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Inclusion Relationships for Certain Classes of p-Valent Functions Involving the Srivastava-Khairnar-More Operator

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Abstract In the present paper, we use the methods of differential subordination and convolution to investigate some inclusion properties for certain classes of p-valent analytic functions in the open unit disk, which are associated with the Srivastava-Khairnar-More operator. The results presented here include several previous known results as their special cases.

Keywords analytic functions; p-valent functions; subordination; Hadmard product (or convolution); Srivastava-Khairnar-More operator.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad p \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \ \text{ and } \ |z| < 1\}.$$

Let $f, g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}.$$

Then the Hadmard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

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For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0)=0 \text{ and } |\omega(z)|<1, \quad z\in\mathbb{U},$$

such that

$$f(z)=g(\omega(z)), \quad z\in\mathbb{U}.$$

We denote this subordination by $f(z)\prec g(z)$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [5, 8] for details, see also [17]):

$$f(z)\prec g(z)\ (z\in\mathbb{U}) \Longleftrightarrow f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U}).$$

Let M be the class of functions $\phi(z)$ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0)=1$ and $\mathrm{Re}[\phi(z)]>0$ for $z\in\mathbb{U}$.

By making use of the principle of subordination between analytic functions, Ma and Minda [7] introduced the subclasses $\mathcal{S}_p^*(\phi),\mathcal{K}_p(\phi)$ and $\mathcal{C}_p(\phi,\psi)$ of the class $\mathcal{A}_p$ for $p\in\mathbb{N}$ and $\phi,\psi\in M$, which are defined by

$$\mathcal{S}_p^*(\phi)=\Big\{f\in\mathcal{A}_p:\ \frac{zf'(z)}{pf(z)}\prec\phi(z) \text{ in } \mathbb{U}\Big\},$$

$$\mathcal{K}_p(\phi)=\Big\{f\in\mathcal{A}_p:\ \frac{1}{p}+\frac{zf''(z)}{pf'(z)}\prec\phi(z) \text{ in } \mathbb{U}\Big\},$$

and

$$\mathcal{C}_p(\phi,\psi)=\Big\{f\in\mathcal{A}_p:\ \exists\, g\in\mathcal{S}_p^*(\phi) \text{ such that } \frac{zf'(z)}{pg(z)}\prec\psi(z) \text{ in } \mathbb{U}\Big\}.$$

In its special case when

$$p=1 \text{ and } \phi(z)=\psi(z)=\frac{1+z}{1-z},$$

we have the familiar classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ of starlike, convex and close-to-convex function in $\mathbb{U}$, respectively. Also, for special choices for the functions ϕ and ψ involved in these definitions, we can obtain other classes investigated many times earlier. For example, the classes

$$\mathcal{S}_p^*\Big(\frac{1+Az}{1+Bz}\Big)=\mathcal{S}_p^*(A,B) \text{ and } \mathcal{K}_p\Big(\frac{1+Az}{1+Bz}\Big)=\mathcal{K}_p(A,B), \quad -1\le B<A\le 1,$$

introduced and studied by Janowski [6].

For parameters

$$a,b\in\mathbb{C} \text{ and } c\in\mathbb{C}\setminus\mathbb{Z}_0^-, \quad \mathbb{Z}_0^-=\{0,-1,-2,\ldots\},$$

the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{1.2}$$

where $(\nu)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$(\nu)_k=\frac{\Gamma(\nu+k)}{\Gamma(\nu)}=\begin{cases}1, & k=0;\nu\in\mathbb{C}\setminus\{0\},\\ \nu(\nu+1)\cdots(\nu+k-1), & k\in\mathbb{N};\nu\in\mathbb{C}.\end{cases}$$

The hypergeometric series in (1.2) converges absolutely for all $z \in \mathbb{U}$, so that it represents an analytic function in $\mathbb{U}$. Dziok and Srivastava [2] (see [3, 4]) considered the generalized hypergeometric function ${}_qF_s$ $(q, s \in \mathbb{N} \cup \{0\})$, which is a certain generalization of (1.2).

