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Higher Order Commutators of Marcinkiewicz Integral Operator on Herz-Morrey Spaces with Variable Exponent

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Abstract In the case of $\Omega \in \mathrm{Lip}_\gamma(\mathbb{S}^{n-1})$ $(0 < \gamma \le 1)$, we prove the boundedness of the Marcinkiewicz integral operator μ_Ω on the variable exponent Herz-Morrey spaces. Also, we prove the boundedness of the higher order commutators $\mu_{\Omega,b}^m$ with $b \in \mathrm{BMO}(\mathbb{R}^n)$ on both variable exponent Herz spaces and Herz-Morrey spaces, and extend some known results.

Keywords Herz-Morrey spaces; Marcinkiewicz integral; commutator; variable exponent.

MR(2010) Subject Classification 42B20; 42B35

1. Introduction

Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ $(n \ge 2)$ with the normalized Lebesgue measure $\mathrm{d}\sigma(x')$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and satisfy the vanishing condition on $\mathbb{S}^{n-1}$, that is,

$$\int_{\mathbb{S}^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0. \tag{1}$$

The Marcinkiewicz integral μ_Ω in higher dimension is defined by

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y.$$

Stein [1] first introduced the operator μ_Ω and proved that μ_Ω is of type (p,p) $(1 < p \le 2)$ and of weak type $(1,1)$ in the case of $\Omega \in \mathrm{Lip}_\gamma(\mathbb{S}^{n-1})$ $(0 < \gamma \le 1)$. Benedek, Calderón and Panzone [2] extended Stein's results, and proved that if $\Omega \in C^1(\mathbb{S}^{n-1})$, then μ_Ω is of type (p,p) $(1 < p < \infty)$. In 2000, Ding, Fan and Pan [6] improved the above results to the case of $\Omega \in H^1(\mathbb{S}^{n-1})$. $H^1(\mathbb{S}^{n-1})$ denotes the Hardy space on $\mathbb{S}^{n-1}$, which is the linear space of distributions $f \in \mathscr{S}'(\mathbb{S}^{n-1})$ with the norm

$$\|f\|_{H^1(\mathbb{S}^{n-1})} = \|P^+ f\|_{L^1(\mathbb{S}^{n-1})} < \infty,$$

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where

$$P^+f(x')=\sup_{0\le r<1}\Big|\int_{\mathbb{S}^{n-1}}f(y')P_{rx'}(y')\mathrm{d}\sigma(y')\Big|,$$

and

$$P_{rx'}(y')=\frac{1-r^2}{|rx'-y'|^n},\quad 0\le r<1,\ x',y'\in\mathbb{S}^{n-1}.$$

Let $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $m\in\mathbb{N}$. The higher order commutators of Marcinkiewicz integral operator $\mu^m_{\Omega,b}$ are defined by

$$\mu^m_{\Omega,b}(f)(x)=\Big(\int_0^\infty|F^m_{\Omega,b,t}(f)(x)|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac12},$$

where

$$F^m_{\Omega,b,t}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))^mf(y)\mathrm{d}y.$$

Obviously, $\mu^1_{\Omega,b}=[b,\mu_\Omega]$, which is the well-known commutator generated by b and the operator μ_Ω. Torchinsky and Wang [10] proved that if $\Omega\in\mathrm{Lip}_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\le1)$, then $[b,\mu_\Omega]$ is bounded on $L^p(\omega)$ $(1<p<\infty)$, $\omega\in A_p$. We refer to [3] for further details on A_p weight. Ding, Lu and Yabuta [7] established the boundedness of the higher order commutators $\mu^m_{\Omega,b}$ on $L^p(\omega)$ $(1<p<\infty)$. Recently, many generalized results about μ_Ω and $\mu^m_{\Omega,b}$ in various other function spaces have been extensively studied [8–9, 18–21].

