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Power Semigroups of a Class of Clifford Semigroups

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Abstract Let $S = \bigcup(G_\alpha : \alpha \in E)$ be a semilattice of groups (i.e., a Clifford semigroup) and n a natural number. E is called an n-element chain of groups if it is an n-element chain. Denote by $\mathcal{C}_n$ the set of all n-element chains of groups. In this paper we shall show that for any natural number n, the class of semigroups $\mathcal{C}_n$ satisfies the strong isomorphism property.

Keywords group; n-element chain of groups; closed subsemigroup; power semigroup.

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1. Introduction

The power semigroup, or global, of a semigroup S is the semigroup $\mathcal{P}(S)$ of all nonempty subsets of S with respect to the operation $\cdot$ defined by

$$A \cdot B = \{ab : a \in A, b \in B\} \text{ for all } A, B \in \mathcal{P}(S).$$

A class $\mathcal{K}$ of semigroups is said to be globally determined if any two members of $\mathcal{K}$ having isomorphic globals must themselves be isomorphic.

Tamura [4] asked in 1967 whether the class of all semigroups is globally determined? The question was negatively answered in the class of all semigroups by Mogiljanskaja [8] in 1973. Crvenković, Dolinka and Vinčić [9] proved that involution semigroups are not globally determined in 2001. But it is known that the following classes are globally determined: groups [5, 6]; rectangular groups [7]; completely 0-simple semigroups [14]; finite semigroups [15]; lattices and semilattices [11, 13], finite simple semigroups and semilattices of torsion groups in which semilattices are finite [10]; completely regular periodic monoid with irreducible identity [12]. However, for an important class, the class of Clifford semigroups, the problem has been left unsolved.

Recall that in [13], Kobayashi considered a stronger property: strong isomorphism property. That is, let $\mathcal{K}$ be a class of semigroups. $\mathcal{K}$ satisfies the strong isomorphism property if for every isomorphism φ from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$, $\varphi|_S$ (the restriction φ to S) is an isomorphism from S onto S' for any $S,\ S' \in \mathcal{K}$. In this statement S (resp., S') is considered to be a subset of $\mathcal{P}(S)$ (resp., $\mathcal{P}(S')$) by identifying an element x of S (resp., S') with the singleton $\{x\}$. He proved that the class of semilattices satisfies the strong isomorphism property.

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A few words on notation and terminology are in order. $\mathcal{N}$ denotes the set of natural numbers, $\underline{n}$ denotes the set $\{1, 2, \ldots, n\}$, $E(S)$ stands for the set of idempotents of a semigroup S and $|A|$ is the cardinal number (or cardinality) of a set A. For each $e \in E(S)$, the maximal subgroup ($\mathcal{H}$-class) of S containing e will be denoted by $H_e(S)$. A singleton member of $\mathcal{P}(S)$ will be identified with the element it contains. Let $a, b \in S$. $a\ \mathcal{H}_S\ b$ means $a\ \mathcal{H}\ b$ in S.

Let $S = \bigcup(G_\alpha : \alpha \in E)$ be a semilattice of groups (i.e., a Clifford semigroup) and n a natural number. E is called an n-element chain of groups if it is an n-element chain. Denote by $\mathcal{C}_n$ the set of all n-element chains of groups. It is clear that a 1-element chain of a group is a group. We follow the usual practice of abbreviating $\{g\}H$ where $g \in S$ and $H \in \mathcal{P}(S)$ by gH. Further, we make the notational convention that e_α denotes the idempotent of $G_\alpha,\ \alpha \in E$ and also e_i denotes the idempotent of $G_{\alpha_i},\ i \in \mathcal{N}$. For $X \in \mathcal{P}(S)$ and $\alpha \in E$ put

$$X_\alpha = X \cap G_\alpha,\ \operatorname{supp} X = \{\alpha \in E : X_\alpha \neq \emptyset\}$$

and

$$\max X = \max(\operatorname{supp} X), \quad \min X = \min(\operatorname{supp} X).$$

In this paper we shall show that for any natural number n, the class semigroups $\mathcal{C}_n$ satisfies the strong isomorphism property.

This paper is divided into four sections. The first section is introduction and preliminary; in the second section, we further characterize the closed subsemigroups of a Clifford semigroup S that will be instrumental in the proof of our main results; in the third section, we will prove that the class of semigroups $\mathcal{C}_2$ satisfies the strong isomorphism property; and in the last section, we will verify that the class of semigroups $\mathcal{C}_n$ satisfies the strong isomorphism property.

The following lemma, which implies that the class of groups also satisfies the strong isomorphism property, is taken from Gould and Iskra [10].

Lemma 1.1 *Let S be a semigroup and $e \in E(S)$. Then $H_e(\mathcal{P}(S)) = H_e(S)$.*

We refer to the books [1–3] for all background information concerning semigroups and universal algebra.

