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On the Normality of Meromorphic Functions

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Abstract In this paper, we deal with the problem of normal families concerning share-values. The results extend and improve some theorems put forward by Miranda, Pang, Chen, Hua and Fang. Moreover, we answer one question posed by Gu, Pang, Fang and so on.

Keywords meromorphic; normality; shared-value.

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1. Introduction and main results

In this paper, it is assumed that the reader is familiar with the notations of Nevanlinna theory of meromorphic functions, for instance, $T(r,f), N(r,f), m(r,f), \overline{N}(r,f), \ldots$. We denote by $S(r,f)$ any function satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to+\infty$, possibly outside of a set with a finite measure in $\mathbf{R}$.

Let f be a non-constant meromorphic function defined in $\mathbf{D}$ and let a be a finite complex number. Set

$$\overline{E}_f(a)=f^{-1}(a)=\{z:f(z)-a=0,z\in\mathbf{D}\}.$$

Let f and g be meromorphic functions in a domain of complex plane. If $\overline{E}_f(a)=\overline{E}_g(a)$, then we say f and g share the value a. If $g(z)=b$ whenever $f(z)=a$, we write $f(z)=a\Rightarrow g(z)=b$. Moreover, if $f(z)=a\Rightarrow g(z)=b$ and $g(z)=b\Rightarrow f(z)=a$, we write $f(z)=a\Leftrightarrow g(z)=b$, as the notation $\overline{E}_f(a)=\overline{E}_g(b)$. When the case is $a\neq b$, if $\overline{E}_f(a)\cup\overline{E}_f(b)=\overline{E}_g(a)\cup\overline{E}_g(b)$, then we say f and g share the set $S(S=\{a,b\})$.

Let $\mathbf{D}$ be a domain in $\mathbb{C}$, and let $\mathfrak{F}$ be a family of meromorphic functions defined in $\mathbf{D}$. The family $\mathfrak{F}$ is said to be normal in $\mathbf{D}$, according to Montel: if each sequence $\{f_n\}\subset\mathfrak{F}$ contains a subsequence $\{f_{n_j}\}$ that converges, spherically locally uniformly in $\mathbf{D}$, to a meromorphic function or ∞ (see [5, 6]).

According to Bloch's Principle, many normality criteria have been obtained by starting to use the conditions known from Picrd-Type theorems.

The following result was proposed by Miranda [9].

Theorem Mi *Let $\mathfrak{F}$ be a family of holomorphic functions in a domain $\mathbf{D}$, k be a positive integer,*

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and a, b be two finite complex numbers in which $b \neq 0$. If, for each $f \in \mathfrak{F}$, $f \neq a, f^{(k)} \neq b$, then $\mathfrak{F}$ is normal in $\mathbf{D}$.

Another approach to normality criteria is to use conditions known from uniqueness theorems. The first attempt at this was made by Schwick who proved in [7] that if there exist three distinct finite values $a_1, a_2, a_3 \in \mathbb{C}$ such that $f(z)$ and $f'(z)$ share a_j $(j = 1, 2, 3)$ for each $f(z) \in \mathfrak{F}$, then $\mathfrak{F}$ is normal in $\mathbf{D}$. The corresponding statement that $f(z)$ and $f'(z)$ share two distinct finite values $a_1, a_2 \in \mathbb{C}$ remains valid [8].

Afterwards, some normality criteria concerning one shared-value were obtained [1, 10].

Theorem HC *Let $\mathfrak{F}$ be a family of holomorphic functions in a domain $\mathbf{D}$, and a be a nonzero finite complex number. If, for each $f \in \mathfrak{F}$, $\overline{E}_f(a) = \overline{E}_{f'}(a) = \overline{E}_{f''}(a)$, then $\mathfrak{F}$ is normal in $\mathbf{D}$.*

Theorem PZ *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathbf{D}$, k be an integer, $b(\neq 0)$ be a complex number, and h be a finite positive number. If, for each $f \in \mathfrak{F}$, all zeros of f have multiplicity at least k, and f satisfies the following conditions:*

(i) $\overline{E}_f(0) = \overline{E}_{f^{(k)}}(b)$,

(ii) $\overline{E}_f(0) \Rightarrow 0 < |\overline{E}_{f^{(k+1)}}(z)| < h$,

then $\mathfrak{F}$ is normal in $\mathbf{D}$.