It is well-known that the main motivation for studying the spaces with variable exponent comes from applications to the modeling for electrorheological fluids, image restoration and PDE with non-standard growth conditions. Since the surveying paper [4] by Kováčik and Rákosník appeared in 1991, the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ have been extensively investigated. During the last two decades, boundedness of some important operators, for example, the Calderón-Zygmund operators, fractional integrals and commutators on $L^{p(\cdot)}(\mathbb{R}^n)$ have been obtained [5, 11, 13]. Izuki [13, 14] introduced the variable exponent Herz spaces and Herz-Morrey spaces, meanwhile, he obtained the boundedness properties of the fractional integral operator and its commutators. In 2012, Wang, Fu and Liu [16] proved that μ_Ω and $\mu^m_{\Omega,b}$ are bounded on the Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Liu and Wang [17] extended the $L^{p(\cdot)}(\mathbb{R}^n)$ boundedness of μ_Ω and $[b,\mu_\Omega]$ to the variable exponent Herz spaces $\dot K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$. Motivated by [13, 14,16, 17], the main purpose of this paper is to establish the boundedness of μ_Ω and $\mu^m_{\Omega,b}$ on $\dot K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and $M\dot K^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$.

For brevity, $|E|$ denotes the Lebesgue measure for a measurable set $E\subset\mathbb{R}^n$, f_E denotes the mean value of f on E ($f_E=|E|^{-1}\int_E f(x)\mathrm{d}x$). The exponent $p'(\cdot)$ means the conjugate of $p(\cdot)$, that is, $1/p(\cdot)+1/p'(\cdot)=1$. C denotes a positive constant, which may have different values even in the same line. Let us first recall some definitions and notations.

Definition 1.1 *For $0<\gamma\le1$, if there exists a constant $L>0$ such that*

$$|\Omega(x')-\Omega(y')|\le L|x'-y'|^\gamma,\quad\forall x',y'\in\mathbb{S}^{n-1},$$

then we say that $\Omega\in\mathrm{Lip}_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\le1)$.

Definition 1.2 *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the* $\mathrm{BMO}(\mathbb{R}^n)$ *is the space of functions f satisfying*

$$\|f\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|}\int_B |f(x) - f_B|\mathrm{d}x < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Definition 1.3 *Let $p(\cdot) : E \to [1,\infty)$ be a measurable function,*

(1) The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \Big\{f \text{ is measurable} : \int_E \Big(\frac{|f(x)|}{\eta}\Big)^{p(x)}\mathrm{d}x < \infty \text{ for some constant } \eta > 0\Big\}.$$

(2) The space with variable exponent $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) = \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E\}.$$

The Lebesgue space $L^{p(\cdot)}(E)$ is a Banach space with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\Big\{\eta > 0 : \int_E \Big(\frac{|f(x)|}{\eta}\Big)^{p(x)}\mathrm{d}x \le 1\Big\}.$$

We denote

$$p_- = \text{ess inf}\{p(x) : x \in E\}, \quad p_+ = \text{ess sup}\{p(x) : x \in E\},$$

$$\mathscr{P}(E) = \{p(\cdot) : \; p_- > 1, \; p_+ < \infty\},$$

and

$$\mathscr{B}(E) = \{p(\cdot) : p(\cdot) \in \mathscr{P}(E), \quad M \text{ is bounded on } L^{p(\cdot)}(E)\},$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} r^{-n}\int_{B(x,r)\cap E} |f(y)|\mathrm{d}y,$$

where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

Proposition 1.1 ([12]) *If $p(\cdot) \in \mathscr{P}(E)$ satisfies*

$$|p(x) - p(y)| \le \frac{-C}{\log(|x-y|)}, \quad |x-y| \le 1/2,$$

$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)}, \quad |y| \ge |x|,$$

then one has $p(\cdot) \in \mathscr{B}(E)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $R_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$.

Definition 1.4 ([13]) *For $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}=\Big\{\sum_{k=-\infty}^{\infty}2^{\alpha kq}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big\}^{\frac{1}{q}}.$$

Definition 1.5 ([14]) *For $\alpha\in\mathbb{R}$, $0<q<\infty$, $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and $0\le\lambda<\infty$. The homogeneous Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ is defined by*

$$M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)=\{f\in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n\backslash\{0\}):\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}=\sup_{L\in\mathbb{Z}}2^{-L\lambda}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha q}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big\}^{\frac{1}{q}}.$$

It is obvious that $M\dot{K}^{\alpha,0}_{q,p(\cdot)}(\mathbb{R}^n)=\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.

Our results in this paper can be stated as follows.

