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# Strong Limit Theorems for Weighted Sums of Widely Orthant Dependent Random Variables

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**Abstract** Strong limit theorems are established for weighted sums of widely orthant dependent (WOD) random variables. As corollaries, the strong limit theorems for weighted sums of extended negatively orthant dependent (ENOD) random variables are also obtained, which extend and improve the related known works in the literature.

**Keywords** widely orthant dependent (WOD) random variables; extended negatively orthant dependent (ENOD) random variables; weighted sums; strong limit theorem.

**MR(2010) Subject Classification** 60F15

## 1. Introduction

Let $\{\Omega, \Im, P\}$ be a probability space. In the following, all random variables are assumed to be defined on $\{\Omega, \Im, P\}$. Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$ and $\{k_n, n \geq 1\}$ be a sequence of positive integers with $k_n \leq Mn$ (where $M$ is a positive constant not depending on n) and let $\{b_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of real numbers. The strong limit theorems of weighted sums $S_{k_n} = \sum_{i=1}^{k_n} b_{ni}X_i$ were studied by many authors. For instance, Thrum [1] obtained the following result

**Theorem A** *If* $\sum_{i=1}^{n} b_{ni}^2 = 1$, *and* $E|X|^p < \infty$ *for some* $p \geq 2$, *then* $S_n/n^{1/p} \to 0$ *a.s.* $n \to \infty$.

Li et al. [2] improved the above result into

**Theorem B** *If* $E|X|^p < \infty$ *for some* $p \geq 2$, $\sup_{n,k} |b_{nk}| < \infty$ *and* $\sum_{i=1}^{k_n} b_{ni}^2 = O(n^{\delta})$ $(0 < \delta < 2/p)$, *then* $S_{k_n}/n^{1/p} \to 0$ *a.s.* $n \to \infty$.

Sung [3] improved Theorem B into

**Theorem C** *Let* $p > 1, \{b_{ni}, 1 \leq i \leq n\}$ *be an array of constants such that*

*(i)* $\max_{1\leq i\leq n} |b_{ni}| = O(1/n^{1/p})$,

*(ii)* $\begin{cases} \sum_{i=1}^{n} |b_{ni}|^{\tau} = O(1/n^{\tau/p-1+\gamma}) \text{ for some } \tau > 0 \text{ and } \gamma > 0, & \text{if } 1 < p < 2, \\ \sum_{i=1}^{n} b_{ni}^2 = O(1/n^{\gamma}) \text{ for some } \gamma > 0, & \text{if } p \geq 2. \end{cases}$

*If* $E|X|^p < \infty$, *then* $S_n \to 0$ *a.s.* $n \to \infty$.

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Bai and Cheng [4] proved that

**Theorem D** *Suppose that $1 < \alpha$, $\beta < \infty$, $1 < p < 2$, and $1/p = 1/\alpha + 1/\beta$. Let $\{b_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real constants such that*

$$A_\alpha =: \limsup_{n\to\infty}(\frac{1}{n}\sum_{k=1}^{n}|b_{nk}|^\alpha)^{1/\alpha} < \infty.$$

*If $E|X|^\beta < \infty$, then $S_n/n^{1/p} \to 0$ a.s. $n \to \infty$.*

Wang et al. [5] introduced the following dependence structure.

**Definition 1.1** *Random variables $Y_1, Y_2, \ldots$ are said to be widely upper orthant dependent (WUOD) if for each $n \ge 1$, there exists some finite positive number $g_U(n)$ such that, for all $y_i \in (-\infty, \infty)$, $i = 1, 2, \ldots, n$,*

$$P(Y_1 > y_1, \ldots, Y_n > y_n) \le g_U(n)\prod_{i=1}^{n}P(Y_i > y_i); \tag{1.1}$$

*they are said to be widely lower orthant dependent (WLOD) if for each $n \ge 1$, there exists some finite positive number $g_L(n)$ such that, for all $y_i \in (-\infty, \infty), i = 1, 2, \ldots, n$,*

$$P(Y_1 \le y_1, \ldots, Y_n \le y_n) \le g_L(n)\prod_{i=1}^{n}P(Y_i \le y_i) \tag{1.2}$$