We now introduce a function $f_{\mu,p}(a,b,c)(z)$ defined by

$$f_{\mu,p}(a,b,c)(z) = (1-\mu)z^p \cdot {}_2F_1(a,b;c;z) + \mu z[z^p \cdot {}_2F_1(a,b;c;z)]', \quad z \in \mathbb{U};\ \mu \geq 0. \tag{1.3}$$

For $p = 1$, we have $f_{\mu,1}(a,b,c)(z) = f_\mu(a,b,c)(z)$, which was studied by Skukla and Skukla [13], and for $\mu = 0$ and $b = 1$, we obtain

$$f_{0,p}(a,1,c)(z) = \phi_p(a,c)(z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p},$$

which was introduced by Saitoh [12].

Next, we introduce the following family of linear operators $\mathcal{I}^\lambda_{\mu,p}(a,b,c):\ \mathcal{A}_p \to \mathcal{A}_p$, defined by

$$\mathcal{I}^\lambda_{\mu,p}(a,b,c)f(z) = f^\lambda_{\mu,p}(a,b,c)(z) * f(z), \quad \lambda > -p;\ \mu \geq 0;\ z \in \mathbb{U}, \tag{1.4}$$

where $f^\lambda_{\mu,p}(a,b,c)(z)$ is the function defined in terms of the Hadamard product (or convolution) as follows:

$$f_{\mu,p}(a,b,c)(z) * f^\lambda_{\mu,p}(a,b,c)(z) = \frac{z^p}{(1-z)^{\lambda+p}}, \quad \lambda > -p;\ \mu \geq 0, \tag{1.5}$$

where $f_{\mu,p}(a,b,c)(z)$ is given by (1.3).

We also note that the operator $\mathcal{I}^\lambda_{\mu,p}(a,b,c)$ generalizes several previously studied familiar operators, and we will show some of the interesting particular cases as follows.

(i) $\mathcal{I}^\lambda_{\mu,1}(a,b,c) = \mathcal{I}^\lambda_\mu(a,b,c)$, where $\mathcal{I}^\lambda_\mu(a,b,c)$ is the Srivastava-Khairnar-More operator [16];

(ii) $\mathcal{I}^\lambda_{0,1}(a,b,c) = \mathcal{I}_\lambda(a,b,c)$, where the operator $\mathcal{I}_\lambda(a,b,c)$ was introduced by Noor [10];

(iii) $\mathcal{I}^\lambda_{0,p}(a,1,c) = \mathcal{I}^\lambda_p(a,c)$, where $\mathcal{I}^\lambda_p(a,c)$ is the Cho-Kwon-Srivastava operator [1];

(iv) $\mathcal{I}^n_{0,1}(a,n+1,a) = \mathcal{I}_n$, where $\mathcal{I}_n$ is the Noor integral operator [9].

Since

$$\frac{z^p}{(1-z)^{\lambda+p}} = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k}{k!} z^{k+p} \quad \lambda > -p;\ z \in \mathbb{U}, \tag{1.6}$$

by using (1.2), (1.3) and (1.6) in (1.5), we get

$$\Big(\sum_{k=0}^{\infty} \frac{((1+\mu(k+p-1))(a)_k(b)_k}{(c)_k} \frac{z^{k+p}}{k!}\Big) * f^\lambda_{\mu,p}(a,b,c)(z) = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k}{k!} z^{k+p}.$$

Therefore, the function $f^\lambda_{\mu,p}(a,b,c)(z)$ has the following explicit form

$$f^\lambda_{\mu,p}(a,b,c)(z) = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k(c)_k}{((1+\mu(k+p-1))(a)_k(b)_k} z^{k+p} \quad (z \in \mathbb{U}). \tag{1.7}$$