Theorem 1.1 *Suppose that $\Omega\in{\rm Lip}_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\le1)$, $b\in{\rm BMO}(\mathbb{R}^n)$, $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. If $0<q<\infty$, $0\le\lambda<\infty$, $0<\delta_1$, $\delta_2<1$ and $-n\delta_1+\lambda<\alpha<n\delta_2$, then the operator μ_Ω is bounded on $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$.*

Theorem 1.2 *Suppose that $\Omega\in{\rm Lip}_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\le1)$, $b\in{\rm BMO}(\mathbb{R}^n)$, $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. If $0<q<\infty$, $0<\delta_1$, $\delta_2<1$ and $-n\delta_1<\alpha<n\delta_2$, then the higher order commutators $\mu^m_{\Omega,b}$ are bounded on $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.*

Theorem 1.3 *Suppose that $\Omega\in{\rm Lip}_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\le1)$, $b\in{\rm BMO}(\mathbb{R}^n)$, $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. If $0<q<\infty$, $0\le\lambda<\infty$, $0<\delta_1$, $\delta_2<1$ and $-n\delta_1+\lambda<\alpha<n\delta_2$, then the higher order commutators $\mu^m_{\Omega,b}$ are bounded on $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$.*

Remark 1.1 The previous main results improve the $L^{p(\cdot)}(\mathbb{R}^n)$ boundedness of the operator μ_Ω and the higher order commutators $\mu^m_{\Omega,b}$ in [16] to the case of Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. If $\lambda=0,m=1$, our conclusions coincide with the corresponding results in [17], so, in this paper we only consider the case $\lambda>0$. Moreover, the same boundedness also holds for the non-homogeneous case [13, 15].

Remark 1.2 In the case of $\Omega\in L^r(\mathbb{S}^{n-1})$ $(1\le r<\infty)$, we note that the boundedness of the operator μ_Ω on $L^{p(\cdot)}$ $(\mathbb{R}^n)$ is still an unsolved problem. Also, the boundedness of the commutators $\mu^m_{\Omega,b}$ with $b\in{\rm Lip}_\gamma(\mathbb{R}^n)$ $(0<\gamma\le1)$ on Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ have not been proven.

2. Proofs of the Theorems

To prove our main results, we need the following Lemmas.

Lemma 2.1 ([4]) *Let $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$. If $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n}|f(x)g(x)|{\rm d}x\le r_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $r_p = 1 + 1/p_- - 1/p_+$.

Lemma 2.2 ([13]) *Let* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. *Then for all balls* B *in* $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

Lemma 2.3 ([13]) *Let* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. *Then there exist constants* $0 < \delta_1,\ \delta_2 < 1$ *such that for all balls* B *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$,

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C(\frac{|S|}{|B|})^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C(\frac{|S|}{|B|})^{\delta_2}.$$

Lemma 2.4 ([16]) *Let* $\Omega \in \mathrm{Lip}_\gamma(\mathbb{S}^{n-1})$ $(0 < \gamma \le 1)$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. *We have*

(1) $\|\mu_\Omega(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$;

(2) $\|\mu^m_{\Omega,b}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|^m_{\mathrm{BMO}}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 2.5 ([15]) *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $k > j$ $(k, j \in \mathbb{N})$. *We have*

(1) $C^{-1}\|b\|^m_{\mathrm{BMO}} \le \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)^m\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|^m_{\mathrm{BMO}}$;

(2) $\|(b - b_{B_j})^m\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(k - j)^m\|b\|^m_{\mathrm{BMO}}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Proof of Theorem 1.1 Let $f \in M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Then, we have

$$\begin{aligned}
\|\mu_\Omega(f)\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} =& \sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha q}\|\mu_\Omega(f)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
\le& C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big(\sum_{j=-\infty}^{k-2}\|\mu_\Omega(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q + \\
& C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big(\sum_{j=k-1}^{k+1}\|\mu_\Omega(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q + \\
& C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha q}\Big(\sum_{j=k+2}^{\infty}\|\mu_\Omega(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q \\
=& U_1 + U_2 + U_3.
\end{aligned}$$

For U_2, we apply Lemma 2.4 (1) and obtain

$$U_2 \le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha q}\|f_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

