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Admissible Linear Estimators of Multivariate Regression Coefficient with Respect to an Inequality Constraint under Balanced Loss Function

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Abstract In this paper, the admissibility of multivariate linear regression coefficient with respect to an inequality constraint under balanced loss function is investigated. Necessary and sufficient conditions for admissible homogeneous and inhomogeneous linear estimators are obtained, respectively.

Keywords admissibility; inequality constraint; balanced loss function; homogeneous (inhomogeneous) linear estimator.

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1. Introduction

We adopt the following notations: for a matrix A, $\mu(A)$, $\mathrm{tr}(A)$, $\mathrm{rank}(A)$, A', A^-, A^+ and $\mathrm{Vec}(A)$ denote the range space, trace, rank, transpose, g-inverse, Moore-Penrose inverse and usual column-stacking of A, respectively. The $n\times n$ identity matrix is denoted by I_n. For non-negative definite matrix A and B, $A\geq B$ ($A>B$) means that $A-B$ is non-negative (positive) definite. $A\otimes B$ denotes the Kronecker product of A and B, $R^{m\times n}$ stands for the set composed of all $m\times n$ real matrices.

Consider the following general multivariate linear model

$$Y=X\Theta+\varepsilon,\quad \mathrm{E}(\varepsilon)=0,\quad \mathrm{Cov}(\mathrm{Vec}(\varepsilon))=\Sigma\otimes V, \tag{1.1}$$

where $Y\in R^{n\times q}$ is observable, $X\in R^{n\times p}$ is a design matrix, ε is an $n\times q$ matrix of random errors with zero mean, $V\geq 0$ is known, Σ and $\Theta\in R^{p\times q}$ are unknown parameter matrices.

In practical situations, there is usually some prior information on the parameters of the model. In other words, the model parameters need to satisfy certain constraint conditions. Common constraints in the literature include ellipsoid constraints and inequality constraints. In

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particular, statistical inference problems with inequality constraints have received much attention recently. In this paper, we discuss the admissibility of linear estimators in model (1.1) with respect to the following inequality constraint

$$\mathcal{T}=\{(\Theta,\Sigma)\mid \operatorname{tr}(H'\Theta)\geq 0,\Sigma\geq 0\},\tag{1.2}$$

where $H\in R^{p\times q}$ is known. It should be pointed out that references [1–7] have considered similar problems with some inequality constraints.

Different from the aforementioned references, the loss function considered in this paper is a kind of balanced loss function. Balanced loss function was originally proposed by Zellner [8], which takes error of estimation and goodness-of-fit into account. Therefore, it is a more comprehensive and reasonable measure of efficiency compared with the usual quadratic loss or matrix loss function.

According to Zellner's thought, we use the following balanced loss function:

$$L(d(Y);\Theta,\Sigma)=\operatorname{tr}[\omega(Y-Xd(Y))'T^{+}(Y-Xd(Y))+(1-\omega)(d(Y)-\Theta)'S(d(Y)-\Theta)],\tag{1.3}$$

where $\omega\in[0,1]$ and $S>0$ are known, $T=V+XX'$, and $d(Y)$ is an estimator of Θ. The corresponding risk is

$$R(d(Y);\Theta,\Sigma)=\mathrm{E}L(d(Y);\Theta,\Sigma).$$

Note that quadratic loss function can be considered as a special case of (1.3), so the results in this paper generalize some related results in [2–5].

In order to estimate Θ, we consider the class of the homogeneous linear estimators and the inhomogeneous linear estimators:

$$\mathcal{LH}=\{AY:A\in R^{p\times n}\},\quad \mathcal{LI}=\{AY+A_1:A\in R^{p\times n},A_1\in R^{p\times q}\}.$$

Definition 1.1 *Let $d(Y)$, $d_1(Y)$ and $d_2(Y)$ be three estimators of Θ. Then $d_1(Y)$ is said to be better than $d_2(Y)$ if*

$$R(d_1(Y);\Theta,\Sigma)\leq R(d_2(Y);\Theta,\Sigma)$$

for all $(\Theta,\Sigma)\in\mathcal{T}$ with strict inequality holding for at least one point. If there does not exist any better estimator which is better than $d(Y)$ in the class of some estimators φ, then $d(Y)$ will be said to be admissible in φ, which is denoted by $d(Y)\overset{\varphi}{\sim}\Theta(\mathcal{T})$.