Recently, Fang and Zalcman [2] proved the following result, which is the complement of Theorem HC.

Theorem FZ *Let $\mathfrak{F}$ be a family of holomorphic functions in a domain $\mathbf{D}$, and a, c be two finite nonzero distinct complex numbers. If each $f \in \mathfrak{F}$ satisfies the following conditions:*

(i) $\overline{E}_f(a) = \overline{E}_{f'}(a)$,

(ii) $\overline{E}_{f'}(c) = \overline{E}_{f''}(c)$,

then $\mathfrak{F}$ is normal in $\mathbf{D}$.

Now, we are interested in what will be stated if $\mathfrak{F}$ is a family of meromorphic functions in a domain $\mathbf{D}$ in Theorem FZ.

Question 1.1 ([11]) Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathbf{D}$ and satisfy the conditions given by Theorem FZ. Does the conclusion hold?

On the other hand, Fang [12] extended Schwick's result in the view of shared set. Actually, he proved the following theorem.

Theorem F *Let $\mathfrak{F}$ be a family of holomorphic functions in a domain $\mathbf{D}$, and let a_1, a_2, and a_3 be three distinct finite complex numbers. If, for each $f(z) \in \mathfrak{F}$, $f(z)$ and $f'(z)$ share the set $S = \{a_1, a_2, a_3\}$, then $\mathfrak{F}$ is normal in $\mathbf{D}$.*

Recently, by generalizing Theorem F from families of holomorphic functions to families of meromorphic functions, Liu and Pang [3] obtained the following result.

Theorem LP *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathbf{D}$, and let a_1, a_2, and a_3 be three distinct finite complex numbers. If, for each $f(z) \in \mathfrak{F}$, $f(z)$ and $f'(z)$ share the set*

$S = \{a_1, a_2, a_3\}$, *then* $\mathfrak{F}$ *is normal in* $\mathbf{D}$.

Naturally, we will think what can be stated if $S = \{a_1, a_2, a_3\}$ is replaced by $S = \{a_1, a_2\}$. However, Example 1 shows that the criterion will not be valid when the set $S = \{a_1, a_2, a_3\}$ in Theorem F and Theorem LP is replaced by $S = \{a_1, a_2\}$.

Example 1.1 Let $S = \{1, -1\}$. Set

$$\mathfrak{F} = \{f_n(z), n = 2, 3, 4, \ldots\}, f_n(z) = \frac{n+1}{2n} e^{nz} + \frac{n-1}{2n} e^{-nz}.$$

Then, for any $f_n \in \mathfrak{F}$, we have

$$n^2[{f_n}^2(z) - 1] = [{f_n}'(z)]^2 - 1.$$

Then f_n and f_n' share the set $S = \{1, -1\}$, but $\mathfrak{F}$ is not normal in $\mathbf{D}$.

In this paper, we have the following result.

Theorem 1.1 *Let* $\mathfrak{F}$ *be a family of meromorphic functions in a domain* $\mathbf{D}$*; let* m *be a positive integer; let* a, b *and* c *be three finite complex numbers in which* $c \neq 0$,$a \neq (m+1)c, b \neq (m+1)c$. *If, for each* $f \in \mathfrak{F}$, $\overline{E}_{f'}(a) \cup \overline{E}_{f'}(b) = \overline{E}_f(a) \cup \overline{E}_f(b)$, *and* $f'' = c$ *whenever* $f' = c$, *then* $\mathfrak{F}$ *is normal in* $\mathbf{D}$.

From the direct result of Theorem 1, we have the following precise results.

