $T_{\bar{\xi}^{s_1}\phi_1} + \cdots + T_{\bar{\xi}^{s_k}\phi_k} = 0$ *if and only if* $T_{\bar{\xi}^{s_i}\phi_i} = 0$ *for each* i.

Analogously to Theorem 4.10 in [21], next theorem describes the quasihomogeneous Toeplitz operators which commute with radial Toeplitz operators.

Theorem 2.10 *Let* $p, q \succ 0$ *and* $\phi, \psi \in \Sigma'$. *Then the following assertions hold.*

(1) If $p \succ q$, $T_\phi T_{\xi^p\bar{\xi}^q\psi} = T_{\xi^p\bar{\xi}^q\psi}T_\phi$ *if and only if* $\psi(1^-) = 0$ *or* $\phi(1^-) = n\int_0^1 r^{2n-1}\phi(r)\mathrm{d}r$;

(2) If $q \succ p$, $T_\phi T_{\xi^p\bar{\xi}^q\psi} = T_{\xi^p\bar{\xi}^q\psi}T_\phi$ *if and only if* $\int_0^1 r^{2n+|q|-1}\psi(r)\mathrm{d}r = 0$ *or* $\phi(1^-) = n\int_0^1 r^{2n-1}\phi(r)\mathrm{d}r$;

(3) If $p \not\succeq q$ *and* $q \not\succeq p$ *or* $p = q$, $T_\phi T_{\xi^p\bar{\xi}^q\psi} = T_{\xi^p\bar{\xi}^q\psi}T_\phi$.

Corollary 2.11 *Let* $p, s \succ 0$ *with* $p \perp s$ *and* $\phi, \psi \in \Sigma'$. *Then* $T_{\phi(r)}T_{\xi^p\bar{\xi}^s\psi(r)} = T_{\xi^p\bar{\xi}^s\psi(r)}T_{\phi(r)}$.

Čučkovic and Rao [6] showed that a Toeplitz operator with radial symbol on the Bergman space of the unit disk may only commute with Toeplitz operators with radial symbols. Zhou and Dong [12] showed it is not true on the Bergman space of the unit ball. By Corollary 2.11, it is not surprise to see that it is neither true on the Dirichlet space of the unit ball.

In the end of this section, we characterize commuting Toeplitz operators with quasihomogeneous symbols.

Theorem 2.12 *Suppose* k_i *has the index decomposition* $k_i = p_i - s_i$ *with* $p_i \perp s_i$ *and* $\phi_i(1-) \ne 0, \phi_i \in \Sigma'$ *for* $i = 1, 2$. *Then* $T_{\xi^{k_1}\phi_1}T_{\xi^{k_2}\phi_2} = T_{\xi^{k_2}\phi_2}T_{\xi^{k_1}\phi_1}$ *if and only if one of the following conditions holds:*

(1) $k_1 = k_2$;

(2) $T_{\xi^{k_1}\phi_1} = \lambda I$;

(3) $T_{\xi^{k_2}\phi_2} = \lambda' I$;

(4) $s_1 = s_2 = 0$, $p_1 \succ 0, p_2 \succ 0$;

(5) $p_1 = s_1 = 0$, $p_2 \succ 0, s_2 \succ 0$;

(6) $p_2 = s_2 = 0$, $p_1 \succ 0, s_1 \succ 0$;

(7) $p_1 = p_2 = 0$, $s_1 \succ 0, s_2 \succ 0$, $\phi_2(1-)\int_0^1 r^{2n+|s_1|-1}\phi_1\mathrm{d}r = \phi_1(1-)\int_0^1 r^{2n+|s_2|-1}\phi_2\mathrm{d}r$;

(8) $p_1 = p_2 = s_1 = 0$, $s_2 \succ 0$, $\int_0^1 r^{2n+|s_2|-1}\phi_2\mathrm{d}r = 0$;

(9) $p_1 = p_2 = s_2 = 0$, $s_1 \succ 0$, $\int_0^1 r^{2n+|s_1|-1}\phi_1\mathrm{d}r = 0$.

