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Reflexivity of Weighted Shifts

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Abstract Let $\{\beta(n)\}_n$ be a sequence of positive numbers such that $\beta(0)=1$ and let $1 \le p < \infty$. We will investigate the reflexivity of all integer powers of the multiplication operator on the Banach spaces of formal Laurent series, $L^p(\beta)$.

Keywords Banach space of Laurent series associated with a sequence β; reflexive operator; weak operator topology.

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1. Introduction

First in the following, we generalize a definition from [1].

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers with $\beta(0)=1$ and $1 \le p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=-\infty}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$$

shall be used whether or not the series converges for any value of z. These are called formal Laurent series. Note that when n ranges on $\mathbb{N} \cup \{0\}$, they are called formal power series and are denoted by $H^p(\beta)$. Let $L^p(\beta)$ denote the space of such formal Laurent series, which is the reflexive Banach space with the norm $\|\cdot\|_\beta$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_{k\in\mathbb{Z}}$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. Now consider M_z, the operator of multiplication by z on $L^p(\beta)$:

$$(M_z f)(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^{n+1}$$

where

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n \in L^p(\beta).$$

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In other words, $(M_z f)\hat{}(n) = \hat{f}(n-1)$ for all $n \in \mathbb{Z}$. Clearly, M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case

$$\|M_z^n\| = \sup_k [\beta(k+n)/\beta(k)]$$

for all $n \in \mathbb{N} \cup \{0\}$. We denote the set of multipliers

$$\{\varphi \in L^p(\beta) : \ \varphi L^p(\beta) \subseteq L^p(\beta)\}$$

by $L^p_\infty(\beta)$ and the linear transformation of multiplication by φ on $L^p(\beta)$ by M_φ. The space $L^p_\infty(\beta)$ is a commutative Banach algebra with the norm $\|\varphi\|_\infty = \|M_\varphi\|$. The set of multipliers on $H^p(\beta)$ is also denoted by $H^p_\infty(\beta)$.

If λ is a complex number, then $e(\lambda)$ denotes the functional of evaluation at λ defined on Laurent polynomials p (that are finite combination of the vectors $\{f_n\}_{n=-\infty}^{\infty}$) by $e(\lambda)(p) = p(\lambda)$. Note that Laurent polynomials are dense in $L^p(\beta)$. We say that λ is a bounded point evaluation on $L^p(\beta)$ if the functional $e(\lambda)$ extends to a bounded linear functional on $L^p(\beta)$.

By the same method used in [2] we can see that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Also if

$$f(z) = \sum_n \hat{f}(n) z^n \in L^p(\beta)$$

and

$$g(z) = \sum_n \hat{g}(n) z^n \in L^q(\beta^{\frac{p}{q}}),$$

then clearly

$$\langle f, g \rangle = \sum_n \hat{f}(n) \overline{\hat{g}(n)} \beta(n)^p.$$

Some sources on formal power series are included in [1–6, 11–13].

If Ω is a domain in the complex plane $\mathbf{C}$, then by $H(\Omega)$ and $H^\infty(\Omega)$ we mean respectively the set of analytic functions and the set of bounded analytic functions on Ω. By $\|\cdot\|_\Omega$ we denote the supremum norm on Ω.

Let X be a Banach space. It is convenient and helpful to introduce the notation $\langle x, x^* \rangle$ to stand for $x^*(x)$, for $x \in X$ and $x^* \in X^*$. Also the set of bounded linear operators on X is denoted by $B(X)$. If $A \in B(X)$, by $\sigma(A)$ we mean the spectrum of A and by $r(A)$ we mean the spectral radius of A.

Recall that if $A \in B(X)$, then $\mathrm{Lat}(A)$ is by definition the lattice of all invariant subspaces of A, and $\mathrm{Alg\,Lat}(A)$ is the algebra of all operators B in $B(X)$ such that $\mathrm{Lat}(A) \subset \mathrm{Lat}(B)$. For the algebra $B(X)$, the weak operator topology (WOT) is the one induced by the family of seminorms

$$p_{x^*,x}(A) = |\langle Ax, x^* \rangle|$$

where $x \in X$, $x^* \in X^*$ and $A \in B(X)$. Hence $A_\alpha \longrightarrow A$ (WOT) if and only if $A_\alpha x \longrightarrow Ax$ weakly. Also similarly $A_\alpha \longrightarrow A$ (SOT) if and only if $A_\alpha x \longrightarrow Ax$ in the norm topology.