Combining (1.1), (1.4), together with (1.7), we have

$$\mathcal{I}^\lambda_{\mu,p}(a,b,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{((1+\mu(k+p-1))(a)_k(b)_k} a_{k+p} z^{k+p} \quad (z \in \mathbb{U}).$$

In particular, we have

$$\mathcal{I}^{\lambda}_{0,p}(a,\lambda+p,a)f(z)=f(z) \text{ and } \mathcal{I}^{1}_{0,p}(a,p,a)f(z)=\frac{zf'(z)}{p}.$$

By using the operator $\mathcal{I}^{\lambda}_{\mu,p}(a,b,c)$ for $\lambda>-p$, $\mu\geq 0$ and $\phi,\psi\in M$, we introduce the subclasses of $\mathcal{A}_p$ as below:

$$\mathcal{S}^{\lambda}_{\mu,p}(a,b,c)(\phi)=\left\{f\in\mathcal{A}_p:\ \mathcal{I}^{\lambda}_{\mu,p}(a,b,c)f(z)\in\mathcal{S}^*_p(\phi)\right\},$$

$$\mathcal{K}^{\lambda}_{\mu,p}(a,b,c)(\phi)=\left\{f\in\mathcal{A}_p:\ \mathcal{I}^{\lambda}_{\mu,p}(a,b,c)f(z)\in\mathcal{K}_p(\phi)\right\},$$

and

$$\mathcal{C}^{\lambda}_{\mu,p}(a,b,c)(\phi,\psi)=\left\{f\in\mathcal{A}_p:\ \mathcal{I}^{\lambda}_{\mu,p}(a,b,c)f(z)\in\mathcal{C}_p(\phi,\psi)\right\}.$$

It is easy to verify that

$$f\in\mathcal{K}^{\lambda}_{\mu,p}(a,b,c)(\phi)\Longleftrightarrow\frac{zf'(z)}{p}\in\mathcal{S}^{\lambda}_{\mu,p}(a,b,c)(\phi). \tag{1.8}$$

As a special case, when $p=1$, we obtain

$$\mathcal{S}^{\lambda}_{\mu,1}(a,b,c)(\phi)=\mathcal{S}^{\lambda}_{\mu}(a,b,c)(\phi),\quad \mathcal{K}^{\lambda}_{\mu,1}(a,b,c)(\phi)=\mathcal{K}^{\lambda}_{\mu}(a,b,c)(\phi),$$

and

$$\mathcal{C}^{\lambda}_{\mu,1}(a,b,c)(\phi,\psi)=\mathcal{C}^{\lambda}_{\mu}(a,b,c)(\phi,\psi),$$

which were introduced and investigated recently by Srivastava et al. [16].

For the sake of convenience, we write

$$\mathcal{S}^{\lambda}_{\mu,p}(a,b,c)\Big(\frac{1+Az}{1+Bz}\Big)=\mathcal{S}^{\lambda}_{\mu,p}(a,b,c;A,B),\quad -1\leq B<A\leq 1,$$

$$\mathcal{K}^{\lambda}_{\mu,p}(a,b,c)\Big(\frac{1+Az}{1+Bz}\Big)=\mathcal{K}^{\lambda}_{\mu,p}(a,b,c;A,B)\quad -1\leq B<A\leq 1,$$

and

$$\mathcal{C}^{\lambda}_{\mu,p}(a,b,c)\Big(\frac{1+Az}{1+Bz};\frac{1+Az}{1+Bz}\Big)=\mathcal{C}^{\lambda}_{\mu,p}(a,b,c;A,B)\quad -1\leq B<A\leq 1.$$

In this paper, we investigate several inclusion properties of the classes $\mathcal{S}^{\lambda}_{\mu,p}(a,b,c)(\phi)$, $\mathcal{K}^{\lambda}_{\mu,p}(a,b,c)(\phi)$ and $\mathcal{C}^{\lambda}_{\mu,p}(a,b,c)(\phi,\psi)$ associated with the operator $\mathcal{I}^{\lambda}_{\mu,p}(a,b,c)$. Also, we point out some new or known consequences of our main results.