For U_1, observe that if $x \in R_k, y \in R_j$ and $j \le k - 2$, then $|x - y| \sim |x| \sim 2^k$ and

$$|\frac{1}{|x-y|^2} - \frac{1}{|x|^2}| \le C\frac{|y|}{|x-y|^3}.$$

Since $\Omega \in \mathrm{Lip}_\gamma(\mathbb{S}^{n-1}) \subset L^\infty(\mathbb{S}^{n-1})$, by Minkowski's inequality and Lemma 2.1, we have

$$\begin{aligned}
|\mu_\Omega(f_j)(x)| \leq & C\Big(\int_0^{|x|} \Big|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)\mathrm{d}y\Big|^2 \frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}} + \\
& C\Big(\int_{|x|}^\infty \Big|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)\mathrm{d}y\Big|^2 \frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}} \\
\leq & C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \int_{R_j} \frac{|f_j(y)|}{|x-y|^{n-1}} \Big(\int_{|x-y|\leq t, |x|\geq t} \frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}} \mathrm{d}y + \\
& C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \int_{R_j} \frac{|f_j(y)|}{|x-y|^{n-1}} \Big(\int_{|x|}^\infty \frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}} \mathrm{d}y \\
& \leq C \int_{R_j} \frac{|f_j(y)|}{|x-y|^{n-1}} \cdot \frac{|y|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} \mathrm{d}y + C \int_{R_j} \frac{|f_j(y)|}{|x-y|^{n-1}} \cdot \frac{1}{|x|} \mathrm{d}y \\
& \leq C 2^{\frac{j-k}{2}} 2^{-kn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} + C 2^{-kn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
& \leq C 2^{-kn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
\|\mu_\Omega(f_j)(x)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} & \leq C 2^{-kn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \leq C \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \\
& \leq C 2^{(j-k)n\delta_2} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we get

$$U_1 \leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \Big(\sum_{j=-\infty}^{k-2} 2^{(j-k)n\delta_2} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q.$$

If $1 < q < \infty$, since $\alpha - n\delta_2 < 0$, Hölder's inequality implies that

$$\begin{aligned}
U_1 & \leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \Big(\sum_{j=-\infty}^{k-2} 2^{j\alpha} 2^{(k-j)(\alpha-n\delta_2)} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q \\
& \leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \Big(\sum_{j=-\infty}^{k-2} 2^{j\alpha q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\alpha-n\delta_2)q/2}\Big) \Big(\sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha-n\delta_2)q'/2}\Big)^{\frac{q}{q'}} \\
& \leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \Big(\sum_{j=-\infty}^{k-2} 2^{j\alpha q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\alpha-n\delta_2)q/2}\Big) \\
& \leq C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L-2} 2^{j\alpha q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=j+2}^{L} 2^{(k-j)(\alpha-n\delta_2)q/2} \\
& \leq C \|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

If $0 < q \leq 1$, using the well-known inequality

$$(\sum_{i=1}^\infty a_i)^q \leq \sum_{i=1}^\infty a_i^q, \quad a_i > 0, i = 1, 2, \ldots, \tag{2}$$

we obtain

$$\begin{aligned}U_1 &\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2} 2^{j\alpha q}2^{(k-j)(\alpha-n\delta_2)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L-2} 2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{L} 2^{(k-j)(\alpha-n\delta_2)q}\\ &\le C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

For U_3, observe that if $x\in R_k, y\in R_j$ and $j\ge k+2$, then $|x-y|\sim|y|\sim 2^j$ and

$$\frac{1}{|x-y|^2}-\frac{1}{|y|^2}|\le C\frac{|x|}{|x-y|^3}.$$

By Minkowski's inequality and Lemma 2.1, we have

$$\begin{aligned}|\mu_\Omega(f_j)(x)| \le& C\Big(\int_0^{|y|}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}+\\ & C\Big(\int_{|y|}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\\ \le& C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\int_{R_j}\frac{|f_j(y)|}{|x-y|^{n-1}}\Big(\int_{|x-y|\le t,|y|\ge t}\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\mathrm{d}y+\\ & C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\int_{R_j}\frac{|f_j(y)|}{|x-y|^{n-1}}\Big(\int_{|y|}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\mathrm{d}y\\ \le& C\int_{R_j}\frac{|f_j(y)|}{|x-y|^{n-1}}\cdot\frac{|x|^{\frac12}}{|x-y|^{\frac32}}\mathrm{d}y+C\int_{R_j}\frac{|f_j(y)|}{|x-y|^{n-1}}\cdot\frac{1}{|y|}\mathrm{d}y\\ \le& C2^{\frac{k-j}{2}}2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}+C2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ \le& C2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\|\mu_\Omega(f_j)(x)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\ &\le C2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus we get