An operator A in $B(X)$ is said to be reflexive if $\mathrm{Alg\,Lat}(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $B(X)$ that contains A and the identity I and is closed in the weak operator topology. For some sources see [7–10, 13]. In [13], for all positive integers k, we investigate the reflexivity of M_{z^k} acting on the spaces of formal Laurent series, and here by a different method we extend this result for all $k \in \mathbf{Z}$.

2. Main result

In this section, we will investigate the reflexivity of the powers of the multiplication operator on $L^p(\beta)$.

In the following theorems we use the notations:

$$\begin{aligned} r_{01} &= \overline{\lim}\beta(-n)^{\frac{-1}{n}}, \\ \Omega_{01} &= \{z \in \mathbf{C} : |z| > r_{01}\}, \\ r_{11} &= \underline{\lim}\beta(n)^{\frac{1}{n}}, \\ \Omega_{11} &= \{z \in \mathbf{C} : |z| < r_{11}\}, \\ \Omega_1 &= \Omega_{01} \cap \Omega_{11}. \end{aligned}$$

Also, for each $\varphi \in H(\Omega_{11}) \cap L^p_\infty(\beta)$, put

$$P_n(\varphi) = \sum_{k=0}^{n}(1 - \frac{k}{n+1})\hat{\varphi}(k)z^k, \quad n \geq 0.$$

Note that $P_n(\varphi)$ is a polynomial and

$$\widehat{P_n(\varphi)}(j) = (1 - \frac{j}{n+1})\hat{\varphi}(j)$$

whenever $j = 0, \ldots, n$ and is 0 else.

The following theorem extends the results obtained by Shields (for the case $p = 2$) in [1] and by similarity we omit the proof.

Theorem 2.1 *If $\varphi \in H(\Omega_{11}) \cap L^p_\infty(\beta)$, then $M_{P_n(\varphi)} \to M_\varphi$ in the weak operator topology.*

Theorem 2.2 *Let M_z be invertible on $L^p(\beta)$. If $r_{01} \leq r_{11}$, then M_{z^k} is reflexive for all integers k.*

Proof Let $A \in \operatorname{Alg}\operatorname{Lat}(M_{z^k})$. Since $\operatorname{Lat}(M_z) \subset \operatorname{Lat}(M_{z^k})$, we have $\operatorname{Lat}(M_z) \subset \operatorname{Lat}(A)$. This implies that $A \in \operatorname{Alg}\operatorname{Lat}(M_z)$. Note that since

$$M_z^* e(\lambda) = \lambda e(\lambda)$$

for all λ in Ω_1, the one dimensional span of $e(\lambda)$ is invariant under M_z^*. Therefore, it is invariant under A^* and we can write

$$A^* e(\lambda) = \varphi(\lambda)e(\lambda), \quad \lambda \in \Omega_1.$$

So

$$\langle Af, e(\lambda)\rangle = \langle f, A^* e(\lambda)\rangle = \varphi(\lambda)f(\lambda)$$

for all $f \in L^p(\beta)$ and $\lambda \in \Omega_1$. This implies that $A = M_\varphi$ and $\varphi \in L^p_\infty(\beta) \cap H^\infty(\Omega_1)$. Set

$$\mathcal{M} = L^p(\beta) \cap H^\infty(\Omega_{11}).$$

Since $L^p(\beta)$ contains the constants, $\mathcal{M} \neq \{0\}$. Clearly, every function in $\mathcal{M}$ is analytic in Ω_{11}. Now we show that $\mathcal{M}$ is a closed invariant subspace of $L^p(\beta)$ that is invariant under M_z. To see this, let $\{\varphi_n\}_n$ be a sequence in $\mathcal{M}$ such that φ_n converges to h in $L^p(\beta)$. Note that $L^p(\beta) \subset H(\Omega_1)$. By applying the Cauchy integral formula we can write $h = h_1 + h_2$ where $h_1 \in H(\Omega_{11})$ and $h_2 \in H_0(\Omega_{01})$ ($H_0(\Omega_{01})$ denotes the space of all functions in $H(\Omega_{01})$ that vanishes at ∞). Now $\varphi_n - h_1$ converges uniformly to h_2 on compact subsets of Ω_1 and so $z^k(\varphi_n - h_1)$ converges uniformly to $z^k h_2$ on compact subsets of Ω_1 $(k = 0, 1, \ldots)$. Since