Corollary 1.1 *Let* $\mathfrak{F}$ *be a family of meromorphic functions in a domain* $\mathbf{D}$*; let* m *be a positive integer; let* a *and* $c \neq 0$ *be two finite complex numbers in which* $a \neq (m+1)c$. *If, for each* $f \in \mathfrak{F}$, $\overline{E}_{f'}(a) = \overline{E}_f(a)$ *and* $f'' = c$ *whenever* $f' = c$, *then* $\mathfrak{F}$ *is normal in* $\mathbf{D}$.

Corollary 1.2 *Let* $\mathfrak{F}$ *be a family of meromorphic functions in a domain* $\mathbf{D}$*; let* a, b *and* $c (c \neq 0)$ *be three distinct finite complex numbers. If, for each* $f \in \mathfrak{F}$, f *and* f' *share* $\{a,\ b\}$ *and* $f'' = c$ *whenever* $f' = c$, *then* $\mathfrak{F}$ *is normal in* $\mathbf{D}$.

Corollary 1.3 *Let* $\mathfrak{F}$ *be a family of meromorphic functions in a domain* $\mathbf{D}$*; let* m *be a positive integer; let* a, b *be two finite distinct complex numbers in which* $a \neq 0$ *and* $b \neq (m+1)a$. *If, for each* $f \in \mathfrak{F}$, f *and* f' *share* $\{a,\ b\}$ *and* $f'' = a$ *whenever* $f' = a$, *then* $\mathfrak{F}$ *is normal in* $\mathbf{D}$.

Corollary 1.4 *Let* $\mathfrak{F}$ *be a family of meromorphic functions in a domain* $\mathbf{D}$*; let* $0, c$ *be two finite distinct complex numbers. If, for each* $f \in \mathfrak{F}$, $\overline{E}_{f'}(0) = \overline{E}_f(0)$, $\overline{E}_{f'}(c) = \overline{E}_{f''}(c)$, *then* $\mathfrak{F}$ *is normal in* $\mathbf{D}$.

Remarks (1) If $a = b = c$, Corollary 1 extends Miranda's criterion and Theorem HX.

(2) If $a = b \neq c$, then the condition (ii) of Theorem FZ, which requires that $\overline{E}_{f'}(b) = \overline{E}_{f''}(b)$, is fairly strong and it is clearly stronger than the boundedness condition on $f' = c$ in Theorem 1 and Corollary 1. As mentioned above, the condition $\overline{E}_{f'}(a) \cup \overline{E}_{f'}(b) = \overline{E}_f(a) \cup \overline{E}_f(b)$ is implicit in Theorem FZ when $a = b$.

(3) If $a \neq b$, Example 1 shows that the condition "$f'' = c$ whenever $f' = c$" in Theorem 1 is necessary.

The following examples illustrate that some conditions of Theorem 1 cannot be omitted. Especially, some conditions are still necessary even if $\mathfrak{F}$ is a family of holomorphic functions.

Examples (1) Let a, c be two nonzero numbers in which $a = (m+1)c$ where m is a positive integer. Set

$$f_n(z) = c(z - \frac{1}{n}) + a + \frac{1}{m(nz-1)^m}, \quad n = 1, 2, \dots,$$

and let $\mathfrak{F} = \{f_n\}, \mathbf{D} = \{z : |z| < 1\}$. Then $f_n'(z) = c - \frac{n}{(nz-1)^{m+1}}$. Thus for each $f \in \mathfrak{F}$, $\overline{E}_{f'}(a) = \overline{E}_f(a)$ and $f'(z) \neq c$. But $\mathfrak{F}$ is not normal in $\mathbf{D}$, which means that "$a \neq (1+m)c$" in Theorem 1 is necessary.

(2) Let $\mathfrak{F} = \{f_n = e^{nz}\}, \mathbf{D} = \{z : |z| < 1\}$, $n = 1, 2, \dots$. Then the spherical derivative $f_n^\sharp(0) = \frac{n}{2} \to \infty$. Thus $\mathfrak{F}$ is not normal in $\mathbf{D}$ by Marty's criterion. However, it is clear that f_n, f_n' and f_n'' share the value 0. It implies that $\mathfrak{F}$ in Theorem 1 is not normal under the condition "$a = b = c = 0$".