2. Preliminary results

In order to establish our main results, we shall require the following lemmas.

Lemma 2.1 *Let $f^{\lambda_i}_{\mu,p}(a,b,c)(z)$, $f^{\lambda}_{\mu,p}(a_i,b,c)(z)$, $f^{\lambda}_{\mu,p}(a,b_i,c)(z)$ and $f^{\lambda}_{\mu,p}(a,b,c_i)(z)$ be defined by (1.7). Then, for $\lambda_i>-p$; $a_i,b_i,c_i\in\mathbb{R}\setminus\mathbb{Z}^-_0$ ($\mathbb{Z}^-_0=\{0,-1,-2,\cdots\}$) $(i=1,2)$ and $\mu\geq 0$,*

$$f^{\lambda_2}_{\mu,p}(a,b,c)(z)=f^{\lambda_1}_{\mu,p}(a,b,c)(z)*\phi_p(\lambda_2+p,\lambda_1+p)(z), \tag{2.1}$$

$$f^{\lambda}_{\mu,p}(a_1,b,c)(z)=f^{\lambda}_{\mu,p}(a_2,b,c)(z)*\phi_p(a_2,a_1)(z), \tag{2.2}$$

$$f^{\lambda}_{\mu,p}(a,b_1,c)(z)=f^{\lambda}_{\mu,p}(a,b_2,c)(z)*\phi_p(b_2,b_1)(z), \tag{2.3}$$

and

$$f_{\mu,p}^{\lambda}(a,b,c_1)(z) = f_{\mu,p}^{\lambda}(a,b,c_2)(z) * \phi_p(c_1,c_2)(z), \tag{2.4}$$

where

$$\phi_p(\alpha,\beta)(z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} z^{k+p}, \quad z \in \mathbb{U}.$$

Proof From (1.7), we have

$$\begin{aligned} f_{\mu,p}^{\lambda_2}(a,b,c)(z) &= \sum_{k=0}^{\infty} \frac{(\lambda_2+p)_k (c)_k}{((1+\mu(k+p-1))(a)_k(b)_k} z^{k+p} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda_1+p)_k (c)_k}{((1+\mu(k+p-1))(a)_k(b)_k} \cdot \frac{(\lambda_2+p)_k}{(\lambda_1+p)_k} z^{k+p} \\ &= f_{\mu,p}^{\lambda_1}(a,b,c)(z) * \phi_p(\lambda_2+p, \lambda_1+p)(z) \end{aligned}$$

and the assertion (2.1) is proved. The proof of (2.2)–(2.4) is similar to that of (2.1) and the details involved may be omitted.

Lemma 2.2 ([11]) *Let $f \in \mathcal{K}$ and $g \in \mathcal{S}^*$. Then, for every analytic function W in $\mathbb{U}$,*

$$\frac{(f * Wg)(\mathbb{U})}{(f * g)(\mathbb{U})} \subset \overline{\text{co}}[W(\mathbb{U})],$$

where $\overline{\text{co}}[W(\mathbb{U})]$ denotes the closed convex hull of $W(\mathbb{U})$.

Lemma 2.3 ([15]) *Let $0 < \alpha \le \beta$. If $\beta \ge 2$ or $\alpha + \beta \ge 3$, then the function*

$$\phi_1(\alpha,\beta)(z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} z^{k+1}, \quad z \in \mathbb{U}$$

belongs to the class $\mathcal{K}$ of convex functions.

3. Main results

Our first main result is contained in Theorem 3.1 as follows.