Proof The inverse implication is clear. We only need to show the necessity. For those multi-indexes α with $\alpha - s_1 - s_2 \succ 0$, by Lemma 2.3 we have

$$T_{\xi^{k_1}\phi_1}T_{\xi^{k_2}\phi_2}z^\alpha = I_1\phi_1(1-)\phi_2(1-)z^{\alpha+p_1+p_2-s_1-s_2},$$
$$T_{\xi^{k_2}\phi_2}T_{\xi^{k_1}\phi_1}z^\alpha = I_2\phi_1(1-)\phi_2(1-)z^{\alpha+p_1+p_2-s_1-s_2},$$

where

$$I_1 = d(\alpha + p_2, \alpha + p_2 - s_2)d(\alpha + p_1 + p_2 - s_2, \alpha + p_1 + p_2 - s_1 - s_2),$$

$$I_2 = d(\alpha+p_1, \alpha+p_1-s_1)d(\alpha+p_1+p_2-s_1, \alpha+p_1+p_2-s_1-s_2).$$

Since $\phi_i(1-) \neq 0$, $T_{\xi^{k_1}\phi_1}T_{\xi^{k_2}\phi_2}z^\alpha = T_{\xi^{k_2}\phi_2}T_{\xi^{k_1}\phi_1}z^\alpha$ implies that $I_1 = I_2$, which is equivalent to

$$d(\alpha+p_1, \alpha+p_1-s_1) = d(\alpha+p_2, \alpha+p_2-s_2)d(\alpha+p_1+p_2-s_2, \alpha+p_1+p_2-s_1).$$

Note that

$$d(\alpha+p_1+p_2-s_2, \alpha+p_1+p_2-s_1) = d(\alpha+p_1+p_2-s_2, \beta)d(\beta, \alpha+p_1+p_2-s_1),$$

for $\beta = \alpha+p_1+p_2$ and

$$\frac{(\alpha+p-s)!}{(\alpha+p)!} = \frac{(\alpha-s)!}{(\alpha)!}, \quad \text{if } \ p \perp s.$$

By the expression of $d(\cdot,\cdot)$ the last equation is equivalent to $I_3 = I_4$, where

$$I_3 = \frac{(\alpha-s_1)!(\alpha+p_2)!(\alpha+p_1-s_2)!}{(\alpha-s_2)!(\alpha+p_1)!(\alpha+p_2-s_1)!},$$
$$I_4 = \frac{(n-1+|\alpha+p_2|)!(n-1+|\alpha+p_1-s_1|)!(n-1+|\alpha+p_1+p_2-s_2|)!}{(n-1+|\alpha+p_1|)!(n-1+|\alpha+p_2-s_2|)!(n-1+|\alpha+p_1+p_2-s_1|)!}.$$

Observe that I_3 depends on α while the I_4 only depends on $|\alpha|$ for fixed p_i, s_i. Since $n > 1$, it follows that $I_3 = I_4 = c$ for some constant c. Recall that I_3, I_4 are both composed of factorial function, c must be 1. That is, $I_3 = I_4 = 1$.

With varying α it is easy to see that $I_3 = 1$ holds if and only if one of the following conditions is fulfilled

(1) $s_1 = s_2, p_1 = p_2$;

(2) $s_1 = s_2 = 0$;

(3) $p_1 = p_2 = 0$;

(4) $s_1 = p_1 = 0$;

(5) $s_2 = p_2 = 0$.

Under each of the above conditions, the equation $I_4 = 1$ holds either. Next, we will discuss the commuting problem under either of the above five conditions, respectively.

Condition (1) contains four cases: $p_i > 0, s_i > 0$; $p_i = 0, s_i = 0$; $p_i > 0, s_i = 0$; $p_i = 0, s_i > 0$. There is no doubt that two Toeplitz operators commute in all cases above by using Corollaries 2.4 and 2.5, Theorems 2.7 and 2.8, respectively.

By Lemma 2.3, direct calculation shows that under condition (2), $T_{\xi^{k_1}\phi_1}$ commutes with $T_{\xi^{k_2}\phi_2}$ if and only if one of the following statements holds

(2.1) $p_1, p_2 \succ 0$;

(2.2) $p_1 = 0, p_2 = 0$;

(2.3) $p_1 = 0, p_2 \succ 0, T_{\xi^{k_1}\phi_1} = \phi_1(1-)I$;

(2.4) $p_1 \succ 0, p_2 = 0, T_{\xi^{k_2}\phi_2} = \phi_2(1-)I$.