2. The closed subsemigroups of a Clifford semigroup

Zhao in [16] introduced the notion of the closed subsemigroup of a semigroup S. A subsemigroup C of a semigroup S is said to be closed if

$$sat,\ sbt \in C \Rightarrow sabt \in C$$

holds for all $a, b \in S,\ s, t \in S^1$, where S^1 denotes the semigroup obtained from S by adjoining an identity if necessary. It is easy to see that every subsemilattice of a semilattice is closed by the definition of closed subsemigroup.

Let S be a semigroup and A a nonempty subset of S. $\overline{A}$ denotes the closed subsemigroup of S generated by A, i.e., the smallest closed subsemigroup of S containing A. In [17], Zhao studied the closed subsemigroups of a Clifford semigroup.

In the remaining part of this section, unless otherwise stated, S always means the Clifford semigroup $\bigcup(G_\alpha : \alpha \in E)$ and we further characterize the closed subsemigroups of S that will be instrumental in the proof of our main results.

Lemma 2.1 ([17, Theorem 2.3]) *Let $A \in \mathcal{P}(S)$. Then $\overline{A} = \bigcup_{\alpha \in \overline{\operatorname{supp} A}} G_\alpha$.*

Corollary 2.2 *Let $A \in \mathcal{P}(S)$. Then $SA = AS = \bigcup_{\gamma \in W} G_\gamma$ is a closed subsemigroup of S, where $W = \{\gamma \in E : (\exists \alpha \in \operatorname{supp} A)\ \gamma \leq \alpha\}$.*

Proof Let $W = \{\gamma \in E : (\exists \alpha \in \operatorname{supp} A)\ \gamma \leq \alpha\}$. Then it is easy to see that W is a subsemilattice of E and so $\bigcup_{\gamma \in W} G_\gamma$ is a closed subsemigroup of S by Lemma 2.1.

Next we will show that $SA = \bigcup_{\gamma \in W} G_\gamma$. In fact, on the one hand, for any $\gamma \in W$ and $g_\gamma \in G_\gamma$, there exists $\alpha \in \operatorname{supp} A$ such that $\gamma \leq \alpha$. Fix an element a_α in A_α. We have

$$g_\gamma = g_\gamma e_\alpha = (g_\gamma a_\alpha^{-1}) a_\alpha \in SA.$$

This implies that $\bigcup_{\gamma \in W} G_\gamma \subseteq SA$. On the other hand, for any $s \in S$ and $a \in A$, we can assume that $s \in G_\beta$ for some $\beta \in E$ and $a \in A_\mu$ for some $\mu \in \operatorname{supp} A$. So we have that $sa \in G_\beta G_\mu \subseteq G_{\beta\mu} \subseteq \bigcup_{\gamma \in W} G_\gamma$, which implies that $SA \subseteq \bigcup_{\gamma \in W} G_\gamma$. Therefore, we have proved that $SA = \bigcup_{\gamma \in W} G_\gamma$.

Similarly, we can show that $AS = \bigcup_{\gamma \in W} G_\gamma$. The proof is completed.

Lemma 2.3 *Let $A \in \mathcal{P}(S)$ and $A^2 = A$. Then the following statements are equivalent:*

(1) $a_\alpha A = b_\alpha A$ *for any* $\alpha \in \operatorname{supp} A$ *and* $a_\alpha, b_\alpha \in G_\alpha$;

(2) $a_\alpha A_\alpha = b_\alpha A_\alpha$ *for any* $\alpha \in \operatorname{supp} A$ *and* $a_\alpha, b_\alpha \in G_\alpha$;

(3) $A_\alpha = G_\alpha$ *for any* $\alpha \in \operatorname{supp} A$.

Proof (1) $\Rightarrow$ (2). Suppose that (1) holds, that is, $a_\alpha A_\alpha \subseteq a_\alpha A = b_\alpha A$ for any $\alpha \in \operatorname{supp} A$ and $a_\alpha,\ b_\alpha \in G_\alpha$. Then for any $c_\alpha \in A_\alpha$, there exists $d \in A$ such that $a_\alpha c_\alpha = b_\alpha d$. Without loss of generality, we can assume that $d = d_\beta \in A_\beta$, where $\beta \in \operatorname{supp} A$. Then

$$a_\alpha c_\alpha = b_\alpha d = b_\alpha d_\beta = b_\alpha e_\alpha d_\beta,$$

and $\alpha = \alpha\beta \leq \beta$ by the above equation.

In addition, choose and fix some $g_\alpha \in A_\alpha$, we have that $e_\alpha A = g_\alpha A \subseteq A$. It follows that $e_\alpha d_\beta \in A$ and so $e_\alpha d_\beta \in A_\alpha$ for $e_\alpha d_\beta \in G_\alpha G_\beta \subseteq G_{\alpha\beta} = G_\alpha$. Thus $a_\alpha c_\alpha = b_\alpha (e_\alpha d_\beta) \in b_\alpha A_\alpha$ and so $a_\alpha A_\alpha \subseteq b_\alpha A_\alpha$ by the arbitrariness of c_α.

Similarly, we can prove that $b_\alpha A_\alpha \subseteq a_\alpha A_\alpha$. So $a_\alpha A_\alpha = b_\alpha A_\alpha$ and (2) holds.