*and they are said to be widely orthant dependent (WOD) if they are both WUOD and WLOD.*

WUOD, WLOD and WOD random variables are called by a joint name widely dependent random variables. Wang et al. [5] pointed out that the widely dependent random variables contain common negatively dependent random variables, some positively dependent random variables and some others by some interesting examples. In the case $g_U(n) = g_L(n) = M$ for all $n \ge 1$ and some finite positive constant $M \ge 1$, inequality (1.1) and (1.2) describe extended negatively upper and lower orthant dependent (ENOUD/ENLOD), respectively. Random variables $Y_1, Y_2, \ldots$ are said to be extended negatively orthant dependent (ENOD) if they are both ENUOD and ENLOD. The concept of general negative dependence was proposed by Liu [6]. More especially, if $M = 1$ in both (1.1) and (1.2), then the random variables $Y_1, Y_2, \ldots$ are called negatively upper orthant dependent (NUOD) and negatively lower orthant dependent (NLOD), respectively, and they are called negatively orthant dependent (NOD) if they are both NUOD and NLOD (see, Block et al. [7], Ebrahimi and Ghosh [8], Wu [9]). Negative association (NA, see, Jing and Liang [10]) is the special case of NOD. A great number of articles for negatively dependent random variables have appeared in literature. For further research on ENOD random variables, we refer to Liu [6], Liu [11], Chen et al. [12], Chen et al. [13], Qiu et al. [14], Yang and Wang [15], and so on. For further research on widely dependent random variables, we refer to Wang et al. [5], Wang and Cheng [16], Liu et al. [17], He et al. [18], and so on.

Jing and Liang [10] extended and improved Theorems C and D to the NA setting, and Wu [9] extended Theorem D to the NOD setting. In this paper, the main purpose is to establish strong limit theorems for weighted sums of WOD random variables. As corollaries, the limit theorems for weighted sums of ENOD random variables are obtained, which extend and improve Theorem C–Theorem D and extend the corresponding results of Jing and Liang [10], Wu [9].

**Definition 1.2** *A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a nonnegative random variable $X$, if there exists a positive constant $D$ such that*

$$P(|X_n| > x) \leq DP(X > x) \text{ for all } x > 0 \text{ and } n \geq 1.$$

*In this case, we write $\{X_n, n \geq 1\} \prec X$.*

In order to prove our main result, we need the following lemmas.

**Lemma 1.3** ([5]) *(1) Let $\{X_n, n \geq 1\}$ be WLOD (WUOD). If $\{f_n, n \geq 1\}$ are nondecreasing, then $\{f_n(X_n), n \geq 1\}$ are still WLOD(WUOD); If $\{f_n, n \geq 1\}$ are nonincreasing, then $\{f_n(X_n), n \geq 1\}$ are still WUOD (WLOD).*

*(2) If $\{X_n, n \geq 1\}$ are nonnegative WUOD, then for each $n \geq 1$,*

$$E(\prod_{j=1}^{n} X_j) \leq g_U(n) \prod_{j=1}^{n} EX_j.$$

*In particular, if $\{X_n, n \geq 1\}$ are WUOD, then for each $n \geq 1$ and any $s > 0$,*

$$E\exp(s\sum_{i=j}^{n} X_j) \leq g_U(n) \prod_{j=1}^{n} E\exp(sX_j).$$

By (2) of Lemma 1.3, we have

**Lemma 1.4** *Let $\{X_n, n \geq 1\}$ be WUOD such that*

$$|X_k| \leq b_k, 1 \leq k \leq n.$$

*Then for each $n \geq 1$ and any $s > 0$,*

$$E\exp(s\sum_{k=1}^{n} X_k) \leq g_U(n)\exp\{s\sum_{k=1}^{n} EX_k + \frac{s^2}{2}\sum_{k=1}^{n} e^{sb_k} EX_k^2\}.$$

Throughout this paper, $\{k_n, n \geq 1\}$ will be a sequence of positive integers with $k_n \leq Mn$, where $M$ is a positive constant not depending on $n$. $C$ will represent positive constant which may change from one place to another, $I(A)$ represent the indicator function of the set $A$.

