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Goodearl-Menal Pairs of Linear Transformations

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Abstract In this short note we discuss the GM property of some special linear transformation pairs over infinite-dimensional vector spaces. In particular, it is proved that if $R = \mathrm{End}(V_D)$ is the endomorphism ring of an infinite-dimensional right vector space V over a division ring D with $|C(D)| > 3$ and $g \in R$, then $(a_0 + a_1g, g)$ is a GM pair for any $a_0, a_1 \in C(D)$. Furthermore, two existing results are obtained as immediate consequences.

Keywords Goodearl-Menal condition; infinite-dimensional vector space; linear transformation; GM pair.

MR(2010) Subject Classification 16S50; 16U99

1. Introduction

Let R be a ring and $U(R)$ the group of units in R. Recall that R is said to satisfy unit 1-stable range if whenever $aR + bR = R$, there exists $u \in U(R)$ such that $a + bu \in U(R)$. This condition has been studied extensively by many authors. In particular, Menal and Mocasi [1] proved that if R satisfies unit 1-stable range, then $K_1(R) = U(R)/V(R)$, where $K_1(R)$ is the K_1 group of R and $V(R)$ is the subgroup of $U(R)$ generated by $\{(ab+1)(ba+1)^{-1} : ab+1 \in U(R)\}$. A ring R satisfies unit 1-stable range provided that for any $x, y \in R$, there exists $u \in U(R)$ such that $x-u, y-u^{-1} \in U(R)$ (see [2]). The latter condition is called the Goodearl-Menal condition by Chen [3], and has been discussed in [2–7]. In the rest of the paper we will use the term GM condition instead of the Goodearl-Menal condition for brevity.

In [7, Corollary 2.9], the authors proved that in general, the endomorphism ring of an infinite-dimensional vector space over a division ring does not satisfy the GM condition. Precisely, they proved that if R is the ring of linear transformations of a right vector space V over a division ring D, then R satisfies the GM condition if and only if V_D is finitely dimensional and V_D is not isomorphic to $(\mathbb{Z}_2)_{\mathbb{Z}_2}$ or $(\mathbb{Z}_3)_{\mathbb{Z}_3}$ or $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)_{\mathbb{Z}_2}$. But some elements in $\mathrm{End}(V_D)$ do have the GM property. In this paper we will discuss the GM property of a class of special linear transformation pairs over an infinite-dimensional vector space V_D. Furthermore, two existing results (see [5, Theorem 5] and [8, Theorem 1(2),(3)]) are obtained as corollaries.

Throughout the paper, all rings are associative with identity and modules are unitary right modules. For a right module M over a ring R, denote the endomorphism ring of M_R by $\mathrm{End}(M_R)$

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or End(M). For a ring R, denote by $C(R)$ the set of central elements in R. Write $\mathbb{Z}_n$ for the ring of integers modulo n and $|X|$ for the cardinal of a set X.

2. Main results

Let R be a ring and $x, y \in R$. (x, y) is called a Goodearl-Menal (or GM) pair of R if there exists $u \in U(R)$ such that $x - u$, $y - u^{-1} \in U(R)$. Obviously, the ring R satisfies the GM condition if and only if for any $x, y \in R$, (x, y) is a GM pair. For a vector space V_D, let $g \in \mathrm{End}(V_D)$ and $a_0, a_1 \in C(D)$. Then $f := a_0 + a_1 g$ is an element of $\mathrm{End}(V_D)$ which is defined by $f(v) = va_0 + g(v)a_1$, $v \in V$. For a countably infinite-dimensional vector space V_D, a linear transformation $f \in \mathrm{End}(V_D)$ is called a shift operator if there exists a basis $\{v_1, v_2, \ldots\}$ of V such that $f(v_i) = v_{i+1}$ for all i.