(2) $\Rightarrow$ (3). Suppose that (2) holds, that is, $A_\alpha = e_\alpha A_\alpha = a_\alpha A_\alpha$ for any $\alpha \in \operatorname{supp} A$ and $a_\alpha, b_\alpha \in G_\alpha$. Choose and fix some $c_\alpha \in A_\alpha$. We have that $c_\alpha^{-1} A_\alpha = A_\alpha$ and so

$$e_\alpha = c_\alpha^{-1} c_\alpha \in c_\alpha^{-1} A_\alpha = A_\alpha.$$

Further, $g_\alpha = g_\alpha e_\alpha \in g_\alpha A_\alpha = A_\alpha$ for any $g_\alpha \in G_\alpha$. Thus $G_\alpha \subseteq A_\alpha$. But it is clear that $A_\alpha = A \cap G_\alpha \subseteq G_\alpha$. Therefore, $A_\alpha = G_\alpha$ and (3) holds.

(3) $\Rightarrow$ (1). Suppose that (3) holds. Then A is a closed subsemigroup of S by Lemma 2.1. It is easy to prove that

$$a_\alpha A = \bigcup_{\beta\in \operatorname{supp} A, \beta\le\alpha} G_\beta = b_\alpha A$$

for any $\alpha\in \operatorname{supp} A$ and $a_\alpha,\ b_\alpha \in G_\alpha$. (1) holds.

By Lemmas 2.1 and 2.3, we have

Theorem 2.4 *Let $A\in\mathcal{P}(S)$. Then A is a closed subsemigroup of S if and only if A satisfies the following two conditions:*

(i) $A^2 = A$,

(ii) $e_\alpha A = g_\alpha A$ *for any* $\alpha\in \operatorname{supp} A$ *and* $g_\alpha\in G_\alpha$.

Lemma 2.5 *Let $A,\ B\in\mathcal{P}(S)$ and $A\ \mathcal{H}\ B$ in $\mathcal{P}(S)$. Then $SA=SB$ and $AS=BS$.*

Proof Since $A\ \mathcal{H}\ B$, there exist $C,\ D,\ U,\ V\in\mathcal{P}^1(S)$ such that

$$A=CB,\ B=DA,\ A=BU,\ B=AV.$$

Thus

$$SA=SCB\subseteq SB,\quad SB=SDA\subseteq SA$$

and so $SA=SB$. Similarly, we can show that $AS=BS$. $\Box$

Theorem 2.6 *Let $S'=\bigcup(G_{\alpha'}:\alpha'\in E')$ be a Clifford semigroup and φ an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. Then $\varphi(S)$ (resp., $\varphi^{-1}(S')$) is a closed subsemigroup of S' (resp., S).*

Proof For any $\alpha'\in E'$ and $e_{\alpha'},\ g_{\alpha'}\in G_{\alpha'}$, we have

$$\begin{aligned} e_{\alpha'}\ \mathcal{H}_{\mathcal{P}(S')}\ g_{\alpha'} &\Longrightarrow \varphi^{-1}(e_{\alpha'})\ \mathcal{H}_{\mathcal{P}(S)}\ \varphi^{-1}(g_{\alpha'})\\ &\Longrightarrow \varphi^{-1}(e_{\alpha'})S=\varphi^{-1}(g_{\alpha'})S \quad \text{(by Lemma 2.5)}\\ &\Longrightarrow e_{\alpha'}\varphi(S)=g_{\alpha'}\varphi(S). \end{aligned}$$

In addition, it is obvious that $\varphi(S)=\varphi(S^2)=\varphi(S)^2$. Therefore, $\varphi(S)$ is a closed subsemigroup of S' by Theorem 2.4.

Applying the above argument to the isomorphism φ^{-1}, we can obtain that

$$e_\alpha\varphi^{-1}(S')=g_\alpha\varphi^{-1}(S')$$

for any $\alpha\in E$, $g_\alpha\in G_\alpha$. Also $(\varphi^{-1}(S'))^2=\varphi^{-1}(S'^2)=\varphi^{-1}(S')$, therefore, $\varphi^{-1}(S')$ is a closed subsemigroup of S by Theorem 2.4.

3. Power semigroups of semigroups in $\mathcal{C}_2$

In this section, we will show that the class of semigroups $\mathcal{C}_2$ satisfies the strong isomorphism property. Let

$$S=G_\alpha\cup G_\beta\ (\alpha<\beta),\quad S'=G_{\alpha'}\cup G_{\beta'}\ (\alpha'<\beta')$$

be both 2-element chains of groups and φ an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. We shall show that $\varphi|_S$ (the restriction of φ to S) is an isomorphism from S onto S'. The following two lemmas are needed.