By Lemma 2.3, direct calculation also shows that under condition (3) $T_{\xi^{k_1}\phi_1}$ commutes with $T_{\xi^{k_2}\phi_2}$ if and only if one of the following statements holds

(3.1) $s_1, s_2 \succ 0$, $\phi_2(1-)\int_0^1 r^{2n+|s_1|-1}\phi_1 dr = \phi_1(1-)\int_0^1 r^{2n+|s_2|-1}\phi_2 \mathrm{d}r$;

(3.2) $s_1 = 0, s_2 = 0$;

(3.3) $s_1 = 0, s_2 \succ 0, T_{\xi^{k_1}\phi_1} = \phi_1(1-)I$ or $\int_0^1 r^{2n+|s_2|-1}\phi_2 \mathrm{d}r = 0$;

(3.4) $s_1 \succ 0, s_2 = 0, T_{\xi^{k_2}\phi_2} = \phi_2(1-)I$ or $\int_0^1 r^{2n+|s_1|-1}\phi_1 \mathrm{d}r = 0$.

By Corollaries 2.4 and 2.5, direct calculation indicates that under condition (4) $T_{\xi^{k_1}\phi_1}$ commutes with $T_{\xi^{k_2}\phi_2}$ if and only if one of the following statements holds

(4.1) $p_2, s_2 \succ 0$;

(4.2) $p_2 = 0, s_2 = 0$;

(4.3) $p_2 = 0, s_2 \succ 0, T_{\xi^{k_1}\phi_1} = \phi_1(1-)I$ or $\int_0^1 r^{2n+|s_2|-1}\phi_2 \mathrm{d}r = 0$;

(4.4) $p_2 \succ 0, s_2 = 0, T_{\xi^{k_1}\phi_1} = \phi_1(1-)I$.

Similarly, under condition (5) $T_{\xi^{k_1}\phi_1}$ commutes with $T_{\xi^{k_2}\phi_2}$ if and only if one of the following statements holds

(5.1) $p_1, s_1 \succ 0$;

(5.2) $p_1 = 0, s_1 = 0$;

(5.3) $p_1 = 0, s_1 \succ 0, T_{\xi^{k_2}\phi_2} = \phi_2(1-)I$ or $\int_0^1 r^{2n+|s_1|-1}\phi_1 \mathrm{d}r = 0$;

(5.4) $p_1 \succ 0, s_1 = 0, T_{\xi^{k_2}\phi_2} = \phi_2(1-)I$.

The proof is finished by careful examination of all the conditions above.

Louhichi and Rao [10] discussed the commutativity of Toeplitz operators on the Bergman space on the unit disk and raised the conjecture: if two Toeplitz operators commute with a third one, none of them being the identity, then they commute with each other.

Zhou and Dong [14] found that the bicommutant conjecture of Louhichi and Rao is not correct on the Bergman space of $\mathbb{B}_n$. Almost at the same time, Vasilevski [13] showed the conjecture is wrong on the weighted Bergman space of the unit ball. In the following, we can show that this conjecture is also wrong when formulated for Toeplitz operators on the Dirichlet space of the unit ball.

Example 2.13 Given $n > 1$, consider the following three symbols

$$f_1 = r^2, \quad f_2 = \xi_1^1 \xi_2^{-1}, \quad f_3 = \xi_1^1 \xi_2^{-2}.$$

By Corollary 2.10, we can prove T_{f_1} commutes with both T_{f_2} and T_{f_3}, while by Theorem 2.12 the operators T_{f_2} and T_{f_3} do not commute.

3. Product of Toeplitz operators with quasihomogeneous symbols

In this section, we first study the finite rank product of Toeplitz operators with quasihomogeneous symbols. Čučkovic and Louhichi [11] proved that if the finite product of quasihomogeneous Toeplitz operators is of finite rank, then one of the symbols must be zero. The following result is a partial answer to the finite product problem for two Toeplitz operators on the Dirichlet Space of $\mathbb{B}_n$.