In this paper, the admissibility for linear estimators of regression coefficient Θ in model (1.1) under the balanced loss function (1.3) is studied, and necessary and sufficient conditions for admissible linear estimators in $\mathcal{LH}$ and $\mathcal{LI}$ are obtained, respectively.

2. Admissible linear estimators in $\mathcal{LH}$

Lemma 2.1 *For the multivariate linear model*

$$Z=(X'T^{+}X)\Theta+\varepsilon,\quad \mathrm{E}(\varepsilon)=0,\quad \operatorname{Cov}(\operatorname{Vec}(\varepsilon))=\sigma^2 I_q\otimes(X'T^{+}X)(I-(X'T^{+}X)),$$

denote $B=\omega X'T^{+}X+(1-\omega)S$, $C=(1-\omega)B^{-1}S$, $L_1=\{AZ:A\in R^{p\times p}\}$, $L_B(d;C\Theta,\sigma^2)=\operatorname{tr}[(d-C\Theta)'B(d-C\Theta)]$ and $\widehat{A}=A-\omega B^{-1}$. Suppose that $C\Theta$ is estimable, then under the loss *function* $L_B(d; C\Theta, \sigma^2)$, $\widehat{A}Z \overset{\mathcal{LH}}{\sim} C\Theta$ *if and only if* $AX'T^+Y \overset{\mathcal{LH}}{\sim} \Theta$ *under the loss function (1.3).*

Proof The proof is similar to Lemma 4 in [9], so we omit it. □

Lemma 2.2 ([5]) *In the class of homogeneous linear estimators* $\mathcal{LH}$*, for model (1.1) and loss function* $L(d;\Theta,\Sigma) = \mathrm{tr}[(d - S\Theta)'(d - S\Theta)]$*, suppose that* $S\Theta$ *is estimable, then* $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$ *if and only if*

(1) $AV = AX(X'T^+X)^-X'T^+V$;

(2) $AX((X'T^+X)^- - I)S' \geq AX((X'T^+X)^- - I)X'A'$;

(3) $\mathrm{rank}[(AX - S)((X'T^+X)^- - I)X'] = \mathrm{rank}(AX - S)$.

Lemma 2.3 *Let* $L_0 = \{AX'T^+Y : A \in R^{p\times p}\}$*. Then under the loss function (1.3),* L_0 *is a class of completeness about* $\mathcal{LH}$*.*

Proof Denote $P_X = X(X'T^+X)X'T^+$, then $AP_XY \in L_0$. For every $(\Theta,\Sigma) \in \mathcal{T}$, we have

$$R(AY;\Theta,\Sigma) - R(AP_XY;\Theta,\Sigma) = \mathrm{tr}[A(I_n - P_X)V(I_n - P_X)'A'B]\cdot \mathrm{tr}(\Sigma) \geq 0.$$

Moreover, the equality holds if and only if $AV = AP_XV$. The proof of Lemma 2.3. is completed. □

Lemma 2.4 *For model (1.1) and loss function (1.3),* $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$ *if and only if* $AY \overset{\mathcal{LH}}{\sim} \Theta$.

Proof The sufficiency is straightforward, so we need only to prove the necessity. In fact, suppose on the contrary that $AY \overset{\mathcal{LH}}{\nsim} \Theta$, then there exists $DY \in \mathcal{LH}$ such that

$$\begin{aligned} &R(AY;\Theta,\Sigma)\\ &\quad= [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(DVD'B) - 2\omega\mathrm{tr}(XDVT^+)]\mathrm{tr}(\Sigma) + \mathrm{tr}[\Theta'(DX - I)'B(DX - I)\Theta]\\ &\quad\leq [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(AVA'B) - 2\omega\mathrm{tr}(XAVT^+)]\mathrm{tr}(\Sigma) + \mathrm{tr}[\Theta'(AX - I)'B(AX - I)\Theta]\\ &\quad= R(DY;\Theta,\Sigma)\end{aligned}$$

for all $(\Theta,\Sigma) \in \mathcal{T}^*$, where $\mathcal{T}^* = \{(\Theta,\Sigma)|\mathrm{tr}(H'\Theta) < 0, \Sigma \geq 0\}$ is the dual cone of $\mathcal{T}$, and the strict inequality holds for at least one point. Let $\Theta_1 = -\Theta$, then $(\Theta_1,\Sigma) \in \mathcal{T}$ and