## 2. Main results and proofs

**Theorem 2.1** *Let $p > 2, \{X_n, n \geq 1\}$ be a sequence of WOD random variables with $EX_n = 0, n \geq 1$, $\{X_n, n \geq 1\} \prec X$, and $EX^p < \infty$. Let $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of constants such that $\max_{1\leq i\leq k_n} |a_{ni}| = O(n^{-1/p})$. Let $c_n = \sum_{i=1}^{k_n} a_{ni}^2$. If*

$$\sum_{n=1}^{\infty} \max\{g_U(n), g_L(n)\}n^{-\theta} < \infty \text{ for some } \theta > 0, \tag{2.1}$$

$$\sum_{n=1}^{\infty} \max\{g_U(n), g_L(n)\} \exp(-u/c_n) < \infty \text{ for any } u > 0. \tag{2.2}$$

*Then*

$$\sum_{i=1}^{n} a_{ni} X_i \to 0 \ \text{ a.s. } n \to \infty. \tag{2.3}$$

**Proof** From the proof of Theorem 3.1 of Li et al. [2], without loss of generality, we assume that $k_n = n$ for every $n \geq 1$. Since $a_{ni} = a_{ni}^+ - a_{ni}^-$, we assume that

$$0 < a_{ni} \leq n^{-1/p}, \ \forall\, 1 \leq i \leq n, n \geq 1. \tag{2.4}$$

We note that $EX^p < \infty$ is equivalent to $\sum_{n=1}^{\infty} P(X \geq \delta n^{1/p}) < \infty$ for any $\delta > 0$, hence we get that $\sum_{n=1}^{\infty} P(|X_n| \geq \delta n^{1/p}) < \infty$. Thus it is possible to construct a sequence $\{b_n, n \geq 1\}$ of real numbers such that $0 < b_n \leq 1, b_n \downarrow 0$, and

$$\sum_{n=1}^{\infty} P(|X_n| > b_n n^{1/p}) < \infty.$$

We choose some small $\rho > 0$ (to be specialized later), and define $d_1 = b_1$, $d_n = \max\{n^{-\rho}, b_n, d_{n-1}(\frac{n-1}{n})^{1/p}\}$ for $n \geq 2$. Then $0 < d_n \leq 1$, $d_n \downarrow 0$, and

$$\sum_{n=1}^{\infty} P(|X_n| > d_n n^{1/p}) < \infty. \tag{2.5}$$

For $\forall\, 1 \leq i \leq n, n \geq 1$, define

$$\begin{aligned}
X_{ni}^{(1)} &= -n^{-\rho} I(a_{ni}X_i < -n^{-\rho}) + a_{ni}X_i I(|a_{ni}X_i| \leq n^{-\rho}) + n^{-\rho} I(a_{ni}X_i > n^{-\rho}),\\
X_{ni}^{(2)} &= (a_{ni}X_i + n^{-\rho}) I(a_{ni}X_i < -d_n) + (a_{ni}X_i - n^{-\rho}) I(a_{ni}X_i > d_n),\\
X_{ni}^{(3)} &= a_{ni}X_i - X_{ni}^{(1)} - X_{ni}^{(2)},\\
T_n^{(l)} &= \sum_{i=1}^{n} X_{ni}^{(l)}, \quad l = 1, 2, 3.
\end{aligned}$$

In order to prove (2.3), it suffices to show that $T_n^{(l)} \to 0$ a.s. $n \to \infty$ for $l = 1, 2, 3$.

For $T_n^{(1)}$, in order to show that $T_n^{(1)} \to 0$ a.s. $n \to \infty$, by the Borel-Cantelli Lemma, it suffices to show that

$$\sum_{n=1}^{\infty} P(|T_n^{(1)}| > \epsilon) < \infty, \quad \forall\, \epsilon > 0. \tag{2.6}$$

First, we show that

$$\sum_{n=1}^{\infty} P(T_n^{(1)} > \epsilon) < \infty, \quad \forall\, \epsilon > 0. \tag{2.7}$$