Theorem 3.1 *Suppose k_i has the index decomposition $k_i = p_i - s_i$ with $p_i \perp s_i$ and $\phi_i \in \Sigma'$ for $i = 1, \ldots, m$. Then the following statements are equivalent.*

(1) $T_{\xi^{k_1}\phi_1} \cdots T_{\xi^{k_m}\phi_m}$ is of finite rank.

(2) $\phi_{i_0}(1-) = 0$ *for some* $1 \le i_0 \le m$.

(3) $T_{\xi^{k_{i_0}}\phi_{i_0}}$ *is of finite rank for some* $1 \le i_0 \le m$.

Proof For simplicity, we denote $T = T_{\xi^{k_1}\phi_1} \cdots T_{\xi^{k_m}\phi_m}$. For multi-index $\alpha \succ s_1 + \cdots + s_m$, by Lemma 2.3 we calculate

$$T(z^\alpha) = C_\alpha \phi_1(1-) \cdots \phi_m(1-) z^{\alpha + k_1 + \cdots + k_m},$$

where C_α is a nonzero constant depending on α and k_i.

Thus the set $\{T(z^\alpha) : \alpha \succ s_1 + \cdots + s_m\}$ is a linearly independent set which is included in the range of T. The finite rank of T implies that $\phi_1(1-) \cdots \phi_m(1-) = 0$, which indicates that (1) is equivalent to (2).

It is easy to see the equivalence of (2) and (3) from Lemma 2.3.

Actually, in the proof of Theorem 3.1, it is obvious that if $T_{\xi^{k_1}\phi_1} \cdots T_{\xi^{k_m}\phi_m}$ is of finite rank, then the rank is less than 1.

By Lemma 2.3, it is easy to obtain the complete characterization of zero Toeplitz operators with quasihomogeneous symbols, which we list for frequent use.

Lemma 3.2 *Suppose* k *has the index decomposition* $k = p - s$ *with* $p \perp s$ *and* $\phi \in \Sigma'$. *Then the following statements hold.*

(1) If $p \succ 0$, $T_{\xi^k\phi} = 0$ *if and only if* $\phi(1-) = 0$;

(2) If $p = 0$, $T_{\xi^k\phi} = 0$ *if and only if* $\phi(1-) = 0$ *and* $\int_0^1 r^{2n+|s|-1}\phi \mathrm{d}r = 0$.

Theorem 3.3 *Suppose* k_i *has the index decomposition* $k_i = p_i - s_i$ *with* $p_i \perp s_i$ *and* $\phi_i \in \Sigma'$ *for* $i = 1, 2$. *Then* $T_{\xi^{k_1}\phi_1} T_{\xi^{k_2}\phi_2} = 0$ *if and only if one of the following statements holds:*

(1) $T_{\xi^{k_1}\phi_1} = 0$;

(2) $T_{\xi^{k_2}\phi_2} = 0$;

(3) $p_1 = 0, p_2 \succ 0, s_1 \not\succeq p_2, \phi_1(1-) = 0$;

(4) $p_2 = 0, s_1 \succ 0, \phi_2(1-) = 0$;

(5) $p_1 = p_2 = s_1 = 0, \phi_1(1-)\phi_2(1-) = 0$ *and* $\int_0^1 r^{2n-1}\phi_1 \mathrm{d}r \int_0^1 r^{2n-1+|s_2|}\phi_2 \mathrm{d}r = 0$.

Proof For simplicity, we denote $T_{\xi^{k_1}\phi_1} = T_1$, $T_{\xi^{k_2}\phi_2} = T_2$. By Theorem 3.1, $T_{\xi^{k_1}\phi_1} T_{\xi^{k_2}\phi_2} = 0$ implies $\phi_1(1-) = 0$ or $\phi_2(1-) = 0$. For two multi-indexes p_1, p_2, to prove this theorem, we need to consider four cases: case 1, $p_1, p_2 \succ 0$; case 2, $p_1 = 0, p_2 \succ 0$; case 3, $p_1 \succ 0, p_2 = 0$; case 4, $p_1 = 0, p_2 = 0$.

Case 1 If $p_1, p_2 \succ 0$, by Lemma 3.2, we have $T_{\xi^{k_1}\phi_1} T_{\xi^{k_2}\phi_2} = 0$ if and only if

$$\phi_1(1-) = 0 \ \text{ or } \ \phi_2(1-) = 0,$$

which is equivalent to that (1) or (2) holds.