Theorem 2.1 *Let $R = \mathrm{End}(V_D)$ and $f, g \in R$, where V is an infinite-dimensional right vector space over a division ring D with $|C(D)| > 3$. If $f = a_0 + a_1 g$, where $a_0, a_1 \in C(D)$, then (f, g) is a GM pair.*

Proof Let $S = \{(W, u) | W$ is an f-and g-invariant subspace of V, u, $f|_W - u$, $g|_W - u^{-1} \in U(\mathrm{End}(W))\}$. It is obvious that $((0), 1) \in S$. Define $(W', u') \le (W, u)$ by $W' \subseteq W$ and $u' = u|_{W'}$. This gives a partial order on the set S, and under this order S is an inductive set. Thus, by Zorn's Lemma, there exists a maximal element $(T, h) \in S$. We need only to prove that $T = V$. Suppose on the contrary that $T \ne V$.

Let $0 \ne x \in V \backslash T$ and write $K = \mathrm{span}\{x, g(x), g^2(x), \ldots\}$. Note that $f = a_0 + a_1 g$. Then K is an f and g-invariant subspace of V. Let $V_0 := T + K$ and write $V_0 = T \oplus N$ for some $0 \ne N \le V_0$. For $v = t + n \in V_0$, $t \in T$, $n \in N$, we define the following homomorphisms:

$$\begin{aligned}
&\overline{f}, \overline{g} : V_0/T \to V_0/T \text{ with } \overline{f}(\overline{v}) = \overline{f(v)},\ \overline{g}(\overline{v}) = \overline{g(v)},\\
&\pi : V_0 \to N \text{ with } \pi(v) = n,\\
&\varphi : V_0/T \to N \text{ with } \varphi(\overline{v}) = \pi(v),\\
&\theta_1 := \varphi \overline{f} \varphi^{-1} : N \to N,\\
&\theta_2 := \varphi \overline{g} \varphi^{-1} : N \to N.
\end{aligned}$$

It follows that $V_0/T = \mathrm{span}\{\overline{x}, \overline{g}(\overline{x}), \overline{g}^2(\overline{x}), \ldots\}$ and

$$\begin{aligned}
N &= \mathrm{span}\{\varphi(\overline{x}), \varphi(\overline{g}(\overline{x})), \varphi(\overline{g}^2(\overline{x})), \ldots\}\\
&= \mathrm{span}\{\varphi(\overline{x}), \theta_2\varphi(\overline{x}), \theta_2^2\varphi(\overline{x}), \ldots\}.
\end{aligned}$$

Claim In the endomorphsim ring $\mathrm{End}(N_D)$ of N_D, (θ_1, θ_2) is a GM pair.

Proof of Claim It suffices to find some $\alpha \in U(\mathrm{End}(N_D))$ such that $\theta_1 - \alpha$, $\theta_2 - \alpha^{-1} \in U(\mathrm{End}(N_D)$. Denote $\omega_i = \theta_2^i \varphi(\overline{x})$, $i = 0, 1, 2, \ldots$. It can be seen that N_D is finitely dimensional or $\theta_2 \in \mathrm{End}(N_D)$ is a shift operator.

If $\dim N_D < \infty$, since $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$, there exists $\alpha \in U(\mathrm{End}(N_D))$ such that $\theta_1 - \alpha$, $\theta_2 - \alpha^{-1} \in$

$U(\mathrm{End}(N_D))$ by [7, Corollary 2.9].

If $\dim N_D = \infty$, then $\{\varphi(\overline{x}), \theta_2\varphi(\overline{x}), \theta_2^2\varphi(\overline{x}), \ldots\}$ is a basis of N and hence θ_2 is a shift operator with respect to this basis. We discuss this situation by two cases.

Case 1 $a_1 = 0$. Since $|C(D)| > 3$, we may take $b \in C(D)$ such that $b \neq 0$ and $b \neq a_0$. Note the fact that $\theta_1 = a_0 + a_1\theta_2$ since $f = a_0 + a_1 g$.