Lemma 3.1 *Let $S = G_\alpha \cup G_\beta$ $(\alpha < \beta)$ and $S' = G_{\alpha'} \cup G_{\beta'}$ $(\alpha' < \beta')$ be both 2-element chains of groups and φ an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. Then $\varphi(e_\beta) = e_{\beta'}$ and $\varphi(G_\alpha) = G_{\alpha'}$.*

Proof This follows from the fact that e_β (resp., G_α) is the identity (resp., zero) of $\mathcal{P}(S)$ and $e_{\beta'}$ (resp., $G_{\alpha'}$) is the identity (resp., zero) of $\mathcal{P}(S')$.

Lemma 3.2 *Let $S = G_\alpha \cup G_\beta$ $(\alpha < \beta)$ and $S' = G_{\alpha'} \cup G_{\beta'}$ $(\alpha' < \beta')$ be both 2-element chains of groups and φ an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. Then $\varphi(S) = S'$.*

Proof Suppose that $A \in \mathcal{P}(S)$ such that $\varphi(A) = S'$. Then by Theorem 2.6 and Lemma 3.1, we have $A = \varphi^{-1}(S')$ is a closed subsemigroup of S and $\beta \in \operatorname{supp} A$. Thus $AS = S$ and so

$$S'\varphi(S) = \varphi(A)\varphi(S) = \varphi(AS) = \varphi(S).$$

It follows from Corollary 2.2 that either $\varphi(S) = G_{\alpha'}$ or $\varphi(S) = S'$.

But $\varphi(S) \neq G_{\alpha'}$ by Lemma 3.1, so we have $\varphi(S) = S'$. $\square$

Theorem 3.3 *The class semigroups $\mathcal{C}_2$ satisfies the strong isomorphism property. Moreover, $\varphi|_{G_\alpha}$ (resp., $\varphi|_{G_\beta}$) is an isomorphism from G_α onto $G_{\alpha'}$ (resp., G_β onto $G_{\beta'}$).*

Proof By Lemmas 1.1 and 3.1, we have $\varphi|_{G_\beta}$ is an isomorphism from G_β onto $G_{\beta'}$. To show that $\varphi|_S$ is an isomorphism from S onto S', it suffices to prove that $\varphi|_{G_\alpha}$ is also an isomorphism from G_α onto $G_{\alpha'}$.

To see that this is so, let $A \in \mathcal{P}(G_\alpha)$. Then $AS = G_\alpha$, which implies by Lemmas 3.1 and 3.2 that

$$\varphi(A)S' = \varphi(A)\varphi(S) = \varphi(AS) = \varphi(G_\alpha) = G_{\alpha'}$$

and so $\varphi(A) \in \mathcal{P}(G_{\alpha'})$.

Hence $\varphi|_{\mathcal{P}(G_\alpha)}$ is an isomorphism from $\mathcal{P}(G_\alpha)$ onto $\mathcal{P}(G_{\alpha'})$ and so $\varphi|_{G_\alpha}$ is also an isomorphism from G_α onto $G_{\alpha'}$. The proof is completed. $\square$

4. Power semigroups of semigroups in $\mathcal{C}_n$

In this section, our goal is to show that the class of semigroups $\mathcal{C}_n$ $(n \geq 3)$ satisfies the strong isomorphism property:

In the following, unless otherwise stated,

$$S = \bigcup_{i \in \underline{n}} G_i, \quad S' = \bigcup_{i \in \underline{n}} G_i'$$

are both n-Clifford semigroups, e_i (resp., e_i') denotes the idempotent of G_i (resp., G_i') and $e_1 < e_2 < \cdots < e_n$, $e_1' < e_2' < \cdots < e_n'$.

Let ψ be an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. We shall show that $\psi|_S$ is an isomorphism from S onto S'.

Lemma 4.1 *Let $S = \bigcup_{i\in\underline{n}} G_i$ and $S' = \bigcup_{i\in\underline{n}} G'_i$ are both n-Clifford semigroups, e_i (resp., e'_i) denotes the idempotent of G_i (resp., G'_i) and $e_1 < e_2 < \cdots < e_n$, $e'_1 < e'_2 < \cdots < e'_n$. Let ψ be an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. Then $\psi(e_n) = e'_n$ and $\psi(G_1) = G'_1$.*

Proof This follows from the fact that e_n (resp., G_1) is the identity (resp., zero) of $\mathcal{P}(S)$ and e'_n (resp., G'_1) is the identity (resp., zero) of $\mathcal{P}(S')$. $\square$

Lemma 4.2 *Let $S = \bigcup_{i\in\underline{n}} G_i$ and $S' = \bigcup_{i\in\underline{n}} G'_i$ are both n-Clifford semigroups, e_i (resp., e'_i) denotes the idempotent of G_i (resp., G'_i) and $e_1 < e_2 < \cdots < e_n$, $e'_1 < e'_2 < \cdots < e'_n$. Let ψ be an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. Then $\psi(S) = S'$.*

Proof Suppose that $A \in \mathcal{P}(S)$ such that $\psi(A) = S'$. Then by Theorem 2.6 and Lemma 4.1, we have $A = \psi^{-1}(S')$ is a closed subsemigroup of S and $\operatorname{supp} A \neq \{1\}$. Let $r = \max A,\ t = \min A$ and $k = \max \psi(S),\ l = \min \psi(S)$. Then $2 \le r \le n$ and $2 \le k \le n$.