Case 2 (a) If $p_1 = 0, p_2 \succ 0, s_1 = 0, s_2 \succeq 0$. For any multi-index $\alpha \succeq s_2$,

$$T_1 T_2 z^\alpha = T_{\phi_1} T_{\xi_2^p \phi_2} z^\alpha = d(\alpha + p_2, \alpha + p_2 - s_2)\phi_1(1-)\phi_2(1-) z^{\alpha + p_2 - s_2}.$$

In view of Corollary 2.4, it follows that $T_1T_2 = 0$ if and only if (2) holds or

$$\phi_1(1-) = 0. \tag{3.1}$$

(b) If $p_1 = 0, p_2 \succ 0, s_1 \succ 0, s_2 = 0$. For any multi-index α,

$$\begin{aligned} T_1T_2z^\alpha &= T_{\bar{\xi}^{s_1}\phi_1}T_{\xi^{p_2}\phi_2}z^\alpha \\ &= \begin{cases} \frac{2n!s_1!}{(n-1+|s_1|)!}\phi_2(1-)\int_0^1 r^{2n+|s_1|-1}\phi_1 \mathrm{d}r, & \alpha + p_2 = s_1; \\ d(\alpha+p_2, \alpha+p_2-s_1)\phi_2(1-)\phi_1(1-)z^{\alpha+p_2-s_1}, & \alpha+p_2 \succ s_1; \\ 0, & \text{else.} \end{cases} \end{aligned}$$

For $s_1 \succeq p_2$, $T_1T_2 = 0$ if and only if $\phi_1(1-) = 0$ and $\int_0^1 r^{2n+|s_1|-1}\phi_1 \mathrm{d}r = 0$ or $\phi_2(1-) = 0$. This is equivalent to that (1) or (2) holds.

For $s_1 \not\succeq p_2$, $T_1T_2 = 0$ if and only if $\phi_1(1-) = 0$ or $\phi_2(1-) = 0$, which is equivalent to that (2) holds or

$$\phi_1(1-) = 0. \tag{3.2}$$

(c) If $p_1 = 0, p_2 \succ 0, s_1 \succ 0, s_2 \succ 0$.

$$\begin{aligned} &T_1T_2z^\alpha = T_{\bar{\xi}^{s_1}\phi_1}T_{\xi_2^{p}\bar{\xi}^{s_2}\phi_2}z^\alpha \\ &= \begin{cases} d(\alpha+p_2, \alpha+k_2-s_1)\phi_1(1-)\phi_2(1-)z^{\alpha+k_2-s_1}, & \alpha+p_2-s_2-s_1 \succ 0 \text{ and } \alpha+p_2-s_2 \succeq 0; \\ \frac{2n!s_1!}{(n-1+|s_1|)!}\phi_2(1-)\int_0^1 r^{2n+|s_1|-1}\phi_1 \mathrm{d}r, & \alpha+p_2-s_2-s_1 = 0 \text{ and } \alpha+p_2-s_2 \succeq 0; \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Since $p_2 \perp s_2$, $s_1 + s_2 - p_2 \succeq 0$ is equivalent to $s_1 - p_2 \succeq 0$.

For $s_1 \succeq p_2$, $T_1T_2 = 0$ if and only if $\phi_1(1-) = 0$ and $\int_0^1 r^{2n+|s_1|-1}\phi_1 \mathrm{d}r = 0$ or $\phi_2(1-) = 0$ if and only if (1) or (2) holds.

For $s_1 \not\succeq p_2$, $T_1T_2 = 0$ if and only if $\phi_1(1-) = 0$ or $\phi_2(1-) = 0$, which is equivalent to that (2) holds or

$$\phi_1(1-) = 0. \tag{3.3}$$

Case 3 (a) If $p_1 \succ 0, p_2 = 0, s_1 = 0, s_2 \succeq 0$. For any multi-index α,

$$\begin{aligned} T_1T_2z^\alpha &= T_{\xi^{p_1}\phi_1}T_{\bar{\xi}^{s_2}\phi_2}z^\alpha \\ &= \begin{cases} \frac{2n!s_2!}{(n-1+|s_2|)!}\int_0^1 r^{2n+|s_2|-1}\phi_2 \mathrm{d}r\phi_1(1-)z^{\alpha+p_1-s_2}, & \alpha = s_2; \\ d(\alpha, \alpha-s_2)\phi_2(1-)\phi_1(1-)z^{\alpha+p_1-s_2}, & \alpha \succ s_2; \\ 0, & \text{else.} \end{cases} \end{aligned}$$

It follows that $T_1T_2 = 0$ if and only if (1) or (2) holds.