By Lemma 1.3, it is clear that for every $n \geq 1$, $\{X_{ni}^{(1)}, i = 1, \ldots, n\}$ is still a sequence of WOD random variables. Since $|X_{ni}^{(1)}| \leq n^{-\rho}$, by Lemma 1.4 and the Markov inequality, we have that

$$\begin{aligned}
P(T_n^{(1)} > \epsilon) &\leq \exp(-u_n\epsilon) E \exp(u_n T_n^{(1)})\\
&\leq g_U(n) \exp\Big( - u_n\epsilon + u_n \sum_{i=1}^{n} EX_{ni}^{(1)} + \frac{u_n^2}{2} e^{u_n n^{-\rho}} \sum_{i=1}^{n} E(X_{ni}^{(1)})^2 \Big), \quad \forall u_n > 0.
\end{aligned} \tag{2.8}$$

By (1.1) and (1.2), we get $g_U(n) \geq 1$ and $g_L(n) \geq 1$ for all $n \geq 1$, thus, by (2.2), we obtain

$$\lim_{n\to\infty} c_n = 0. \tag{2.9}$$

In view of $EX_n = 0$ $(n \geq 1)$, $EX^p < \infty$, $p > 2$, (2.4), and (2.9), choose $\rho$ small enough such that $1 - 2/p - \rho p > 0$, we get that

$$\begin{aligned}\left|\sum_{i=1}^n EX_{ni}^{(1)}\right| &\leq \sum_{i=1}^n \left\{E|a_{ni}X_i|I(|a_{ni}X_i| > n^{-\rho}) + n^{-\rho}P(|a_{ni}X_i| > n^{-\rho})\right\} \\ &\leq 2\sum_{i=1}^n n^{\rho(p-1)}E|a_{ni}X_i|^p \\ &\leq Cn^{\rho(p-1)}(\max_{1\leq i\leq n} a_{ni})^{p-2}\sum_{i=1}^n a_{ni}^2 \\ &\leq Cn^{\rho(p-1)-1+2/p}c_n \to 0, \quad n \to \infty.\end{aligned} \tag{2.10}$$

We note that $EX^p < \infty$ and $p > 2$ imply that $EX^2 < \infty$. So

$$\sum_{i=1}^n E(X_{ni}^{(1)})^2 \leq \sum_{i=1}^n a_{ni}^2 EX_i^2 \leq Cc_n. \tag{2.11}$$

Let $u_n \in (0, n^\rho]$, from (2.8), (2.10) and (2.11), for sufficiently large $n$, we have that

$$P(T_n^{(1)} > \epsilon) \leq g_U(n)\exp(-\frac{\epsilon}{2}u_n + Cu_n^2 c_n). \tag{2.12}$$

Let $u_n = \min\{\epsilon/(4Cc_n), n^\rho\}$. If $\epsilon/(4Cc_n) \geq n^\rho$, for sufficiently large $n$, from (2.12), we get that $P(T_n^{(1)} > \epsilon) \leq g_U(n)\exp(-\epsilon n^\rho/4)$. If $\epsilon/(4Cc_n) < n^\rho$, then $P(T_n^{(1)} > \epsilon) \leq g_U(n)\exp(-\epsilon^2/(16Cc_n))$. Thus, from (2.1) and (2.2), (2.7) holds. In a similar way, we have $\sum_{n=1}^\infty P(-T_n^{(1)} > \epsilon) < \infty$. Therefore, (2.6) holds, we have proved that $T_n^{(1)} \to 0$ a.s. $n \to \infty$.

For $T_n^{(3)}$, choose positive integer $N$ such that $(1 - 2/p - p\rho)N > \theta$. For any fixed $\epsilon > 0$, and for sufficiently large $n$ such that $n^{-\rho} \leq d_n < \epsilon/N$, from the definition of $X_{ni}^{(3)}$, we get that: if $a_{ni}X_i \leq n^{-\rho}$, then $X_{ni}^{(3)} \leq 0$; if $a_{ni}X_i > n^{-\rho}$, then $X_{ni}^{(3)} \leq d_n < \epsilon/N$. So we have that