Claim 1 We have $r = k$ and $\psi(\bigcup_{i\in\underline{j}} G_i) = \bigcup_{i\in\underline{j}} G'_i$ for any $j = 1, 2, \ldots, r$.

Indeed, for any $j = 1, 2, \ldots, r$, we have

$$\begin{aligned}(\bigcup_{i\in\underline{j}} G_i)A = \bigcup_{i\in\underline{j}} G_i &\Longrightarrow \psi(\bigcup_{i\in\underline{j}} G_i)\psi(A) = \psi(\bigcup_{i\in\underline{j}} G_i)\\ &\Longrightarrow \psi(\bigcup_{i\in\underline{j}} G_i)S' = \psi(\bigcup_{i\in\underline{j}} G_i)\\ &\Longrightarrow \psi(\bigcup_{i\in\underline{j}} G_i) \in \{\bigcup_{i\in\underline{m}} G'_i : m = 1, 2, \ldots, n\}\end{aligned}$$

and

$$\begin{aligned}(\bigcup_{i\in\underline{j}} G_i)S = \bigcup_{i\in\underline{j}} G_i &\Longrightarrow \psi(\bigcup_{i\in\underline{j}} G_i)\psi(S) = \psi(\bigcup_{i\in\underline{j}} G_i)\\ &\Longrightarrow \max \psi(\bigcup_{i\in\underline{j}} G_i) \le \max \psi(S) = k,\end{aligned}$$

which shows that

$$\{\psi(\bigcup_{i\in\underline{j}} G_i) : j = 1, 2, \ldots, r\} \subseteq \{\bigcup_{i\in\underline{m}} G'_i : m = 1, 2, \ldots, k\}.$$

Thus $r \le k$. Similarly, applying the above argument to the isomorphism ψ^{-1}, we can show that $k \le r$. Therefore, $r = k$ and

$$\{\psi(\bigcup_{i\in\underline{j}} G_i) : j = 1, 2, \ldots, r\} = \{\bigcup_{i\in\underline{m}} G'_i : m = 1, 2, \ldots, r\}. \tag{*}$$

Since ψ is an isomorphism, from the above equation $(*)$, it is easy to see that $\psi(\bigcup_{i\in\underline{j}} G_i) = \bigcup_{i\in\underline{j}} G'_i$ for any $j = 1, 2, \ldots, r$. The claim is proved.

Next, consider the following cases:

Case 1 $r = n$. By Claim 1, we have $\psi(S) = S'$.

Case 2 $r \leq n-1,\ t \geq 2$.

We have

$$\begin{aligned} G_2A = G_2 &\Longrightarrow \psi(G_2)S' = \psi(G_2)\psi(A) = \psi(G_2) \\ &\Longrightarrow \psi(G_2) = \bigcup_{i \in \underline{j}} G_i' \ \text{ for some } \ r+1 \leq j \leq n \end{aligned}$$

and

$$\begin{aligned} G_2S = G_1 \cup G_2 &\Longrightarrow \psi(G_2)\psi(S) = \psi(G_1 \cup G_2) \\ &\Longrightarrow \psi(G_2)\psi(S) = G_1' \cup G_2' \quad \text{(by Claim 1)} \\ &\Longrightarrow r = 2, \end{aligned}$$

which shows that $A = G_2$. Also $\psi(S) = G_2'$ by Claim 1. Now we have

$$\psi(G_2) = S',\ \psi(S) = G_2' \ \text{ and } \ \psi(G_1 \cup G_2) = G_1' \cup G_2'.$$

Claim 2 We have $\psi(B) \subseteq G_2'$ for any $B \in \mathcal{P}(S)$ satisfying $\operatorname{supp} B = \{1, n\}$.

In fact, by the proof of Theorem 2.6, we have $e_1G_2 = e_1\psi^{-1}(S') = g_1\psi^{-1}(S') = g_1G_2$ for any $g_1 \in G_1$, which implies that for any $C \in \mathcal{P}(G_1)$,

$$CG_2 = e_1G_2 = G_1G_2 = G_1.$$

So for any $B \in \mathcal{P}(S)$ satisfying $\operatorname{supp} B = \{1, n\}$, we have

$$\begin{aligned} G_2B = G_1 \cup G_2 &\Longrightarrow \psi(G_2)\psi(B) = \psi(G_1 \cup G_2) \Longrightarrow S'\psi(B) = G_1' \cup G_2' \\ &\Longrightarrow \operatorname{supp}\psi(B) \subseteq \{1, 2\} \end{aligned}$$

and

$$SB = S \Longrightarrow \psi(S)\psi(B) = \psi(S) \Longrightarrow G_2\psi(B) = G_2' \Longrightarrow 1 \notin \operatorname{supp}\psi(B).$$

Thus $\operatorname{supp}\psi(B) = \{2\}$ and so $\psi(B) \subseteq G_2'$, the Claim is proved.