For $k \geq 1$, define $\alpha : N \to N$ by

$$\alpha(\omega_{2k-1}) = \omega_{2k-1}b - \omega_{2k}b^2, \quad \alpha(\omega_{2k}) = \omega_{2k}b.$$

Then $\alpha \in U(\mathrm{End}(N_D))$ with

$$\alpha^{-1}(\omega_{2k-1}) = \omega_{2k-1}b^{-1} + \omega_{2k}, \quad \alpha^{-1}(\omega_{2k}) = \omega_{2k}b^{-1},\ k \geq 1.$$

It follows that $\theta_1 - \alpha$, $\theta_2 - \alpha^{-1} \in U(\mathrm{End}(N_D))$. In fact, it can be computed that for $k \geq 1$,

$$(\theta_1 - \alpha)(\omega_{2k-1}) = \omega_{2k-1}(a_0 - b) + \omega_{2k}b^2,$$
$$(\theta_1 - \alpha)(\omega_{2k}) = \omega_{2k}(a_0 - b)$$

with

$$(\theta_1 - \alpha)^{-1}(\omega_{2k-1}) = \omega_{2k-1}(a_0 - b)^{-1} - \omega_{2k}(a_0 - b)^{-1}b^2(a_0 - b)^{-1},$$
$$(\theta_1 - \alpha)^{-1}(\omega_{2k}) = \omega_{2k}(a_0 - b)^{-1}.$$

Also,

$$(\theta_2 - \alpha^{-1})(\omega_{2k-1}) = -\omega_{2k-1}b^{-1},$$
$$(\theta_2 - \alpha^{-1})(\omega_{2k}) = -\omega_{2k}b^{-1} + \omega_{2k+1}$$

with

$$(\theta_2 - \alpha^{-1})^{-1}(\omega_{2k-1}) = -\omega_{2k-1}b,$$
$$(\theta_2 - \alpha^{-1})^{-1}(\omega_{2k}) = -\omega_{2k}b - \omega_{2k+1}b^2.$$

Case 2 $a_1 \neq 0$. Since $|C(D)| > 3$, we may take $b \in C(D)$ such that $b \neq 0$, $b \neq a_0$ and if in addition $a_0 \neq 0$, let $b \neq -a_0^{-1}a_1$. It follows that $a_1b^{-1} \neq 0$ and $a_0 + a_1b^{-1} \neq 0$.

For $k \geq 1$, define $\alpha : N \to N$ by

$$\alpha(\omega_{2k-1}) = \omega_{2k-1}b + \omega_{2k}a_1, \quad \alpha(\omega_{2k}) = -\omega_{2k}a_1b^{-1}.$$

Then $\alpha \in U(\mathrm{End}(N_D))$ with

$$\alpha^{-1}(\omega_{2k-1}) = \omega_{2k-1}b^{-1} + \omega_{2k}, \quad \alpha^{-1}(\omega_{2k}) = -\omega_{2k}ba_1^{-1}.$$

It follows that $\theta_1 - \alpha$, $\theta_2 - \alpha^{-1} \in U(\mathrm{End}(N_D))$. In fact, it can be computed that for $k \geq 1$,

$$(\theta_1 - \alpha)(\omega_{2k-1}) = \omega_{2k-1}(a_0 - b),$$
$$(\theta_1 - \alpha)(\omega_{2k}) = \omega_{2k}(a_0 + a_1b^{-1}) + \omega_{2k+1}a_1$$

and

$$(\theta_1 - \alpha)^{-1}(\omega_{2k-1}) = \omega_{2k-1}(a_0 - b)^{-1},$$

$$(\theta_1-\alpha)^{-1}(\omega_{2k})=\omega_{2k}(a_0+a_1b^{-1})^{-1}-\omega_{2k+1}(a_0-b)^{-1}a_1(a_0+a_1b^{-1})^{-1}.$$

Also,

$$(\theta_2-\alpha^{-1})(\omega_{2k-1})=-\omega_{2k-1}b^{-1},$$
$$(\theta_2-\alpha^{-1})(\omega_{2k})=\omega_{2k}ba_1^{-1}+\omega_{2k+1}$$

with

$$(\theta_2-\alpha^{-1})^{-1}(\omega_{2k-1})=-\omega_{2k-1}b,$$
$$(\theta_2-\alpha^{-1})^{-1}(\omega_{2k})=\omega_{2k}a_1b^{-1}+\omega_{2k+1}ba_1b^{-1}.$$

Thus the claim is proved. □

Let $s:V_0\to V_0$ be given by $s(t+n)=h(t)+\alpha(n)$, $t\in T$, $n\in N$, where α is given as in the proof of Claim accordingly. Then $s\in U(\mathrm{End}(V_0))$. We next show that $f-s$, $g-s^{-1}\in U(\mathrm{End}(V_0))$.