$$\begin{aligned}P(T_n^{(3)} > \epsilon) &\leq P \text{ (there are at least } N \text{ values of } i \text{ such that } a_{ni}X_i > n^{-\rho}) \\ &\leq \sum_{1\leq i_1<\cdots<i_N\leq n} P(a_{ni_1}X_{i_1} > n^{-\rho}, \cdots, a_{ni_N}X_{i_N} > n^{-\rho}) \\ &\leq \sum_{1\leq i_1<\cdots<i_N\leq n} g_U(n)P(a_{ni_1}X_{i_1} > n^{-\rho})\cdots P(a_{ni_N}X_{i_N} > n^{-\rho}) \\ &\leq g_U(n)\Big(\sum_{i=1}^n P(a_{ni}X_i > n^{-\rho})\Big)^N \leq g_U(n)\Big(n^{\rho p}\sum_{i=1}^n a_{ni}^p E|X_i|^p\Big)^N \\ &\leq g_U(n)\Big(n^{\rho p}\max_{1\leq i\leq n} a_{ni}^{p-2}\sum_{i=1}^n a_{ni}^2\Big)^N \\ &\leq g_U(n)n^{-(1-2/p-p\rho)N}(c_n)^N.\end{aligned} \tag{2.13}$$

Therefore, we get $\sum_{n=1}^\infty P(T_n^{(3)} > \epsilon) < \infty$ by (2.1) and (2.13). In a similar way, we have $\sum_{n=1}^\infty P(-T_n^{(3)} > \epsilon) < \infty$. Thus $\sum_{n=1}^\infty P(|T_n^{(3)}| > \epsilon) < \infty$, which implies $T_n^{(3)} \to 0$ a.s. $n \to \infty$.

For $T_n^{(2)}$, by (2.5) and the Borel-Cantelli Lemma, we have $P(|X_n| > d_n n^{1/p} \text{ i.o. }) = 0$. Thus $\exists$ event $\Omega_0 \subset \Omega$ such that $P(\Omega_0) = 1$ and for $\forall\ \omega \in \Omega_0, \exists$ positive interger $N(\omega)$ satisfying $|X_k| \le d_k k^{1/p}, \forall\ k > N(\omega)$. So, from (2.4) and the definition of $d_n$, we have that $I(|a_{nk}X_k| > d_n) \le I(|X_k| > d_n n^{1/p}) = 0$ for $\forall\ \omega \in \Omega_0, k \in (N(\omega), n]$. Then we have by the definition of $X_{ni}^{(2)}$ that

$$\begin{aligned}\left|T_n^{(2)}\right| = \left|\sum_{i=1}^{N(\omega)} X_{ni}^{(2)}\right| &\le \sum_{i=1}^{N(\omega)} |a_{ni}X_i| I(|a_{ni}X_i| > d_n) \\ &\le n^{-1/p} \sum_{i=1}^{N(\omega)} |X_i| \to 0, \quad n \to \infty,\ \omega \in \Omega_0. \qquad (2.14)\end{aligned}$$

Therefore, (2.3) holds. $\Box$

**Corollary 2.2** *Let $p, \{X_n, n \ge 1\}$ and $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ be as in Theorem 2.1. If (2.1) holds and*

$$\sum_{i=1}^{k_n} a_{ni}^2 = o\left((\log n)^{-1}\right), \qquad (2.15)$$

*then (2.3) holds.*

**Proof** Since (2.15) and (2.1) imply (2.2), the result follows from Theorem 2.1. $\Box$

**Corollary 2.3** *Let $p > 2, \{X_n, n \ge 1\}$ be a sequence of ENOD random variables with $EX_n = 0, n \ge 1, \{X_n, n \ge 1\} \prec X$, and $EX^p < \infty$, $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ be as in Theorem 2.1. If*

$$\sum_{n=1}^{\infty} \exp\left(-u/c_n\right) < \infty, \quad \forall u > 0, \qquad (2.16)$$

*then (2.3) holds.*

**Proof** Since $g_U(n) = g_L(n) = M$ for all $n \ge 1$ and some finite positive constant $M \ge 1$, (2.1) is satisfied automatically for $\theta > 1$ and (2.16) implies (2.2). Hence the result follows from Theorem 2.1. $\Box$

**Corollary 2.4** *Let $p, \{X_n, n \ge 1\}$ and $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ be as in Corollary 2.3. If (2.15) holds, then (2.3) holds.*