$$\begin{aligned} &R(AY;\Theta,\Sigma)\\ &\quad= [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(DVD'B) - 2\omega\mathrm{tr}(XDVT^+)]\mathrm{tr}(\Sigma) + \mathrm{tr}[\Theta_1'(DX - I)'B(DX - I)\Theta_1]\\ &\quad\leq [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(AVA'B) - 2\omega\mathrm{tr}(XAVT^+)]tr(\Sigma) + \mathrm{tr}[\Theta_1'(AX - I)'B(AX - I)\Theta_1]\\ &\quad= R(DY;\Theta,\Sigma)\end{aligned}$$

with strict inequality holding for at least one point, which contradicts $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$. Therefore, $AY \overset{\mathcal{LH}}{\sim} \Theta$. □

Theorem 2.1 *In the class of homogeneous linear estimators* $\mathcal{LH}$*, for model (1.1) and loss function (1.3),* $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$ *if and only if*

(1) $AV = AP_XV$;

(2) $(1-\omega)\widetilde{A}X[(X'T^+X)^- - I]SB^{-1} \geq \widetilde{A}X[(X'T^+X)^- - I]X'\widetilde{A}'$;

(3) $\mathrm{rank}[(AX-I)((X'T^+X)^- - I)X'T^+X] = \mathrm{rank}(AX-I)$,
where $P_X = X(X'T^+X)^-X'T^+$, $B = \omega'T^+X + (1-\omega)S$, $\widetilde{A} = A - \omega B^{-1}X'T^+$.

Proof First, by Lemma 2.1, under the loss function (1.3), $AX'T^+Y \overset{\mathcal{LH}}{\sim} \Theta$ holds if and only if $\widehat{A}Z \overset{\mathcal{LH}}{\sim} C\Theta$ under the loss function $L_B(d;C\Theta,\sigma^2) = \mathrm{tr}[(d-C\Theta)'B(d-C\Theta)]$. Then from Lemma 2.2 in [10], we get that under the loss function $L_B(d;C\Theta,\sigma^2) = \mathrm{tr}[(d-C\Theta)'B(d-C\Theta)]$, $\widehat{A}Z \overset{\mathcal{LH}}{\sim} C\Theta$ holds if and only if $\widehat{A}Z \overset{\mathcal{LH}}{\sim} C\Theta$ under the loss function $L(d;C\Theta,\sigma^2) = \mathrm{tr}[(d-C\Theta)'(d-C\Theta)]$.

By Lemma 2.2, under the loss function (1.3), $AX'T^+Y \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$ holds if and only if

$$\widehat{A}(X'T^+X)(I-(X'T^+X)) = \widehat{A}X'T^+X(I-(X'T^+X)), \tag{2.1}$$

$$\widehat{A}X'T^+X((X'T^+X)^- - I)C' \geq \widehat{A}X'T^+X((X'T^+X)^- - I)X'T^+X\widehat{A}', \tag{2.2}$$

$$\mathrm{rank}(\widehat{A}X'T^+X - C)((X'T^+X)^- - I)X'T^+X = \mathrm{rank}(\widehat{A}X'T^+X - C). \tag{2.3}$$

Note that $\widehat{A} = A - \omega B^{-1}$ and $C = (1-\omega)B^{-1}S$, thus (2.2) and (2.3) become

$$\begin{aligned}&(1-\omega)(AX'T^+ - \omega B^{-1}X'T^+)X((X'T^+X)^- - I)SB^{-1}\\ &\quad \geq (AX'T^+ - \omega B^{-1}X'T^+)X((X'T^+X)^- - I)X'(AX'T^+ - \omega B^{-1})'\end{aligned} \tag{2.4}$$

and

$$\mathrm{rank}[(AX'T^+X - I)((X'T^+X)^- - I)X'T^+X] = \mathrm{rank}(AX'T^+X - I). \tag{2.5}$$

From Lemma 2.3, $R(AY;\Theta,\Sigma) = R(AP_XY;\Theta,\Sigma)$ if and only if $AV = AP_XV$, and then by Lemma 2.4, we obtain that under the loss function (1.3), $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$ holds if and only if (1), (2) and (3) hold simultaneously. This completes the proof of Theorem 2.1. □

3. Admissible linear estimators in $\mathcal{LI}$

Lemma 3.1 ([5]) *If* $\mathrm{tr}(K'\Theta) \geq 0$ *for all* $\Theta \in \{\Theta | \mathrm{tr}(H'\Theta) \geq 0\}$, *then there exists a real constant* $\lambda \geq 0$, *such that* $K = \lambda H$.