(b) If $p_1 \succ 0, p_2 = 0, s_1 \succ 0, s_2 = 0$. For any multi-index α, by Corollary 2.11 we have

$$T_1T_2z^\alpha = T_{\xi^{p_1}\bar{\xi}^{s_1}\phi_1}T_{\phi_2}z^\alpha = T_{\phi_2}T_{\xi^{p_1}\bar{\xi}^{s_1}\phi_1}z^\alpha.$$

Lemma 2.3 shows that $T_1T_2 = 0$ if and only if (1) holds or

$$\phi_2(1-) = 0. \tag{3.4}$$

(c) If $p_1 \succ 0, p_2 = 0, s_1 \succ 0, s_2 \succ 0$.

$$T_1T_2z^\alpha = T_{\xi^{p_1}\bar{\xi}^{s_1}\phi_1}T_{\bar{\xi}^{s_2}\phi_2}z^\alpha$$
$$= \begin{cases} d(\alpha,\alpha-s_2)d(\alpha-s_2+p_1,\gamma_\alpha)\phi_1(1-)\phi_2(1-)z^{\gamma_\alpha}, & \alpha \succ s_2 \text{ and } \gamma_\alpha \succ 0; \\ d(\alpha,\alpha-s_2)\phi_2(1-)\frac{2n!s_1!}{(n-1+|s_1|)!}\int_0^1 r^{2n-1+|s_1|}\phi_1 \mathrm{d}r, & \alpha \succ s_2 \text{ and } \gamma_\alpha = 0; \\ 0, & \text{else}, \end{cases}$$

where $\gamma_\alpha = \alpha - s_2 + p_1 - s_1$.

Note that $p_1 \perp s_1$, for multi-index β, $\beta - s_1 + p_1 \succ 0$ if and only if $\beta \succeq s_1$. Therefore, for multi-index α, $\alpha \succ s_2$ and $\gamma_\alpha \succ 0$ if and only if $\alpha \succeq s_1 + s_2$. On the other hand, it is impossible for multi-index α to satisfy both $\alpha \succ s_2$ and $\gamma_\alpha = 0$. That is,

$$T_1T_2z^\alpha = \begin{cases} d(\alpha,\alpha-s_2)d(\alpha-s_2+p_1,\gamma_\alpha)\phi_1(1-)\phi_2(1-)z^{\gamma_\alpha}, & \alpha \succeq s_1+s_2; \\ 0, & \text{else}, \end{cases}$$

which means that $T_1T_2 = 0$ if and only if (1) holds or

$$\phi_2(1-) = 0. \tag{3.5}$$

Case 4 $p_1 = 0, p_2 = 0$. With Lemma 2.3, by direct computation, we can get the following results. If $p_1 = 0, p_2 = 0, s_1 = 0, s_2 \succeq 0$. $T_1T_2 = 0$ if and only if (5) holds.

If $p_1 = 0, p_2 = 0, s_1 \succ 0, s_2 = 0$. $T_1T_2 = 0$ if and only if (1) holds or

$$\phi_2(1-) = 0. \tag{3.6}$$

If $p_1 = 0, p_2 = 0, s_1 \succ 0, s_2 \succ 0$. Theorem 2.9 shows that $T_1T_2 = 0$ if and only if (1) holds or

$$\phi_2(1-) = 0. \tag{3.7}$$

As we know, conditions (3.1)–(3.3) are equivalent to (3), while conditions (3.4)–(3.7) are equivalent to (4). The proof is completed. □

Finally, we consider when the product of two quasihomogeneous Toeplitz operators equals another such Toeplitz operator. Note that if $\phi \in \Sigma'$ and $\phi(1-) = 0$, the rank of $T_{\xi^k\phi}$ is less than 1. We only discuss Toeplitz operators with symbols $\phi(1-) \neq 0$.