**Theorem 2.5** *Let $0 < p \le 2, \{X_n, n \ge 1\}$ be a sequence of WOD random variables with $\{X_n, n \ge 1\} \prec X$, and $EX^p < \infty$. Let $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of constants such that $\max_{1 \le i \le k_n} |a_{ni}| = O(n^{-1/p})$. If $p > 1$, moreover, we assume that $EX_n = 0, n \ge 1$. If (2.1) holds and*

$$\sum_{i=1}^{k_n} |a_{ni}|^p = O(n^{-\delta}) \text{ for some } \delta > 0, \qquad (2.17)$$

*then (2.3) holds.*

**Proof** The proof is similar to that of Theorem 2.1. Define $k_n, a_{ni}, d_n, X_{ni}^{(l)}$ $(l = 1, 2, 3)$ as in

Theorem 2.1. If $1 < p \le 2$, in view of $EX_n = 0$, we have that

$$\begin{aligned}\left|\sum_{i=1}^{n} EX_{ni}^{(1)}\right| &\le \sum_{i=1}^{n}\left\{E|a_{ni}X_i|I(|a_{ni}X_i| > n^{-\rho}) + n^{-\rho}P(|a_{ni}X_i| > n^{-\rho})\right\}\\ &\le 2n^{\rho(p-1)}\sum_{i=1}^{n} E|a_{ni}X_i|^p \le Cn^{\rho(p-1)-\delta}.\end{aligned}$$

Choose $\rho$ small enough such that $\rho p - \delta < 0$, then $\sum_{i=1}^{n} EX_{ni}^{(1)} \to 0$, $n \to \infty$. If $0 < p \le 1$,

$$\begin{aligned}\left|\sum_{i=1}^{n} EX_{ni}^{(1)}\right| &\le \sum_{i=1}^{n} E|a_{ni}X_i|I(|a_{ni}X_i| \le n^{-\rho}) + n^{-\rho}\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{-\rho})\\ &\le 2n^{\rho(p-1)}\sum_{i=1}^{n} E|a_{ni}X_i|^p\\ &\le Cn^{\rho(p-1)-\delta} \to 0, \quad n \to \infty, \ \forall\, \rho > 0.\end{aligned}$$

On the other hand, by the definition of $X_{ni}^{(1)}$ and (2.17), we have

$$\begin{aligned}n^{\rho}\sum_{i=1}^{n} E(X_{ni}^{(1)})^2 &\le n^{\rho}\sum_{i=1}^{n} E\left\{a_{ni}^2X_i^2I(|a_{ni}X_i| \le n^{-\rho}) + n^{-2\rho}I(|a_{ni}X_i| > n^{-\rho})\right\}\\ &\le Cn^{-\delta-\rho(1-p)} \to 0, \quad n \to \infty.\end{aligned}$$

Then, take $u_n = n^{\rho}$, from (2.8), for sufficiently large $n$, we obtain

$$P(T_n^{(1)} > \epsilon) \le g_U(n)\exp(-n^{\rho}\epsilon/2).$$

Therefore, (2.7) remains true. Similarly $\sum_{n=1}^{\infty} P(-T_n^{(1)} > \epsilon) < \infty$. Hence $T_n^{(1)} \to 0$ a.s. $n \to \infty$.

Similarly to (2.13) and (2.14), we have $T_n^{(l)} \to 0$ a.s. $n \to \infty, l = 2, 3$. □

**Corollary 2.6** *Let $0 < p \le 2, \{X_n, n \ge 1\}$ be a sequence of ENOD random variables with $\{X_n, n \ge 1\} \prec X$, and $EX^p < \infty$. Let $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ be as in Theorem 2.5. If $p > 1$, moreover, we assume that $EX_n = 0, n \ge 1$. If (2.17) holds, then (2.3) holds.*

**Remark 2.7** When $p > 2$, the condition (2.15) is weaker than the condition in Theorem C. When $1 < p \le 2$, from Lemma 1.4 of Sung [3], the condition (2.17) is equivalent to the condition in Theorem C. Therefore Corollaries 2.4 and 2.6 not only extend the Theorem C from i.i.d. to ENOD random variables, but also extend the corresponding results of Jing and Liang [10] from NA to ENOD random variables.