Theorem 3.1 *Let $-p < \lambda_2 \le \lambda_1$, $\mu \ge 0$ and $\phi \in M$ with*

$$\text{Re}(\phi(z)) > \frac{p-1}{p}, \quad p \in \mathbb{N};\ z \in \mathbb{U}. \tag{3.1}$$

If $\lambda_1 \ge 2 - p$ or $\lambda_1 + \lambda_2 \ge 3 - 2p$, then

$$\mathcal{S}_{\mu,p}^{\lambda_1}(a,b,c)(\phi) \subset \mathcal{S}_{\mu,p}^{\lambda_2}(a,b,c)(\phi). \tag{3.2}$$

Proof Let $f \in \mathcal{S}_{\mu,p}^{\lambda_1}(a,b,c)(\phi)$. Then, by the definition of the class $\mathcal{S}_{\mu,p}^{\lambda_1}(a,b,c)(\phi)$, we have

$$\frac{z[\mathcal{I}_{\mu,p}^{\lambda_1}(a,b,c)f(z)]'}{p\mathcal{I}_{\mu,p}^{\lambda_1}(a,b,c)f(z)} = \phi[\omega(z)], \quad z \in \mathbb{U}, \tag{3.3}$$

where ϕ is convex univalent with $\text{Re}[\phi(z)] > 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ with $\omega(0) = 0 = \phi(0) - 1$. Therefore,

$$\frac{z[z^{1-p}(\mathcal{I}_{\mu,p}^{\lambda_1}(a,b,c)f(z))]'}{z^{1-p}(\mathcal{I}_{\mu,p}^{\lambda_1}(a,b,c)f(z))} = p[\phi(\omega(z)) - 1] + 1 \prec \frac{1+z}{1-z}. \tag{3.4}$$

Applying (1.4), (2.1), (3.3) and the properties of convolution, we get

$$\frac{z[\mathcal{I}^{\lambda_2}_{\mu,p}(a,b,c)f(z)]'}{p\mathcal{I}^{\lambda_2}_{\mu,p}(a,b,c)f(z)} = \frac{z[(f^{\lambda_2}_{\mu,p}(a,b,c)*f)(z)]'}{p[(f^{\lambda_2}_{\mu,p}(a,b,c)*f)(z)]}$$
$$= \frac{z[(f^{\lambda_1}_{\mu,p}(a,b,c)*\phi_p(\lambda_2+p,\lambda_1+p)*f)(z)]'}{p[(f^{\lambda_1}_{\mu,p}(a,b,c)*\phi_p(\lambda_2+p,\lambda_1+p)*f)(z)]}$$
$$= \frac{\phi_p(\lambda_2+p,\lambda_1+p)(z)*z[\mathcal{I}^{\lambda_1}_{\mu,p}(a,b,c)f(z)]'}{p[\phi_p(\lambda_2+p,\lambda_1+p)(z)*\mathcal{I}^{\lambda_1}_{\mu,p}(a,b,c)f(z)]}$$
$$= \frac{\phi_p(\lambda_2+p,\lambda_1+p)(z)*p\phi[\omega(z)]\mathcal{I}^{\lambda_1}_{\mu,p}(a,b,c)f(z)}{p\phi_p(\lambda_2+p,\lambda_1+p)(z)*\mathcal{I}^{\lambda_1}_{\mu,p}(a,b,c)f(z)}. \tag{3.5}$$

It follows from (3.4) that $z^{1-p}\mathcal{I}^{\lambda_1}_{\mu,p}(a,b,c)f(z)\in\mathcal{S}^*$. Also, by Lemma 2.3, we see that $z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)(z)\in\mathcal{K}$. Thus, an application of Lemma 1 to (3.5) yields

$$\frac{\{[z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)]*\phi[\omega(z)]z^{1-p}\mathcal{I}^{\lambda_1}_{\mu,p}(a,b,c)f\}(\mathbb{U})}{\{[z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)]*z^{1-p}\mathcal{I}^{\lambda_1}_{\mu,p}(a,b,c)f\}(\mathbb{U})}\subset\overline{\mathrm{co}}\phi[\omega(\mathbb{U})], \tag{3.6}$$

because ϕ is convex univalent function.