Claim 3 We have $\psi^{-1}(e_2')B = B$ for any $B \in \mathcal{P}(S)$ satisfying $\operatorname{supp} B = \{1, n\}$.

In fact, by Claim 2, we have $e_2'\psi(B) = \psi(B)$ and so $\psi^{-1}(e_2')B = B$.

Claim 4 $\operatorname{supp}\psi^{-1}(e_2') = \{1, n\}$.

Indeed,

$$\begin{aligned} G_2'e_2' = G_2' &\Longrightarrow \psi^{-1}(G_2')\psi^{-1}(e_2') = \psi^{-1}(G_2') \Longrightarrow S\psi^{-1}(e_2') = S \\ &\Longrightarrow n \in \operatorname{supp}\psi^{-1}(e_2') \end{aligned}$$

and

$$\begin{aligned} S'e_2' = G_1' \cup G_2' &\Longrightarrow \psi^{-1}(S')\psi^{-1}(e_2') = \psi^{-1}(G_1' \cup G_2') \\ &\Longrightarrow G_2\psi^{-1}(e_2') = G_1 \cup G_2 \\ &\Longrightarrow 1 \in \operatorname{supp}\psi^{-1}(e_2'). \end{aligned}$$

Also, by Claim 3, we have $\psi^{-1}(e_2')(G_1 \cup G_n) = G_1 \cup G_n$. It follows that $\operatorname{supp}\psi^{-1}(e_2') = \{1, n\}$. The Claim is proved.

Claim 5 $\psi(\{e_1,\ e_n\}) = e_2'$.

Indeed, for any $a_n \in \psi^{-1}(e_2') \cap G_n$, by Claims 3 and 4, we have

$$a_n = a_n e_n \in \psi^{-1}(e_2')(\{e_1,\ e_n\}) = \{e_1,\ e_n\}$$

and so $a_n = e_n$.

Similarly, we can prove that $a_1 = e_1$ for any $a_1 \in \psi^{-1}(e_2') \cap G_1$. Thus $\psi^{-1}(e_2') = \{e_1, e_n\}$, that is $\psi(\{e_1,\ e_n\}) = e_2'$. The Claim is proved.

Claim 6 $|G_1| = 1$.

Suppose by way of contradiction that $|G_1| \geq 2$. Choose and fix some $b_1 \in G_1$ such that $b_1 \neq e_1$, then

$$\psi^{-1}(e_2')\{e_n, b_1\} = \{e_1, e_n\}\{e_n, b_1\} = \{e_1, b_1, e_n\} \neq \{e_n, b_1\},$$

contradicting Claim 3.

Similarly, applying the entire argument above to the isomorhism ψ^{-1}, we can claim that $\psi^{-1}(\{e_1',\ e_n'\}) = e_2$, i.e., $\psi(e_2) = \{e_1',\ e_n'\}$ and $|G_1'| = 1$. Thus by Lemma 4.1, we have $\psi(e_1) = e_1'$. Also, since

$$\psi(\{e_2, e_n\}) \subseteq \psi(\{e_2, e_n\})\{e_1',\ e_n'\} = \psi(\{e_2, e_n\}e_2) = \psi(e_2) = \{e_1',\ e_n'\},$$

we have

$$\psi(\{e_2, e_n\}) = \{e_1'\} \ \text{ or } \ \{e_n'\} \ \text{ or } \ \{e_1',\ e_n'\}.$$

But then $\{e_2, e_n\} = \{e_1\}$ or $\{e_n\}$ or $\{e_2\}$, a contradiction.

Case 3 $r \leq n-1$, $l \geq 2$. Applying the proof of Case 2 to the isomorphism ψ^{-1} also leads to a contradiction.

Case 4 $t = l = 1,\ r \leq n-1$.

Let $M = \{m \in \underline{n} : m \notin \operatorname{supp} A\}$. Then $M \neq \emptyset$ since $n \in M$. Thus M contains a least element. Denote by u the least element of M. By Claim 1, it is easy to see that $1 < u < r$. Thus $A = G_1 \cup G_2 \cup \cdots \cup G_{u-1} \cup G_{j_1} \cup \cdots \cup G_{js}$, for some $\{j_1, \ldots, j_s\} \subseteq \{u+1, \ldots, r\}$.

Let $D = G_1 \cup G_2 \cup \cdots \cup G_{u-1} \cup G_{u+1}$. Then we have

$$\begin{aligned} DA = D &\Longrightarrow \psi(D)\psi(A) = \psi(D) \Longrightarrow \psi(D)S' = \psi(D) \\ &\Longrightarrow \psi(D) \in \{\bigcup_{i \in \underline{j}} G_i' : j = 1, 2, \ldots, n\}. \end{aligned}$$

In addition, by Claim 1, we have

$$\psi(D) = \bigcup_{i \in \underline{j}} G_i' \ \text{ for some } \ r+1 \leq j \leq n.$$

Also since $DS = \bigcup_{i\in\underline{u+1}} G_i$, we have $\psi(D)\psi(S) = \bigcup_{i\in\underline{u+1}} G'_i$ by Claim 1. Note that max $\psi(S) = r$ and max $\psi(D) = j \geq r+1$, we have

$$\max(\psi(D)\psi(S)) = r = u + 1.$$

Therefore, $A = G_1 \cup G_2 \cup \cdots \cup G_{r-2} \cup G_r = D$.