For $t\in T$, $n\in N$, $(f-s)(t+n)=(f-s)(t)+[f(n)-\alpha(n)]$. Applying π to both sides of the equation, we get

$$\begin{aligned}\pi(f-s)(t+n)&=\pi[f(n)-\alpha(n)]=\pi f(n)-\pi\alpha(n)\\&=\varphi(\overline{f}(\overline{n}))-\alpha(n)=\varphi\overline{f}(\overline{n})-\alpha(n)\\&=\theta_1\varphi(\overline{n})-\alpha(n)=\theta_1\pi(n)-\alpha(n)\\&=(\theta_1-\alpha)(n).\end{aligned}$$

We now prove that $f-s$ is an isomorphism of V_0.

To see that $f-s$ is a monomorphism, let $(f-s)(t+n)=0$. Then $(\theta_1-\alpha)(n)=0$. Since $\theta_1-\alpha\in U(\mathrm{End}(N))$, $n=0$. This gives $(f-s)(t)=0$, and hence $t=0$ since $(f-s)|_T=f|_T-h\in U(\mathrm{End}(T))$.

To see that $f-s$ is an epimorphism, note that $T\subseteq \mathrm{Im}(f-s)$. For any $\omega\in N$, there exists $n\in N$ such that $\omega=(\theta_1-\alpha)(n)=\pi(f-s)(t+n)\in\mathrm{Im}(f-s)$ since $T\subseteq\mathrm{Im}(f-s)$. Thus $V_0=T\oplus N\subseteq\mathrm{Im}(f-s)$.

Hence $f-s$ is an isomorphism.

Similarly, we have

$$(g-s^{-1})(t+n)=(g-s^{-1})(t)+[g(n)-\alpha^{-1}(n)],$$
$$\pi(g-s^{-1})(t+n)=(\theta_2-\alpha^{-1})(n),$$

and we can prove that $g-s^{-1}\in U(\mathrm{End}(V_0))$.

Thus, $(V_0,s)\in S$, $(T,h)\le(V_0,s)$ and $(T,h)\ne(V_0,s)$, which is a contradiction. This implies that $V=T$ and hence the proof is complete. □

Following [8], a ring R is said to satisfy condition (P) if for any $a\in R$, there exists $u\in U(R)$ such that $a+u$, $a-u^{-1}\in U(R)$. Let $a_0=0$ and $a_1=-1$, that is, let $f=-g$ in Theorem 2.1. Since $-1\in C(D)$, by letting $b=-1$ in the proof of Case 2 in Theorem 2.1, we get that (f,g) is a GM pair. In this situation, we need only to assume that $D\ncong\mathbb{Z}_2$ or $\mathbb{Z}_3$ rather than $|C(D)|>3$.

Thus, we have [5, Theorem 5] and [8, Theorem 1(2)] as corollaries.

Corollary 2.2 *Let $R = \mathrm{End}(V_D)$, where V is a vector space over a division ring D with $D \ncong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Then R satisfies condition (P).*

Following [8], a ring R is said to satisfy condition (Q) if for any $a \in R$, there exists $u \in U(R)$ such that $a - u$, $a - u^{-1} \in U(R)$. Let $a_0 = 0$ and $a_1 = 1$, that is, let $f = g$ in Theorem 2.1. Since $1 \in C(D)$, by letting $b = 1$ in the proof of Case 2 in Theorem 2.1, we get that (f, g) is a GM pair. In this situation, we need only to assume that $D \ncong \mathbb{Z}_2$ rather than $|C(D)| > 3$. In fact, if $\dim N_D < \infty$ in the proof of Claim in Theorem 2.1 and $D \ncong \mathbb{Z}_2$, then there exists $\alpha \in U(\mathrm{End}(N_D))$ such that $\theta_1 - \alpha$, $\theta_2 - \alpha^{-1} \in U(\mathrm{End}(N_D))$ by [7, Proposition 4.2]. Thus, we have [8, Theorem 1(3)] as a corollary.