**Theorem 2.8** *Let $0 < p < 2, 1/p = 1/\alpha + 1/\beta$ for $0 < \alpha, \beta < \infty$, and let $\{X_n, n \ge 1\}$ be a sequence of WOD random variables with $\{X_n, n \ge 1\} \prec X$, and $EX_n = 0, n \ge 1, EX^{\beta} < \infty$. Let $\{b_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants such that*

$$\sum_{i=1}^{n} |b_{ni}|^{\alpha} = O(n). \tag{2.18}$$

*If (2.1) holds, then*

$$n^{-1/p}\sum_{i=1}^{n}b_{ni}X_i\to 0 \quad \text{a.s.} \quad n\to\infty. \tag{2.19}$$

**Proof** We shall apply Corollary 2.2 and Theorem 2.5 to prove Theorem 2.8. Define $a_{ni}=b_{ni}/n^{1/p}$ for $1\le i\le n$ and $n\ge 1$. Since $\max_{1\le i\le n}|b_{ni}|\le n^{1/\alpha}(\frac{1}{n}\sum_{i=1}^{n}|b_{ni}|^{\alpha})^{1/\alpha}$, assumption (2.18) implies that

$$\max_{1\le i\le n}|a_{ni}|=O(n^{-1/\beta}).$$

By (2.18) and the Hölder's inequality and the $C_r$-inequality, respectively, we have that

$$\sum_{i=1}^{n}|b_{ni}|^{\gamma}=\Big(\sum_{i=1}^{n}|b_{ni}|^{\alpha}\Big)^{\gamma/\alpha}\Big(\sum_{i=1}^{n}1\Big)^{1-\gamma/\alpha}\le Cn,\quad \forall 0<\gamma<\alpha; \tag{2.20}$$

$$\sum_{i=1}^{n}|b_{ni}|^{\gamma}\le\sum_{i=1}^{n}(|b_{ni}|^{\alpha})^{\gamma/\alpha}\le\Big(\sum_{i=1}^{n}|b_{ni}|^{\alpha}\Big)^{\gamma/\alpha}\le Cn^{\gamma/\alpha},\quad \forall\gamma\ge\alpha. \tag{2.21}$$

For the case $\beta>2$, we show that the condition (2.15) of Corollary 2.2 holds. If $0<\alpha\le 2$, by (2.21), we have that

$$\sum_{i=1}^{n}|a_{ni}|^{2}\le Cn^{2/\alpha}n^{-2/p}=Cn^{-2/\beta}. \tag{2.22}$$

If $\alpha>2$, by (2.20), we have that

$$\sum_{i=1}^{n}|a_{ni}|^{2}\le Cn^{1-2/p}. \tag{2.23}$$

From (2.22), (2.23) and $0<p<2$, we obtain (2.15).

For the case $0<\beta\le 2$, we prove that condition (2.17) of Theorem 2.5 holds. If $\beta<\alpha$, by (2.20), we have that

$$\sum_{i=1}^{n}|a_{ni}|^{\beta}\le Cnn^{-\beta/p}=Cn^{-\beta/\alpha}. \tag{2.24}$$

If $\beta\ge\alpha$, by (2.21), we have

$$\sum_{i=1}^{n}|a_{ni}|^{\beta}\le Cn^{\beta/\alpha}n^{-\beta/p}=Cn^{-1}. \tag{2.25}$$

Hence, from (2.24) and (2.25), (2.17) holds. Thus, (2.19) holds by Corollary 2.2 and Theorem 2.5. $\square$

**Corollary 2.9** *Let $0<p<2, 1/p=1/\alpha+1/\beta$ for $0<\alpha,\beta<\infty$, and let $\{X_n,n\ge 1\}$ be a sequence of ENOD random variables with $\{X_n,n\ge 1\}\prec X$, and $EX_n=0, n\ge 1, EX^{\beta}<\infty$. Let $\{b_{ni},1\le i\le n,n\ge 1\}$ be as in Theorem 2.8. Then (2.19) holds.*

**Remark 2.10** Corollary 2.9 extends Theorem D (see also Corollary 3 of Sung [3]) on i.i.d. case to ENOD random variables, and extends the corresponding results of Jing and Liang [10] and Wu [9].

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