Theorem 3.4 *Suppose k_i has the index decomposition $k_i = p_i - s_i$ with $p_i \perp s_i$ and $\phi_i \in \Sigma'$ with $\phi_i(1-) \neq 0$ for $i = 1, 2, 3$. Then $T_{\xi^{k_1}\phi_1} \cdot T_{\xi^{k_2}\phi_2} = T_{\xi^{k_3}\phi_3}$ if and only if the following statements hold:*

(1) $\phi_3(1-) = \phi_1(1-)\phi_2(1-)$;

(2) $k_3 = k_1 + k_2, p_i \perp s_j$ for $1 \le i, j \le 3$;

(3) Multi-indexes s_i, p_j satisfy one of the conditions below:

(i) $s_1 + s_2 \succ 0, s_2 = 0$;

(ii) $s_1 + s_2 \succ 0, p_1 = 0$;

(iii) $s_1 = s_2 = 0, p_2 \succ 0$;

(iv) $s_1 = s_2 = 0, p_2 = 0, p_1 \succ 0$ and $T_{\xi^{k_2}\phi_2} = \phi_2(1-)I$;

(v) $s_1 = s_2 = 0, p_1 = p_2 = 0$ and $\tilde{\phi}_1(2n)\tilde{\phi}_2(2n) = \tilde{\phi}_3(2n)$.

Proof First suppose s_1, s_2 are not both 0. For those multi-indexes α with $\alpha - s_1 - s_2 \succ 0$, by Lemma 2.3 we have

$$\begin{aligned} T_{\xi^{k_1}\phi_1} T_{\xi^{k_2}\phi_2} z^\alpha &= I_1 \phi_1(1-)\phi_2(1-) z^{\alpha+p_1+p_2-s_1-s_2}, \\ T_{\xi^{k_3}\phi_3} z^\alpha &= I_2 \phi_3(1-) z^{\alpha+p_3-s_3}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= d(\alpha+p_2, \alpha+p_2-s_2) d(\alpha+p_1+p_2-s_2, \alpha+p_1+p_2-s_1-s_2), \\ I_2 &= d(\alpha+p_3, \alpha+p_3-s_3). \end{aligned}$$

If $T_1 T_2 = T_3$, we have the following

$$p_1+p_2-s_1-s_2 = p_3 - s_3 \tag{3.8}$$

$$I_1 \phi_1(1-)\phi_2(1-) = I_2 \phi_3(1-). \tag{3.9}$$

By the definition of function $d(\cdot,\cdot)$, (3.9) is equivalent to

$$I_3 = C I_4, \tag{3.10}$$

where

$$\begin{aligned} I_3 &= \frac{(p_2+\alpha)!(p_1+p_2+\alpha-s_2)!}{(p_2+\alpha-s_2)!(p_3+\alpha)!}, \quad C = \frac{\phi_3(1-)}{\phi_1(1-)\phi_2(1-)}, \\ I_4 &= \frac{(n-1+|p_2+\alpha|)!(n-1+|p_1+p_2+\alpha-s_2|)!}{(n-1+|p_2+\alpha-s_2|)!(n-1+|p_3+\alpha|)!}. \end{aligned}$$

Observe that I_3 depends on α while I_4 only depends on $|\alpha|$, it follows from (3.10) that $I_3 = CI_4 = c$ for some constant c. Since both I_3, I_4 are composed of factorial function, it follows that

$$I_3 = I_4 = C = 1.$$

It is easy to see that $I_4 = 1$ holds if and only if

(i) $s_2 = 0$, $p_3 = p_1 + p_2, s_3 = s_1$; or

(ii) $|p_2| = |p_3|, p_1 = 0$.

Together with $I_3 = 1$ and (3.8), they are equivalent to

(i′) $s_2 = 0$, $p_3 = p_1 + p_2, s_3 = s_1$; or

(ii′) $p_1 = 0, p_3 = p_2, s_3 = s_1 + s_2$.

Next, we will discuss the above two cases, respectively.