Corollary 2.3 *Let $R = \mathrm{End}(V_D)$, where V is a vector space over a division ring D with $D \ncong \mathbb{Z}_2$. Then R satisfies condition (Q).*

A ring R is called 2-good [9] if every element of R can be written as a sum of two units. By Corollary 2.3, the following result is obvious.

Corollary 2.4 *Let $R = \mathrm{End}(V_D)$, where V is a vector space over a division ring D with $D \ncong \mathbb{Z}_2$. Then R is a 2-good ring.*

3. Discussion for D with $C(D) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$

For D with $C(D) \cong \mathbb{Z}_2$, we do not know if Theorem 2.1 is true. But if V is a countably infinite-dimensional right vector space over D and g is a shift operator, Theorem 2.1 is true (See Proposition 3.1). Thus, if $C(D) \cong \mathbb{Z}_2$ and $\dim N \geq 3$, Theorem 2.1 holds by the above statement and [7, Corollary 2.9].

Proposition 3.1 *Let $R = \mathrm{End}(V)$ and $g \in R$ be a shift operator, where V is a countably infinite-dimensional right vector space over a division ring D with $C(D) \cong \mathbb{Z}_2$. If $f = a_0 + a_1 g$, where a_0, $a_1 \in C(D)$, (f, g) is a GM pair.*

Proof Let $\{\omega_1, \omega_2, \ldots\}$ be a basis of V. Since $2 = 0$ in $C(D) \cong \mathbb{Z}_2$, the proof of Theorem 2.1 is not working. We discuss this situation by three cases. If $f = 1$, for $k \geq 1$, define $\alpha \in U(\mathrm{End}(V))$ as

$$\begin{aligned}
\alpha(\omega_{3k-2}) &= \omega_{3k-1} + \omega_{3k}, \\
\alpha(\omega_{3k-1}) &= \omega_{3k-2} + \omega_{3k-1} + \omega_{3k}, \\
\alpha(\omega_{3k}) &= \omega_{3k-2} + \omega_{3k}
\end{aligned}$$

with its inverse defined by

$$\begin{aligned}
\alpha^{-1}(\omega_{3k-2}) &= \omega_{3k-2} + \omega_{3k-1}, \\
\alpha^{-1}(\omega_{3k-1}) &= \omega_{3k-1} + \omega_{3k}, \\
\alpha^{-1}(\omega_{3k}) &= \omega_{3k-2} + \omega_{3k-1} + \omega_{3k}.
\end{aligned}$$

It follows that

$$\begin{aligned}(\theta_1-\alpha)(\omega_{3k-2}) &= \omega_{3k-2}+\omega_{3k-1}+\omega_{3k},\\ (\theta_1-\alpha)(\omega_{3k-1}) &= \omega_{3k-2}+\omega_{3k},\\ (\theta_1-\alpha)(\omega_{3k}) &= \omega_{3k-2}\end{aligned}$$

with its inverse given by

$$\begin{aligned}(\theta_1-\alpha)^{-1}(\omega_{3k-2}) &= \omega_{3k},\\ (\theta_1-\alpha)^{-1}(\omega_{3k-1}) &= \omega_{3k-2}+\omega_{3k-1},\\ (\theta_1-\alpha)^{-1}(\omega_{3k}) &= \omega_{3k-1}+\omega_{3k}\end{aligned}$$

and

$$\begin{aligned}(\theta_2-\alpha^{-1})(\omega_{3k-2}) &= \omega_{3k-2},\\ (\theta_2-\alpha^{-1})(\omega_{3k-1}) &= \omega_{3k-1},\\ (\theta_2-\alpha^{-1})(\omega_{3k}) &= \omega_{3k-2}+\omega_{3k-1}+\omega_{3k}+\omega_{3k+1}\end{aligned}$$