(3) Let $\mathfrak{F} = \{f_n = e^{nz} - a/n + a\}, \mathbf{D} = \{z : |z| < 1\}$. Thus for each $f \in \mathfrak{F}$, $\overline{E}_{f'}(a) = \overline{E}_f(a)$, and $f' \neq 0$. But $\mathfrak{F}$ is not normal in $\mathbf{D}$. This means that "$c \neq 0$" in Theorem 1 is necessary.

2. Some lemmas

Lemma 2.1 ([1]) *Let k be a positive integer and let $\mathfrak{F}$ be a family of meromorphic functions on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathfrak{F}$. Then if $\mathfrak{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) a number $0 < r < 1$,

(b) points z_n, $|z_n| < r$,

(c) functions $f_n \in \mathfrak{F}$, and

(d) positive numbers $\rho_n \to 0$

such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^\alpha} = g_n(\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$. Moreover, g has order at most two. In particular, if $\mathfrak{F}$ is a family of holomorphic functions, then g has order at most one.

Here, usually, $g^\sharp(z) = |g'(z)|/(1 + |g(z)|^2)$ is the spherical derivative.

Lemma 2.2 ([15]) *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + \frac{p(z)}{q(z)}$, where $a_0, a_1, \dots, a_n (\neq 0), c (\neq 0)$ are constants, and $p(z)$ and $q(z)$ are two coprime polynomials with $\deg p(z) < \deg q(z)$, and let k be a positive integer. If $f^{(k)} \neq c$, then*

$$f(z) = \frac{c}{k!} z^k + \cdots + a_0 + \frac{1}{(az+b)^m},$$

where $a (\neq 0), b$ are constants, $m \in \mathbf{N}$.

Lemma 2.3 ([16]) *Let f be a transcendental meromorphic function with finite order, all of whose zeros are of multiplicity at least k. Let K be a positive number and c be a nonzero finite complex number. If $|f^{(k)}(z)| \le K$ whenever $f(z) = 0$, then for each l $(1 \le l \le k)$, $f^{(l)}(z)$ assumes any finite nonzero value infinitely often.*

Lemma 2.4 ([13]) *Let f be a finite order meromorphic function on $\mathbb{C}$ and b, d be two nonzero complex constants. If $f = 0 \Rightarrow f' = b$, and $f' \neq d$, then $f(z) = b(z - C)$, or $f(z) = d(z - C) + \frac{A}{m(z-C)^m}$, where $b = (m+1)d$ and $C, A(\neq 0) \in \mathbb{C}$.*

3. Proof of Theorem 1.1

We may assume that $\mathbf{D} = \Delta$, the unit disc. Suppose that $\mathfrak{F}$ is not normal on Δ, then by Lemma 2.1 we can find $f_n \in \mathfrak{F}$, $z_n \in \Delta$, and $\rho_n \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta) - a}{\rho_n} \Rightarrow g(\zeta),$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, such that $g^{\sharp}(\zeta) \le g^{\sharp}(0) = (\max\{|a|, |b|\}) + 1$.

We claim that

(i) $g(\zeta) = 0 \Leftrightarrow g'(\zeta) \in S$, and

(ii) $g'(\zeta) \neq c$.

Suppose that there exists ζ_0 such that $g(\zeta_0) = 0$, then by Hurwitz's Theorem there exists a sequence $\zeta_n, \zeta_n \to \zeta_0$, for sufficiently large n, such that

$$0 = g_n(\zeta_n) = \frac{f_n(z_n + \rho_n \zeta_n) - a}{\rho_n},$$

which implies that $f_n(z_n + \rho_n \zeta_n) = a$, thus $f'(z_n + \rho_n \zeta_n) \in S$. Then without loss of generality, we can assume that there exists a or b, such that $f'(z_n + \rho_n \zeta_n) = a$, so

$$g'(\zeta_0) = \lim_{n\to\infty} g_n'(\zeta_n) = \lim_{n\to\infty} f'(z_n + \rho_n \zeta_n) \in S,$$

proving $g(\zeta) = 0 \Rightarrow g'(\zeta) \in S$.