Theorem 3.1 *In the class of inhomogeneous linear estimators* $\mathcal{LI}$, *for model (1.1) and loss function (1.3), if* $AY + A_1 \overset{\mathcal{LI}}{\sim} \Theta(\mathcal{T})$, *then*

(1) $A_1 \in \mu(AX-I)$;
(2) $\mathrm{tr}(H'(AX-I)^+A_1) \leq 0$ *or* $\mu(H) \nsubseteq \mu[(B^{1/2}(AX-I))']$;
(3) $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$,

where $B = \omega X'T^+X + (1-\omega)S$.

Proof (1) Let P be an orthogonal projection matrix onto $\mu(B^{1/2}(AX-I))$, and $C = B^{-1/2}PB^{1/2}A_1$, then $C \in \mu(AX-I)$. Note that

$$\begin{aligned}&R(AY + A_1;\Theta,\Sigma) - R(AY + C;\Theta,\Sigma)\\ &\quad = \mathrm{tr}[((AX-I)\Theta + A_1)'B((AX-I)\Theta + A_1) - ((AX-I)\Theta + C)'B((AX-I)\Theta + C)]\\ &\quad = \mathrm{tr}[A_1'BA_1 - C'BC + A_1'B(AX-I)\Theta - C'B(AX-I)\Theta]\\ &\quad = \mathrm{tr}(A_1'BA_1 - C'BC) = \mathrm{tr}(A_1'B^{1/2}(I-P)B^{1/2}A_1) \geq 0.\end{aligned} \tag{3.1}$$

Suppose $A_1 \notin \mu(AX-I)$, then the strict inequality in (3.1) holds. In fact, if $A_1 \notin \mu(AX-I)$, then $A_1 = A_{11} + A_{12}$, where $A_{11} \in \mu(AX-I)$, $A_{12} \in \mu^{\perp}(AX-I)$ and $A_{12} \neq 0$. Thus, $B^{1/2}A_{11} \in \mu(B^{1/2}(AX-I))$, $A_{12} \in \mu^{\perp}(B^{1/2}(AX-I))$, and (3.1) becomes $\mathrm{tr}(A_{12}'B^{1/2}(I-P)B^{1/2}A_{12}) > 0$. It follows that $AY+C$ is better than $AY+A_1$ in the class of $\mathcal{LI}$, which contradicts $AY+A_1 \overset{\mathcal{LI}}{\sim} \Theta(\mathcal{T})$. Hence, $A_1 \in \mu(AX-I)$.

(2) Suppose, by contraction, that H is such that $\mathrm{tr}(H'(AX-I)^+A_1) > 0$ and $\mu(H) \subseteq \mu[(B^{1/2}(AX-I))']$. Write $H = (B^{1/2}(AX-I))'(B^{1/2}(AX-I))H_0$ for some H_0, let $C = (AX-I)^+A_1 - \lambda H_0$, where $\lambda > 0$. Then for all $(\Theta, \Sigma) \in \mathcal{T}$, we have

$$\begin{aligned} &R(AY+(AX-I)C;\Theta,\Sigma) - R(AY+A_1;\Theta,\Sigma) \\ &\quad = \lambda^2\mathrm{tr}[H_0'(AX-I)'B(AX-I)H_0] - 2\lambda\mathrm{tr}(H'\Theta) - 2\lambda\mathrm{tr}[H'(AX-I)^+A_1]. \end{aligned} \tag{3.2}$$

Since $\mathrm{tr}(H'(AX-I)^+A_1) > 0$ and $\mathrm{tr}(H'\Theta) \geq 0$, it follows that for λ sufficiently small,

$$R(AY+(AX-I)C;\Theta,\Sigma) - R(AY+A_1;\Theta,\Sigma) < 0, \quad \forall(\Theta,\Sigma) \in \mathcal{T}.$$

Thus, $AY+(AX-I)C$ is better than $AY+A_1$, which contradicts the assumption. Hence, we have $\mathrm{tr}(H(AX-I)^+A_1) \leq 0$ or $\mu(H) \nsubseteq \mu[(B^{1/2}(AX-I))']$.