$$U_3\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q.$$

If $1<q<\infty$, we have

$$\begin{aligned}U_3 \le& C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\Big(\sum_{j=k+2}^{L}2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q+\\ & C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\Big(\sum_{j=L+1}^{\infty}2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q\end{aligned}$$

$$=U_{31}+U_{32}.$$

For U_{31}, since $\alpha+n\delta_1>0$, by Hölder's inequality, we obtain

$$\begin{aligned}
U_{31}&=C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=k+2}^{L}2^{j\alpha}2^{(k-j)(\alpha+n\delta_1)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q\\
&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=k+2}^{L}2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\alpha+n\delta_1)q/2}\Big)\Big(\sum_{j=k+2}^{L}2^{(k-j)(\alpha+n\delta_1)q'/2}\Big)^{\frac{q}{q'}}\\
&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=k+2}^{L}2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\alpha+n\delta_1)q/2}\Big)\\
&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=-\infty}^{j-2}2^{(k-j)(\alpha+n\delta_1)q/2}\\
&\le C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For U_{32}, we note that $\alpha+n\delta_1-\lambda>0$ and derive

$$\begin{aligned}
U_{32}&=C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=L+1}^{\infty}2^{j\alpha}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\alpha+n\delta_1+\lambda)/2}2^{(k-j)(\alpha+n\delta_1-\lambda)/2}\Big)^q\\
&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=L+1}^{\infty}2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\alpha+n\delta_1+\lambda)q/2}\Big)\Big(\sum_{j=L+1}^{\infty}2^{(k-j)(\alpha+n\delta_1-\lambda)q'/2}\Big)^{\frac{q}{q'}}\\
&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=L+1}^{\infty}2^{(k-j)(\alpha+n\delta_1+\lambda)q/2}2^{j\lambda q}2^{-j\lambda q}\Big(\sum_{\ell=-\infty}^{j}2^{\ell\alpha q}\|f_\ell\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\
&\le C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\lambda q}\sum_{j=L+1}^{\infty}2^{(k-j)(\alpha+n\delta_1-\lambda)q/2}\\
&\le C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Consequently, we get

$$U_3\le U_{31}+U_{32}\le C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

If $0<q\le1$, since $-n\delta_1+\lambda<\alpha<n\delta_2$, by inequality (2) and using the method in Izuki [14, P. 353], we have

$$\begin{aligned}
U_3\le&C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\
\le&C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\sum_{j=k+2}^{L}2^{(k-j)n\delta_1q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}+\\
&C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\sum_{j=L+1}^{\infty}2^{(k-j)n\delta_1q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
\leq & C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=-\infty}^{j-2}2^{(k-j)(\alpha+n\delta_1)q}+\\
& C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\sum_{j=L+1}^{\infty}2^{(k-j)n\delta_1 q}2^{-j\alpha q}2^{j\lambda q}2^{-j\lambda q}\Big(\sum_{\ell=-\infty}^{j}2^{\ell\alpha q}\|f_\ell\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\
\leq & C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}+C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\sum_{j=L+1}^{\infty}2^{(k-j)n\delta_1 q}2^{-j\alpha q}2^{j\lambda q}\\
\leq & C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}+C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{(\alpha+n\delta_1)kq}\sum_{j=L}^{\infty}2^{(\lambda-\alpha-n\delta_1)jq}\\
\leq & C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}+C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}\sup_{L\in\mathbb{Z}}2^{-L\lambda q}2^{(\alpha+n\delta_1)Lq}2^{(\lambda-\alpha-n\delta_1)Lq}\\
\leq & C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Combining the estimates for U_1, U_2 and U_3 completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 Let $f\in\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$. Write

$$
f(x)=\sum_{j=-\infty}^{\infty}f(x)\chi_j(x)=\sum_{j=-\infty}^{\infty}f_j(x).
$$