On the other hand, suppose that there exists ζ_0 such that $g'(\zeta_0) = a$. We obtain $g' \not\equiv a$. Indeed, if $g'(\zeta) \equiv a \neq 0$, then g will be a polynomial of exact degree 1, so $g = a(\zeta - \zeta_0)$. Then a simple calculation shows that

$$|g^{\sharp}(0)| \le \begin{cases} 1, & |\zeta_1| \ge 1, \quad (1a) \\ |a|, & |\zeta_1| < 1. \quad (1b) \end{cases}$$

So we have $g^{\sharp}(0) < (\max\{|a|, |b|\}) + 1$, which contradicts $g^{\sharp}(0) = (\max\{|a|, |b|\}) + 1$. Since $g'(\zeta_0) = a$ and $g'(\zeta) \not\equiv a$, by Hurwitz's Theorem there exists a sequence $\zeta_n, \zeta_n \to \zeta_0$, for sufficiently large n, such that $g_n'(\zeta_n) = f'(z_n + \rho_n \zeta_n) = a$. This leads to $f(z_n + \rho_n \zeta_n) \in S$. If there exists a positive number N, for any $n > N$, $f_n(z_n + \rho_n \zeta_n) \neq a$, then

$$g(\zeta_0) = \lim_{n\to\infty} \frac{f_n(z_n + \rho_n \zeta_n) - a}{\rho_n} = \infty,$$

which is a contradiction by our hypotheses. Then there exists a subsequence of f_n, which we also denote by f_n, such that $f_n(z_n+\rho_n\zeta_n)=a$, and, so

$$g(\zeta_0)=\lim_{n\to\infty}\frac{f_n(z_n+\rho_n\zeta_n)-a}{\rho_n}=0,$$

proving $g(\zeta)=0 \Leftarrow g'(\zeta)=a$. Similarly, we can obtain $g(\zeta)=0 \Leftarrow g'(\zeta)=b$.

Suppose that there exists a ζ_0 such that $g'(\zeta_0)=c$, then by claim(i) we have $g'(\zeta)\not\equiv c$. By Hurwitz's Theorem there exists a sequence $\zeta_n, \zeta_n\to\zeta_0$, for sufficiently large n, such that $g_n'(\zeta_n)=f'(z_n+\rho_n\zeta_n)=c$, then $f''(z_n+\rho_n\zeta_n)=c$. Thus we obtain

$$g''(\zeta_0)=\lim_{n\to\infty}|g_n''(\zeta_n)|=\lim_{n\to\infty}\rho_n|f''(z_n+\rho_n\zeta_n)|=0,$$

which implies $g'(\zeta)=c\Rightarrow g''(\zeta)=0$.

Next, we will prove that $g'(\zeta)\neq c$. Suppose that there exists a ζ_0 such that $g'(\zeta_0)=c$. Thus ζ_0 is a zero point of $g'(\zeta_0)-c$ with multiplicity $k(\geq 2)$, then $g^{(k+1)}(\zeta_0)\neq 0$, and there exists a $\delta(>0)$ such that

$$g(\zeta)\neq 0,\ g'(\zeta)\neq 0,\ g^{(k)}(\zeta)\neq 0 \ \text{ in } \ 0<|\zeta-\zeta_0|<\delta.$$

The Argument Principle shows the existence of $k(\geq 2)$ sequences $\{\zeta_n^{(j)}\}$ $(j=1,2,\dots,k)$ each tending to ζ_0, such that, for n sufficiently large,

$$g_n'(\zeta_n^{(1)})=\cdots=g_n'(\zeta_n^{(j)})=c,$$

which implies

$$f'(z_n+\rho_n\zeta_n^{(1)})=\cdots=f'(z_n+\rho_n\zeta_n^{(j)})=c. \tag{*}$$

Now $(*)$ means that

$$g''(\zeta_0)=0,\quad g_n''(\zeta_n^{(j)})=\rho_n f''(z_n+\rho_n\zeta_n^{(j)})\neq 0,\quad \text{for } \ j=1,2,\dots,k,$$

so each zero of $g_n'-c$ is simple. This rules out the possibility that any two sequences of $\{\zeta_n^{(j)}\}$ $(j=1,2,\dots,k)$ might coincide.