(3) According to part (1), we can write $A_1 = (AX-I)A_0$ for some A_0. Suppose that DY is better than AY. Similarly to Lemma 2.2 in [2], we can get

$$\mathrm{tr}(DVD'B) - 2\omega\mathrm{tr}(XDVT^+) \leq \mathrm{tr}(AVA'B) - 2\omega\mathrm{tr}(XAVT^+),$$

$$\mathrm{tr}[(DX-I)'B(DX-I)] \leq \mathrm{tr}[(AX-I)'B(AX-I)].$$

Therefore, for all $(\Theta,\Sigma) \in \mathcal{T}$,

$$\begin{aligned} &R(DY+(DX-I)A_0;\Theta,\Sigma) \\ &\quad = [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(DVD'B) - 2\omega\mathrm{tr}(XDVT^+)]\mathrm{tr}(\Sigma) + \\ &\qquad \mathrm{tr}[(\Theta+A_0)'(DX-I)'B(DX-I)(\Theta+A_0)] \\ &\quad \leq [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(AVA'B) - 2\omega\mathrm{tr}(XAVT^+)]\mathrm{tr}(\Sigma) + \\ &\qquad \mathrm{tr}[(\Theta+A_0)'(AX-I)'B(AX-I)(\Theta+A_0)] \\ &\quad = R(AY+A_1;\Theta,\Sigma). \end{aligned} \tag{3.3}$$

Since $AY+A_1 \overset{\mathcal{LI}}{\sim} \Theta(\mathcal{T})$, the equality in (3.3) holds for all $(\Theta,\Sigma) \in \mathcal{T}$. Replacing Σ by $\lambda^2\Sigma$ and Θ by $\lambda\Theta$ ($\lambda > 0$), then multiplying $1/\lambda^2$ to both sides of (3.3) and letting $\lambda \to \infty$, we have

$$\begin{aligned} &R(DY;\Theta,\Sigma) \\ &\quad = [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(DVD'B) - 2\omega\mathrm{tr}(XDVT^+)]\mathrm{tr}(\Sigma) + \mathrm{tr}[\Theta'(DX-I)'B(DX-I)\Theta] \\ &\quad = [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(AVA'B) - 2\omega\mathrm{tr}(XAVT^+)]\mathrm{tr}(\Sigma) + \mathrm{tr}[\Theta'(AX-I)'B(AX-I)\Theta] \\ &\quad = R(AY;\Theta,\Sigma). \end{aligned}$$

This means that there exists no homogeneous linear estimator which is better than AY on $\mathcal{T}$. Therefore, $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$. $\square$

Theorem 3.2 *In the class of inhomogeneous linear estimators $\mathcal{LI}$, for model (1.1) and loss function (1.3), $AY + A_1 \overset{\mathcal{LI}}{\sim} \Theta(\mathcal{T})$ holds if and only if*

(1) $A_1 \in \mu(AX - I)$;

(2) $\mathrm{tr}[H'(AX - I)^+ A_1] \le 0$ *or* $\mu(H) \nsubseteq \mu[(B^{1/2}(AX - I))']$;

(3) $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$,

where B is the same as that in Theorem 3.1.

Proof By Theorem 3.1, we need only to establish the sufficiency. Suppose that $DY+(DX-I)D_0$ is better than $AY + A_1 = AY + (AX - I)A_0$, then for all $(\Theta, \Sigma) \in \mathcal{T}$,

$$\begin{aligned}
&R(DY + (DX - I)D_0; \Theta, \Sigma) \\
&\quad = [\omega\mathrm{tr}(VT^+) + \mathrm{tr}(AVA'B) - 2\omega\mathrm{tr}(XAVT^+)]\mathrm{tr}(\Sigma)+ \\
&\qquad \mathrm{tr}[((AX - I)\Theta + A_1)'B((AX - I)\Theta + A_1)] \\
&\quad \le [\omega\mathrm{tr}(VT^+) + tr(AVA'B) - 2\omega\mathrm{tr}(XAVT^+)]\mathrm{tr}(\Sigma)+ \\
&\qquad \mathrm{tr}[((AX - I)\Theta + A_1)'B((AX - I)\Theta + A_1)]. \\
&\quad = R(AY + A_1; \Theta, \Sigma).
\end{aligned} \tag{3.4}$$