Then, we have

$$
\begin{aligned}
\|\mu^m_{\Omega,b}(f)\|^q_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}=&\sum_{k=-\infty}^{\infty}2^{\alpha kq}\|\mu^m_{\Omega,b}(f)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\
\leq & C\sum_{k=-\infty}^{\infty}2^{\alpha kq}\Big(\sum_{j=-\infty}^{k-2}\|\mu^m_{\Omega,b}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q+\\
& C\sum_{k=-\infty}^{\infty}2^{\alpha kq}\Big(\sum_{j=k-1}^{k+1}\|\mu^m_{\Omega,b}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q+\\
& C\sum_{k=-\infty}^{\infty}2^{\alpha kq}\Big(\sum_{j=k+2}^{\infty}\|\mu^m_{\Omega,b}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q\\
=&V_1+V_2+V_3.
\end{aligned}
$$

For V_2, by Lemma 2.4 (2), we obtain

$$
V_2\leq C\sum_{k=-\infty}^{\infty}2^{k\alpha q}\|f_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\leq C\|f\|^q_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}.
$$

For V_1, observe that if $x\in R_k, y\in R_j$ and $j\leq k-2$, then

$$
\begin{aligned}
|\mu^m_{\Omega,b}(f_j)(x)|\leq & C\Big(\int_0^{|x|}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]^m f_j(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}}+\\
& C\Big(\int_{|x|}^{\infty}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]^m f_j(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\int_{R_j}\frac{|b(x)-b(y)|^m|f_j(y)|}{|x-y|^{n-1}}\Big(\int_{|x-y|\le t,|x|\ge t}\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\mathrm{d}y+\\
&\quad C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\int_{R_j}\frac{|b(x)-b(y)|^m|f_j(y)|}{|x-y|^{n-1}}\Big(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\mathrm{d}y\\
&\le C\int_{R_j}\frac{|b(x)-b(y)|^m|f_j(y)|}{|x-y|^{n-1}}\cdot\frac{|y|^{\frac12}}{|x-y|^{\frac32}}\mathrm{d}y+C\int_{R_j}\frac{|b(x)-b(y)|^m|f_j(y)|}{|x-y|^{n-1}}\cdot\frac{1}{|x|}\mathrm{d}y\\
&\le C2^{\frac{j-k}{2}}2^{-kn}\int_{R_j}|b(x)-b(y)|^m|f_j(y)|\mathrm{d}y+C2^{-kn}\int_{R_j}|b(x)-b(y)|^m|f_j(y)|\mathrm{d}y\\
&\le C2^{-kn}\int_{R_j}|b(x)-b(y)|^m|f_j(y)|\mathrm{d}y\\
&\le C2^{-kn}\sum_{i=0}^{m}C_m^i|b(x)-b_{B_j}|^{m-i}\int_{R_j}|b_{B_j}-b(y)|^i|f_j(y)|\mathrm{d}y\\
&\le C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i=0}^{m}C_m^i|b(x)-b_{B_j}|^{m-i}\|(b_{B_j}-b)^i\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

An application of Lemmas 2.2, 2.3 and 2.5 gives

$$
\begin{aligned}
&\|\mu_{\Omega,b}^m(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i=0}^{m}C_m^i\|(b(x)-b_{B_j})^{m-i}\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|(b_{B_j}-b)^i\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i=0}^{m}C_m^i(k-j)^{m-i}\|b\|_{\mathrm{BMO}}^{m-i}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|b\|_{\mathrm{BMO}}^{i}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C(k-j+1)^m2^{-kn}\|b\|_{\mathrm{BMO}}^m\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C(k-j+1)^m\|b\|_{\mathrm{BMO}}^m\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}\\
&\quad\le C(k-j+1)^m2^{(j-k)n\delta_2}\|b\|_{\mathrm{BMO}}^m\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus we get

$$
V_1\le C\|b\|_{\mathrm{BMO}}^{mq}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{\alpha j}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}(k-j+1)^m2^{(k-j)(\alpha-n\delta_2)}\Big)^q.
$$