$$z^k(\varphi_n - h_1) \in H(\Omega_{11}),$$

we have

$$\int_{\gamma_0} \xi^k(\varphi_n(\xi) - h_1(\xi))\mathrm{d}\xi = 0, \quad k = 0, 1, 2, \ldots; \; n = 1, 2, \ldots$$

where

$$\gamma_0 = \{z : |z| = r_{11} - \varepsilon > r_{01}\}$$

for some $\varepsilon > 0$. Choose the circle γ_0' sufficiently close to γ_0 with smaller radius so that γ_0 lies in $\mathrm{ext}(\gamma_0')$. We can write

$$h_2(z) = \sum_{n=-\infty}^{-1} a_n z^n, \quad z \in \mathrm{ext}(\gamma_0')$$

$$a_n = \frac{1}{2\pi i} \int_{\gamma_0} h_2(\xi)\mathrm{d}\xi / \xi^{n+1}, \quad n < 0.$$

But

$$\int_{\gamma_0} \xi^k h_2(\xi)\mathrm{d}\xi = 0, \quad k = 0, 1, 2, \ldots.$$

From this it follows that $h_2(z) = 0$, $z \in \mathrm{ext}(\gamma_0')$. Hence $h \equiv 0$. Therefore, $h = h_0$ is analytic on Ω_{11} and so $\mathcal{M}$ is closed and is invariant under M_z. Since $A\mathcal{M} \subset \mathcal{M}$ and $1 \in \mathcal{M}$, we see that

$$A1 = \varphi \in \mathcal{M} \subset H^\infty(\Omega_{11}).$$

Hence

$$\varphi \in L^p_\infty(\beta) \cap H^\infty(\Omega_{11}) \subset L^p_\infty(\beta) \cap H(\Omega_{11}).$$

Thus by Theorem 2.1, $M_{P_n(\varphi)} \to M_\varphi$ in the weak operator topology. Now let $\mathcal{M}_k$ be the closed linear span of the set $\{f_{nk} : n \geq 0\}$. We have

$$M_{z^k} f_{nk} = f_{(n+1)k} \in \mathcal{M}_k$$

for all $n \geq 0$. Thus $\mathcal{M}_k \in \mathrm{Lat}(M_{z^k})$, and so $\mathcal{M}_k \in \mathrm{Lat}(M_\varphi)$. Let

$$\varphi(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n.$$

Since $1 \in \mathcal{M}_k$,

$$M_\varphi 1 = \varphi \in \mathcal{M}_k.$$

Hence $\hat{\varphi}(i) = 0$ for all $i \neq nk$, $n \geq 0$. Now, by the particular construction of $P_n(\varphi)$ relative to φ, each $P_n(\varphi)$ should be a polynomial in z^k, i.e., $P_n(\varphi)(z) = q_n(z^k)$ for some polynomial q_n. Thus

$$M_{P_n(\varphi)} = P_n(\varphi)(M_z) = q_n(M_{z^k}) \to A$$

in the weak operator topology. Hence $A \in W(M_{z^k})$. Thus M_{z^k} is reflexive for all positive integers k.

Note that since $M_z f_m = f_{m+1}$, we have $M_z^{-1} f_m = f_{m-1}$ for all m. Let $f'_m = f_{-m}$. Then $M_z^{-1} f'_m = f'_{m+1}$ for all m. So $\{f'_m\}$ is shifted (forward) by M_z^{-1}. Hence by the above discussion M_z^{-k} is reflexive for all $k \geq 1$. But the identity operator is also reflexive, thus indeed M_{z^k} is reflexive for all integers k. Now the proof is completed. □

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