Similarly, applying the entire argument above to the isomorhism ψ^{-1}, we can obtain $\psi(S) = G'_1 \cup G'_2 \cup \cdots \cup G'_{r-2} \cup G'_r$. Now, we have

$$\psi(G_1 \cup G_2 \cup \cdots \cup G_{r-2} \cup G_r) = S', \quad \psi(S) = G'_1 \cup G'_2 \cup \cdots \cup G'_{r-2} \cup G'_r$$

and

$$\psi(\bigcup_{i\in\underline{j}} G_i) = \bigcup_{i\in\underline{j}} G'_i \quad \text{for any} \quad j = 1, 2, \ldots, r.$$

Claim 7 We have $\operatorname{supp}\psi(H) \subseteq \{1, 2, \ldots, r-2, r\}$ for any $H \in \mathcal{P}(S)$ satisfying $\{r-1,\ n\} \subseteq \operatorname{supp} H$.

In fact, by the proof of Theorem 2.6, we have $e_{r-1}A = g_{r-1}A$ for any $g_{r-1} \in G_{r-1}$ and so $e_{r-1}G_r = g_{r-1}G_r$, which implies that for any $C \in \mathcal{P}(G_{r-1})$,

$$CG_r = e_{r-1}G_r = G_{r-1}G_r = G_{r-1}.$$

So for any $H \in \mathcal{P}(S)$ satisfying $\{r-1,\ n\} \subseteq \operatorname{supp} H$, we have

$$\begin{aligned} HA = \bigcup_{i\in\underline{r}} G_i &\Longrightarrow \psi(H)\psi(A) = \psi(\bigcup_{i\in\underline{r}} G_i) \\ &\Longrightarrow \psi(H)S' = \bigcup_{i\in\underline{r}} G'_i \quad \text{(by Claim 1)} \\ &\Longrightarrow \operatorname{supp}\psi(H) \subseteq \{1, 2, \ldots, r\} \end{aligned}$$

and

$$\begin{aligned} SH = S &\Longrightarrow \psi(S)\psi(H) = \psi(S) \\ &\Longrightarrow (G'_1 \cup \cdots \cup G'_{r-2} \cup G'_r)\psi(H) = G'_1 \cup \cdots \cup G'_{r-2} \cup G'_r \\ &\Longrightarrow r-1 \notin \operatorname{supp}\psi(H). \end{aligned}$$

Thus $\operatorname{supp}\psi(H) \subseteq \{1, 2, \ldots, r-2, r\}$, the Claim is proved.

Claim 8 We have $\psi^{-1}(e'_r)H = H$ for any $H \in \mathcal{P}(S)$ satisfying $\{r-1, n\} \subseteq \operatorname{supp} H$.

In fact, by Claim 7, we have $e'_r\psi(H) = \psi(H)$ and so $\psi^{-1}(e'_r)H = H$.

Claim 9 $\operatorname{supp}\psi^{-1}(e'_r) = \{r-1, n\}$.

Indeed,

$$\psi(S)e'_r = \psi(S) \Longrightarrow S\psi^{-1}(e'_r) = S \Longrightarrow n \in \operatorname{supp}\psi^{-1}(e'_r)$$

and

$$S'e'_r = \bigcup_{i\in\underline{r}} G'_i \Longrightarrow \psi^{-1}(S')\psi^{-1}(e'_r) = \psi^{-1}(\bigcup_{i\in\underline{r}} G'_i)$$

$$\Longrightarrow (G_1 \cup \cdots \cup G_{r-2} \cup G_r)\psi^{-1}(e'_r) = \bigcup_{i \in \underline{r}} G_i$$

$$\Longrightarrow r-1 \in \operatorname{supp} \psi^{-1}(e'_r).$$

Also, by Claim 8, we have $\psi^{-1}(e'_r)(G_{r-1} \cup G_n) = G_{r-1} \cup G_n$. It follows that $\operatorname{supp} \psi^{-1}(e'_r) = \{r-1, n\}$. The Claim is proved.

Claim 10 $\psi(\{e_{r-1}, e_n\}) = e'_r$.

Indeed, for any $a_n \in \psi^{-1}(e'_r) \cap G_n$, by Claims 8 and 9, we have

$$a_n = a_n e_n \in \psi^{-1}(e'_r)(\{e_{r-1},\ e_n\}) = \{e_{r-1},\ e_n\}$$

and so $a_n = e_n$.

Similarly, we can prove that $a_{r-1} = e_{r-1}$ for any $a_{r-1} \in \psi^{-1}(e'_r) \cap G_{r-1}$. Thus $\psi^{-1}(e'_r) = \{e_{r-1}, e_n\}$, that is $\psi(\{e_{r-1},\ e_n\}) = e'_r$. The Claim is proved.