If $1<q<\infty$, since $\alpha-n\delta_2<0$, Hölder's inequality implies that

$$
\begin{aligned}
V_1&\le C\|b\|_{\mathrm{BMO}}^{mq}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{\alpha jq}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q2^{(k-j)(\alpha-n\delta_2)q/2}\Big)\times\\
&\quad\Big(\sum_{j=-\infty}^{k-2}(k-j+1)^{mq'}2^{(k-j)(\alpha-n\delta_2)q'/2}\Big)^{q/q'}\\
&\le C\|b\|_{\mathrm{BMO}}^{mq}\sum_{j=-\infty}^{\infty}2^{\alpha jq}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\sum_{k=j+2}^{\infty}2^{(k-j)(\alpha-n\delta_2)q/2}\\
&\le C\|b\|_{\mathrm{BMO}}^{mq}\|f\|_{\dot K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)}^q.
\end{aligned}
$$

If $0 < q \leq 1$, by inequality (2), we have

$$\begin{aligned}
V_1 &\leq C\|b\|_{\mathrm{BMO}}^{mq} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} 2^{\alpha j q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q (k-j+1)^{mq} 2^{(k-j)(\alpha - n\delta_2)q} \\
&\leq C\|b\|_{\mathrm{BMO}}^{mq} \sum_{j=-\infty}^{\infty} 2^{\alpha j q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=j+2}^{\infty} (k-j+1)^{mq} 2^{(k-j)(\alpha - n\delta_2)q} \\
&\leq C\|b\|_{\mathrm{BMO}}^{mq} \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)}^q.
\end{aligned}$$

For V_3, if $x \in R_k$, $y \in R_j$ and $j \geq k+2$, with the same arguments as for V_1, we can obtain that

$$\begin{aligned}
|\mu_{\Omega,b}^m(f_j)(x)| \leq& C\Big(\int_0^{|y|} \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)]^m f_j(y) \mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}} + \\
& C\Big(\int_{|y|}^{\infty} \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)]^m f_j(y) \mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}} \\
\leq& C \int_{R_j} \frac{|b(x)-b(y)|^m |f_j(y)|}{|x-y|^{n-1}} \cdot \frac{|x|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} \mathrm{d}y + C \int_{R_j} \frac{|b(x)-b(y)|^m |f_j(y)|}{|x-y|^{n-1}} \cdot \frac{1}{|y|} \mathrm{d}y \\
\leq& C 2^{\frac{k-j}{2}} 2^{-jn} \int_{R_j} |b(x)-b(y)|^m |f_j(y)| \mathrm{d}y + C 2^{-jn} \int_{R_j} |b(x)-b(y)|^m |f_j(y)| \mathrm{d}y \\
\leq& C 2^{-jn} \int_{R_j} |b(x)-b(y)|^m |f_j(y)| \mathrm{d}y \\
\leq& C 2^{-jn} \sum_{i=0}^m C_m^i |b(x) - b_{B_k}|^{m-i} \int_{R_j} |b_{B_k} - b(y)|^i |f_j(y)| \mathrm{d}y \\
\leq& C 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{i=0}^m C_m^i |b(x) - b_{B_k}|^{m-i} \|(b_{B_k} - b)^i \chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

By Lemmas 2.2, 2.3 and 2.5, we obtain

$$\begin{aligned}
&\|\mu_{\Omega,b}^m(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\leq C 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{i=0}^m C_m^i \|(b(x) - b_{B_k})^{m-i} \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|(b_{B_k} - b)^i \chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad\leq C 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{i=0}^m C_m^i (j-k)^i \|b\|_{\mathrm{BMO}}^{m-i} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|b\|_{\mathrm{BMO}}^i \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad\leq C(j-k+1)^m 2^{-jn} \|b\|_{\mathrm{BMO}}^m \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad\leq C(j-k+1)^m \|b\|_{\mathrm{BMO}}^m \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\
&\leq C(j-k+1)^m 2^{(k-j)n\delta_1} \|b\|_{\mathrm{BMO}}^m \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we get

$$V_3 \leq C\|b\|_{\mathrm{BMO}}^{mq} \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} 2^{\alpha j} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} (j-k+1)^m 2^{(k-j)(\alpha + n\delta_1)} \Big)^q.$$

Similarly to the estimate for V_1, in both cases $1<q<\infty$ and $0<q\le 1$, we have

$$V_3\le C\|b\|_{\mathrm{BMO}}^{mq}\|f\|^q_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for V_1, V_2 and V_3, consequently we have proved Theorem 1.2. Since the proof of Theorem 1.3 is similar to those of Theorems 1.1 and 1.2, we omit the details here. □

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