with its inverse given by

$$(\theta_2-\alpha^{-1})^{-1}=\theta_2-\alpha^{-1}.$$

Similarly, define α as $\alpha(\omega_{2k-1})=\omega_{2k-1}+\omega_{2k}$, $\alpha(\omega_{2k})=\omega_{2k}$ with its inverse given by $\alpha^{-1}=\alpha$ for $k\ge 1$ when $f=g$ and define α as $\alpha(\omega_{3k-2})=\omega_{3k-2}$, $\alpha(\omega_{3k-1})=\omega_{3k-2}+\omega_{3k-1}+\omega_{3k}$, $\alpha(\omega_{3k})=\omega_{3k-2}+\omega_{3k-1}$ with its inverse given by $\alpha^{-1}(\omega_{3k-2})=\omega_{3k-2}$, $\alpha^{-1}(\omega_{3k-1})=\omega_{3k-2}+\omega_{3k}$, $\alpha^{-1}(\omega_{3k})=\omega_{3k-1}+\omega_{3k}$ for $k\ge 1$ when $f=1+g$. $\square$

For D with $C(D)\cong\mathbb{Z}_3$, we do not know if Theorem 2.1 is true. But if V is a countably infinite-dimensional right vector space over D and g is a shift operator, then Theorem 2.1 is true (see Proposition 3.2). Thus, if $C(D)\cong\mathbb{Z}_3$ and $\dim N\ge 2$, Theorem 2.1 holds by the above statement and [7, Corollary 2.9].

Proposition 3.2 *Let $R=\mathrm{End}(V)$ and $g\in R$ be a shift operator, where V is a countably infinite-dimensional right vector space over a division ring D with $C(D)\cong\mathbb{Z}_3$. If $f=a_0+a_1g$, where a_0, $a_1\in C(D)$, $(f,\, g)$ is a GM pair.*

Proof Let $\{\omega_1,\omega_2,\ldots\}$ be a basis of V. Without loss of generality, we may assume that $C(D)=\mathbb{Z}_3$. By the proof of Theorem 2.1, we need only to prove the situation when $a_0=1$ or 2 and $a_1=1$, that is, $f=1+g$ or $f=2+g$. Take $\alpha\in U(\mathrm{End}(V))$ given by

$$\begin{aligned}\alpha(\omega_{3k-2}) &= \omega_{3k-2},\\ \alpha(\omega_{3k-1}) &= \omega_{3k-2}+\omega_{3k-1}+\omega_{3k},\\ \alpha(\omega_{3k}) &= \omega_{3k-2}+\omega_{3k-1}\end{aligned}$$

with its inverse given by

$$\alpha^{-1}(\omega_{3k-2})=\omega_{3k-2},$$

$$\alpha^{-1}(\omega_{3k-1}) = 2\omega_{3k-2} + \omega_{3k},$$
$$\alpha^{-1}(\omega_{3k}) = \omega_{3k-1} + 2\omega_{3k}.$$

Then it can be verified that $f - \alpha$ and $g - \alpha^{-1}$ are units of $\mathrm{End}(N)$, and hence (f, g) is a GM pair. $\Box$

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References

[1] P. MENAL, J. MONCASI. *K_1 of von Neumann regular rings*. J. Pure Appl. Algebra, 1984, **33**(3): 295–312.
[2] K. R. GOODEARL, P. MENAL. *Stable range one for rings with many units*. J. Pure Appl. Algebra, 1988, **54**(2-3): 261–287.
[3] Huanyin CHEN. *Units, idempotents, and stable range conditions*. Comm. Algebra, 2001, **29**(2): 703–717.
[4] Huanyin CHEN. *On SB-rings*. J. Korean Math. Soc., 2008, **45**(3): 741–756.
[5] Huanyin CHEN. *Decompositions of countable linear transformations*. Glasg. Math. J., 2010, **52**(3): 427–433.
[6] Huanyin CHEN. *Rings Related to Stable Range Conditions*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
[7] Chunna LI, Lu WANG, Yiqiang ZHOU. *On rings with the Goodearl-Menal condition*. Comm. Algebra, 2012, **40**(12): 4679–4692.
[8] Lu WANG, Yiqiang ZHOU. *Decomposing linear transformations*. Bull. Aust. Math. Soc., 2011, **83**(2): 256–261.
[9] P. VÁMOS. *2-good rings*. Q. J. Math., 2005, **56**(3): 417–430.