Replacing Σ by $\lambda\Sigma(\lambda > 0)$, and letting $\lambda \to \infty$, we get

$$\omega\mathrm{tr}(VT^+) + \mathrm{tr}(DVD'B) - 2\omega\mathrm{tr}(XDVT^+) \le \omega\mathrm{tr}(VT^+) + \mathrm{tr}(AVA'B) - 2\omega\mathrm{tr}(XAVT^+).$$

On the other hand, replacing Σ by $\lambda\Sigma(\lambda > 0)$ and Θ by $k\Theta(k > 0)$, then letting $\lambda \to 0$ and $k \to \infty$, we have

$$\mathrm{tr}[(DX - I)'B(DX - I)] \le \mathrm{tr}[(AX - I)'B(AX - I)].$$

Since $AY \overset{\mathcal{LH}}{\sim} \Theta(\mathcal{T})$, we conclude that

$$R(DY; \Theta, \Sigma) = R(AY; \Theta, \Sigma).$$

Consequently,

$$\begin{aligned}
\omega\mathrm{tr}(VT^+) + \mathrm{tr}(DVD'B) - 2\omega\mathrm{tr}(XDVT^+) &= \omega\mathrm{tr}(VT^+) + \mathrm{tr}(AVA'B) - 2\omega\mathrm{tr}(XAVT^+), \\
\mathrm{tr}[(DX - I)'B(DX - I)] &= \mathrm{tr}[(AX - I)'B(AX - I)].
\end{aligned} \tag{3.5}$$

It follows from (3.4) that, for all $(\Theta, \Sigma) \in \mathcal{T}$

$$\begin{aligned}
&\mathrm{tr}[D_0'(AX - I)'B(AX - I)D_0] - \mathrm{tr}(A_1'BA_1) \\
&\quad \le -2\mathrm{tr}[\Theta'(AX - I)'B(AX - I)(D_0 - (AX - I)^+A_1)].
\end{aligned} \tag{3.6}$$

Let $\Theta = 0$. Then

$$\mathrm{tr}[D_0'(AX - I)'B(AX - I)D_0] - \mathrm{tr}(A_1'BA_1) \le 0 \tag{3.7}$$

and for all $(\Theta, \Sigma) \in \mathcal{T}$,

$$-2\mathrm{tr}[\Theta'(AX - I)'B(AX - I)(D_0 - (AX - I)^+A_1)] \ge 0.$$

By Lemma 3.1, there exists $\lambda \ge 0$, such that

$$(AX - I)'B(AX - I)(D_0 - (AX - I)^+A_1) = -\lambda H. \tag{3.8}$$

If $\lambda = 0$, then $(AX-I)'B(AX-I)D_0 = (AX-I)'BA_1$, and

$$(AX-I)D_0 = A_1. \tag{3.9}$$

If $\lambda > 0$, then from (3.8) we get $\mu(H) \subseteq \mu[(B^{1/2}(AX-I))']$. By condition (2),

$$\mathrm{tr}[H'(AX-I)^+A_1] \le 0. \tag{3.10}$$

Furthermore, it follows from (3.8) that

$$\begin{aligned} &-\lambda[\mathrm{tr}(H'D_0) - \mathrm{tr}(H'(AX-I)^+A_1)] \\ &\quad = \mathrm{tr}[(D_0-(AX-I)^+A_1)'(AX-I)'B(AX-I)(D_0-(AX-I)^+A_1)] \ge 0. \end{aligned}$$

Thus

$$\mathrm{tr}(H'D_0) \le \mathrm{tr}[H'(AX-I)^+A_1]. \tag{3.11}$$

On the other hand, combining with (3.7), we have

$$-\lambda[\mathrm{tr}(H'D_0) + \mathrm{tr}(H'(AX-I)^+A_1)] = \mathrm{tr}[D_0'(AX-I)'B(AX-I)D_0] - \mathrm{tr}(A_1'BA_1) \le 0.$$