If $s_2 = 0, p_3 = p_1 + p_2, s_3 = s_1$. Since $p_3 \perp s_3$ and $p_1 \perp s_3$, it follows that $p_2 \perp s_3$. Note that $s_1 + s_2 \succ 0$, by Lemma 2.3 for $\alpha \succeq s_1$, we get

$$\begin{aligned} T_1 T_2 z^\alpha &= T_{\xi^{p_1}\bar{\xi}^{s_1}\phi_1} T_{\xi^{p_2}\phi_2} z^\alpha \\ &= \phi_1(1-)\phi_2(1-) d(\alpha+p_1+p_2, \alpha+p_1+p_2-s_1) z^{\alpha+p_1+p_2-s_1} \\ &= \phi_3(1-) d(\alpha+p_3, \alpha+p_3-s_3) z^{\alpha+p_3-s_3} \\ &= T_3 z^\alpha, \end{aligned}$$

which implies that $T_{\xi^{k_1}\phi_1} \cdot T_{\xi^{k_2}\phi_2} = T_{\xi^{k_3}\phi_3}$. Thus we get (i).

If $p_1=0, p_3=p_2, s_3=s_1+s_2$. Since $p_2\perp s_3$ and $p_2\perp s_2$, it follows that $p_2\perp s_1$. Using Lemma 2.3 again, for $\alpha\succeq s_1+s_2$, we obtain

$$\begin{aligned}T_1T_2z^\alpha &= T_{\bar{\xi}^{s_1}\phi_1}T_{\xi^{p_2}\bar{\xi}^{s_2}\phi_2}z^\alpha\\ &=\phi_1(1-)\phi_2(1-)d(\alpha+p_2,\alpha+p_2-s_1-s_2)z^{\alpha+p_2-s_1-s_2}\\ &=\phi_3(1-)d(\alpha+p_3,\alpha+p_3-s_3)z^{\alpha+p_3-s_3}\\ &=T_3z^\alpha,\end{aligned}$$

which also implies that $T_{\xi^{k_1}\phi_1}\cdot T_{\xi^{k_2}\phi_2}=T_{\xi^{k_3}\phi_3}$. So (ii) is also obtained.

Now suppose s_1, s_2 are both 0. By the assumption, we have $T_{\xi^{p_1}\phi_1}T_{\xi^{p_2}\phi_2}=T_{\xi^{p_3}\bar{\xi}^{s_3}\phi_3}$. It is easy to see that $s_3=0$ and $p_1+p_2=p_3$.

If $p_2\succ 0$. It follows from Lemma 2.3 that $T_{\xi^{p_1}\phi_1}T_{\xi^{p_2}\phi_2}=T_{\xi^{p_3}\bar{\xi}^{s_3}\phi_3}$ if and only if $\phi_1(1-)\phi_2(1-) = \phi_3(1-)$, which leads to (iii).

If $p_2=0, p_1\succ 0$. The equation $T_1T_2=T_3$ turns into

$$T_{\xi^{p_1}\phi_1}T_{\phi_2}=T_{\xi^{p_1}\phi_3},$$

which is equivalent to $\phi_3(1-)=\phi_1(1-)\phi_2(1-)=\phi_1(1-)2n\int_0^1 r^{2n-1}\phi_2\mathrm{d}r$. That is, $T_1T_2=T_3$ if and only if $T_{\phi_2}=\phi_2(1-)I$ and $\phi_3(1-)=\phi_1(1-)\phi_2(1-)$. It comes to (iv).

If $p_2=0, p_1=0$. By Corollary 2.5, $T_1T_2=T_3$ if and only if $\tilde{\phi_1}(2n)\tilde{\phi_2}(2n)=\tilde{\phi_3}(2n)$ and $\phi_3(1-)=\phi_1(1-)\phi_2(1-)$. Therefore, (v) is achieved. The proof is completed. □

Example 3.5 Given $n>1$, consider the following three symbols

$$\begin{aligned}&k_1=(1,-1,0,\ldots,0),\ \phi_1=r^2,\\ &k_2=(1,0,0,\ldots,0),\ \phi_2=(r+1),\\ &k_3=(2,-1,0,\ldots,0),\ \phi_3=r(r+1).\end{aligned}$$

Then by Theorem 3.4, it is easy to see that $T_{\xi^{k_1}\phi_1}\cdot T_{\xi^{k_2}\phi_2}=T_{\xi^{k_3}\phi_3}$.

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