When n is sufficiently large, $\{\zeta_n^{(i)}\}\neq\{\zeta_n^{(j)}\}$ $(i\neq j\in\{1,2,\dots,k\})$, so it shows that ζ_0, as a zero of $g'-c$, is a zero of $g''(\zeta)$ with multiplicity k. But this contradicts the statement that ζ_0 is a zero of $g'(\zeta_0)-c$ with multiplicity k. Therefore, $g'(\zeta)\neq c$ for all $\zeta\in\mathbb{C}$.

Case 1 $a\neq b$. We will discuss it based on the following two subcases.

Subcase 1.1 If $c\notin\{a,b\}$, then by Lemmas 2.2 and 2.3, we have $g(\zeta)=c\zeta+t+\frac{n}{(\zeta-m)^l}$ or $g(\zeta)=\alpha\zeta+\beta$, where $\alpha(\neq 0,c),\beta,t,m$ are finite complex numbers, and l is a positive integer. Then we will discuss the two cases, respectively.

Subcase 1.1.1 If $g(\zeta)=c\zeta+t+\frac{n}{(\zeta-m)^l}$, then by Claim (i), (ii) and Nevanlinna-Second-Fundamental-Theorem we get

$$\begin{aligned}2T(r,g')&\leq\overline{N}(r,g')+\overline{N}(r,\frac{1}{g'-a})+\overline{N}(r,\frac{1}{g'-b})+\overline{N}(r,\frac{1}{g'-c})+S(r,g')\\&\leq\overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+S(r,g').\end{aligned}$$

Note that $g(\zeta)$ is a rational function, so $S(r,g') = S(r,g) = O(1)$, and

$$\begin{aligned}
&\overline{N}(r,g) = \ln r + O(1),\\
&\overline{N}(r,\frac{1}{g}) \le (l+1)\ln r + O(1),\\
&T(r,g') = (l+1)\ln r + O(1).
\end{aligned}$$

Thus we get

$$2(l+1)\ln r \le \ln r + (l+1)\ln r + O(1),$$

which implies $l = 0$, but this contradicts the fact that l is a positive integer.

Subcase 1.1.2 If $g(\zeta) = \alpha\zeta + \beta$, then $g(\frac{-\beta}{\alpha}) = 0$, $g'(\zeta) = \alpha$, and particularly note that $g'(0) = g'(\frac{-\beta}{\alpha}) = \alpha, |\alpha| \le \max\{|a|,|b|\}$. Thus

$$|\alpha| = |g'(0)| \ge g^{\sharp}(0) = \max\{|a|,|b|\} + 1 > |\alpha|$$

is a contradiction.

Subcase 1.2 If $c \in \{a,b\}$, we will discuss it based on the following two subcases.

Subcase 1.2.1 If $c = a$, then by Claim (i), (ii), we obtain $g = 0 \Leftrightarrow g' = b$, and $g' \ne c$. Thus by Lemma 2.4 and $b \ne (1+m)c$, we obtain $g(\zeta) = b(\zeta - \zeta_0)$, which is a contradiction by the above proof.

Subcase 1.2.2 If $c = b$, then the process of proof is the same as that of subcase 1.2.1.

Case 2 If $a = b$, then by Claims (i) and (ii), we obtain $g = 0 \Leftrightarrow g' = a$, and $g' \ne c$. Thus by Lemma 2.4 and $a \ne (1+m)c$, we obtain $g(\zeta) = a(\zeta - \zeta_0)$, which is a contradiction by the above proof.

The proof of Theorem 1.1 is completed.

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