In a similar way, applying the entire argument above to the isomorhism ψ^{-1}, we can claim that $\psi^{-1}(\{e'_{r-1},\ e'_n\}) = e_r$, i.e., $\psi(e_r) = \{e'_{r-1},\ e'_n\}$. Thus

$$\psi(\{e_r, e_n\}) \subseteq \psi(\{e_r, e_n\})\{e'_{r-1},\ e'_n\} = \psi(\{e_r, e_n\}e_r) = \psi(e_r) = \{e'_{r-1},\ e'_n\}$$

and so

$$\psi(\{e_r, e_n\}) = \{e'_{r-1}\} \text{ or } \{e'_n\} \text{ or } \{e'_{r-1},\ e'_n\}.$$

But if $\psi(\{e_r, e_n\}) = \{e'_n\}$ or $\{e'_{r-1},\ e'_n\}$, then $\{e_r, e_n\} = \{e_n\}$ or $\{e_r\}$, a contradiction. Thus we get the next Claim:

Claim 11 $\psi(\{e_r, e_n\}) = \{e'_{r-1}\}$.

However, by Claims 10 and 11, we have

$$\{e'_{r-1}\} = \{e'_r\} \cdot \{e'_{r-1}\} = \psi(\{e_{r-1}, e_n\}) \cdot \psi(\{e_r, e_n\}) = \psi(\{e_{r-1}, e_r, e_n\}),$$

contradicting Claim 11. This contradiction concludes the proof. $\Box$

By Lemma 4.2 and Claim 1 in Lemma 4.2, we have

Lemma 4.3 *Let $S = \bigcup_{i \in \underline{n}} G_i$ and $S' = \bigcup_{i \in \underline{n}} G'_i$ are both n-Clifford semigroups, e_i (resp., e'_i) denotes the idempotent of G_i (resp., G'_i) and $e_1 < e_2 < \cdots < e_n$, $e'_1 < e'_2 < \cdots < e'_n$. Let ψ be an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. Then $\psi(\bigcup_{i \in \underline{j}} G_i) = \bigcup_{i \in \underline{j}} G'_i$ for any $j = 1, 2, \ldots, n$.*

Lemma 4.4 *Let $S = \bigcup_{i \in \underline{n}} G_i$ and $S' = \bigcup_{i \in \underline{n}} G'_i$ are both n-Clifford semigroups, e_i (resp., e'_i) denotes the idempotent of G_i (resp., G'_i) and $e_1 < e_2 < \cdots < e_n$, $e'_1 < e'_2 < \cdots < e'_n$. Let ψ be an isomorphism from $\mathcal{P}(S)$ onto $\mathcal{P}(S')$. Then $\psi|_{\mathcal{P}(\bigcup_{i \in \underline{n-1}} G_i)}$ is an isomorphism from $\mathcal{P}(\bigcup_{i \in \underline{n-1}} G_i)$ to $\mathcal{P}(\bigcup_{i \in \underline{n-1}} G'_i)$.*

Proof Indeed, for any $K \in \mathcal{P}(\bigcup_{i \in \underline{n-1}} G_i)$, let $j = \max K$. Then $j \le n-1$. Also, by Corollary 2.2 and Lemma 4.3, we have

$$KS = \bigcup_{i \in \underline{j}} G_i \Longrightarrow \psi(K)S' = \psi(K)\psi(S) = \bigcup_{i \in \underline{j}} G'_i$$

$$\Longrightarrow \max\psi(K) = j \leq n-1$$
$$\Longrightarrow \psi(K) \in \mathcal{P}(\bigcup_{i\in \underline{n-1}} G_i).$$

The Lemma is proved. □

Theorem 4.5 *For any $n \in \mathcal{N}$, the class of semigroups $\mathcal{C}_n$ satisfies the strong isomorphism property.*

Proof We prove it by induction on n. First the result is true for $n = 1$ since the class of groups satisfies the strong isomorphism property. For $n = 2$, Theorem 3.3 shows that the result is true.

Next, assume that $n \geq 3$ and the result is true for $n-1$. We shall show that the result is true for n. In fact, by Lemma 4.4, we have $\psi|_{\mathcal{P}(\bigcup_{i\in\underline{n-1}} G_i)}$ is an isomorphism from $\mathcal{P}(\bigcup_{i\in\underline{n-1}} G_i)$ to $\mathcal{P}(\bigcup_{i\in\underline{n-1}} G'_i)$. By the hypothesis, $\psi|_{\bigcup_{i\in\underline{n-1}} G_i}$ is an isomorphism from $\bigcup_{i\in\underline{n-1}} G_i$ to $\bigcup_{i\in\underline{n-1}} G'_i$.

Also, by Lemmas 1.1 and 4.1, $\psi|_{G_n}$ is an isomorphism from G_n to G'_n. Therefore, $\psi|_S$ is an isomorphism from S to S'. The proof is completed. □

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