Consequently,

$$-\mathrm{tr}(H'D_0) \le \mathrm{tr}[H'(AX-I)^+A_1]. \tag{3.12}$$

Together with (3.10), (3.11) and (3.12), we have

$$\mathrm{tr}(H'D_0) = \mathrm{tr}[H'(AX-I)^+A_1] = 0$$

when $\lambda > 0$. Therefore,

$$\begin{aligned} &\mathrm{tr}[D_0'(AX-I)'B(AX-I)D_0] - \mathrm{tr}[A_0'(AX-I)'B(AX-I)A_0] \\ &\quad = -\lambda[\mathrm{tr}(H'D_0) + \mathrm{tr}(H'(AX-I)^+A_1)] = 0, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\mathrm{tr}[D_0'(AX-I)'B(AX-I)D_0] - \mathrm{tr}[D_0'(AX-I)'B(AX-I)A_0] \\ &\quad = -\lambda \mathrm{tr}(D_0'H) = 0. \end{aligned} \tag{3.14}$$

It follows that

$$\mathrm{tr}[((AX-I)D_0 - (AX-I)A_0)'B((AX-I)D_0 - (AX-I)A_0)] = 0,$$

which implies that (3.9) also holds for the case $\lambda > 0$. Therefore, (3.9) holds for all $\lambda \ge 0$. Thus there exists no estimator of Θ which is better than $AY + A_1$ in $\mathcal{LI}$. Consequently, $AY + A_1 \overset{\mathcal{LI}}{\sim} \Theta(\mathcal{T})$. Thus, the proof of Theorem 3.2 is completed. □

Together with Theorems 2.1 and 3.2, we obtain

Corollary 3.1 *For model (1.1) and loss function (1.3), $AY + A_1 \overset{\mathcal{LI}}{\sim} \Theta(\mathcal{T})$ holds if and only if*

(1) $A_1 \in \mu(AX-I)$;

(2) $\mathrm{tr}[H(AX-I)^+A_1] \le 0$ *or* $\mu(H) \not\subseteq \mu[(B^{1/2}(AX-I))']$;

(3) $AV = AP_XV$;

(4) $(1-\omega)\widetilde{A}X[(X'T^+X)^- - I]SB^{-1} \ge \widetilde{A}X[(X'T^+X)^- - I]X'\widetilde{A}'$;

(5) $\mathrm{rank}[(AX-I)((X'T^+X)^- - I)X'T^+X] = \mathrm{rank}(AX-I)$,
where P_X*,* B *and* $\widetilde{A}$ *are the same as that in Theorem 2.1.*

So far, we have obtained the necessary and sufficient conditions for AY to be admissible for Θ in $\mathcal{LH}$ and the necessary and sufficient conditions for $AY + A_1$ to be admissible for Θ in $\mathcal{LI}$, where the parameter space is restricted on $\mathcal{T}$.

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References

[1] Changyu LU. *The theory of admissibility of parameters estimation in linear models.* Chinese J. Appl. Probab. Statist., 2001, **17**(2): 203–212. (in Chinese)
[2] Changyu LU, Ningzhong SHI. *Admissible linear estimators in linear models with respect to inequality constraints.* Linear Algebra Appl., 2002, **354**: 187–194.
[3] Changyu LU, Jianhong WU, Binyan ZHANG. *Admissible linear estimators with respect to inequality constraints.* Communications in Statistics-Theory and Methods, 2007, **36**(5): 961–966.
[4] Jianhong WU. *Admissibility of linear estimators in multivariate linear models with respect to inequality constraints.* Linear Algebra Appl., 2008, **428**(8-9): 2040–2048.
[5] Daojiang HE, Dawei GUO. *Admissibility of linear estimators in a multivariate linear model with respect to an inequality restriction.* J. Jilin Univ. Sci., 2007, **45**(5): 738–742. (in Chinese)
[6] Daojiang HE, Dawei GUO. *Admissibility of linear estimators with respect to inequality constraints under matrix loss function.* Linear Algebra Appl., 2009, **430**(1): 241–250.
[7] Liwen Xu. *Admissible linear predictors in the superpopulation model with respect to inequality constraints.* Comm. Statist. Theory Methods, 2009, **38**(13-15): 2528–2540.
[8] A. ZELLNER. *Bayesian and Non-Bayesian Estimation Using Balanced Loss Functions.* Springer, New York, 1994.
[9] Mingxiang CAO. *ϕ admissibility for linear estimators on regression coefficients in a general multivariate linear model under balanced loss function.* J. Statist. Plann. Inference, 2009, **139**(9): 3354–3360.
[10] Liming DONG, Qiguang WU. *Necessary and sufficient conditions for linear estimators of stochastic regression coefficients and parameters to be admissible under quadratic loss.* Acta Math. Sinica, 1988, **31**(2): 145–157. (in Chinese)