Thus, from the definition of subordination and (3.6), we have

$$\frac{z[\mathcal{I}^{\lambda_2}_{\mu,p}(a,b,c)f(z)]'}{p\mathcal{I}^{\lambda_2}_{\mu,p}(a,b,c)f(z)}\prec\phi(z)\quad(z\in\mathbb{U}),$$

and so $f\in\mathcal{S}^{\lambda_2}_{\mu,p}(a,b,c)(\phi)$. The proof of Theorem 3.1 is completed. □

Theorem 3.2 *Let $0<a_2\le a_1$, $\lambda>-p$, $\mu\ge 0$ and $\phi\in M$ with (3.1) holding. If $a_1\ge 2$ or $a_1+a_2\ge 3$, then*

$$\mathcal{S}^{\lambda}_{\mu,p}(a_2,b,c)(\phi)\subset\mathcal{S}^{\lambda}_{\mu,p}(a_1,b,c)(\phi).$$

Proof Let $f\in\mathcal{S}^{\lambda}_{\mu,p}(a_2,b,c)(\phi)$. Then $z^{1-p}\mathcal{I}^{\lambda}_{\mu,p}(a_2,b,c)f(z)\in\mathcal{S}^*$. Using (2.2) and the same techniques as in the proof of Theorem 3.1, we get

$$\frac{z[\mathcal{I}^{\lambda}_{\mu,p}(a_1,b,c)f(z)]'}{p\mathcal{I}^{\lambda}_{\mu,p}(a_1,b,c)f(z)} = \frac{z[(f^{\lambda}_{\mu,p}(a_1,b,c)*f)(z)]'}{p[(f^{\lambda}_{\mu,p}(a_1,b,c)*f)(z)]}$$
$$= \frac{z[(f^{\lambda}_{\mu,p}(a_2,b,c)*\phi_p(a_2,a_1)*f)(z)]'}{p[(f^{\lambda}_{\mu,p}(a_2,b,c)*\phi_p(a_2,a_1)*f)(z)]}$$
$$= \frac{\phi_p(a_2,a_1)(z)*z[\mathcal{I}^{\lambda}_{\mu,p}(a_2,b,c)f(z)]'}{p[\phi_p(a_2,a_1)(z)*\mathcal{I}^{\lambda}_{\mu,p}(a_2,b,c)f(z)]}$$
$$= \frac{\phi_p(a_2,a_1)(z)*p\phi[\omega(z)]\mathcal{I}^{\lambda}_{\mu,p}(a_2,b,c)f(z)}{p[\phi_p(a_2,a_1)(z)*\mathcal{I}^{\lambda}_{\mu,p}(a_2,b,c)f(z)]}$$
$$= \frac{\phi_p(a_2,a_1)(z)*\phi[\omega(z)]\mathcal{I}^{\lambda}_{\mu,p}(a_2,b,c)f(z)}{\phi_p(a_2,a_1)(z)*\mathcal{I}^{\lambda}_{\mu,p}(a_2,b,c)f(z)}. \tag{3.7}$$

In view of Lemma 2.3, we have $z^{1-p}\phi_p(a_2,a_1)(z)\in\mathcal{K}$, and by applying Lemma 2.2 to (3.7), we conclude that $f\in\mathcal{S}^{\lambda}_{\mu,p}(a_1,b,c)(\phi)$. □

By means of (2.3) and (2.4), and using the similar method of the proof of Theorem 3.2, we get the following results.

Theorem 3.3 *(i) Let $0 < b_2 \leq b_1,\ \lambda > -p,\ \mu \geq 0$ and $\phi \in M$ with (3.1) holding. If $b_1 \geq 2$ or $b_1 + b_2 \geq 3$, then*

$$\mathcal{S}^{\lambda}_{\mu,p}(a, b_2, c)(\phi) \subset \mathcal{S}^{\lambda}_{\mu,p}(a, b_1, c)(\phi).$$

(ii) Let $0 < c_1 \leq c_2,\ \lambda > -p,\ \mu \geq 0$ and $\phi \in M$ with (3.1) holding. If $c_2 \geq 2$ or $c_1 + c_2 \geq 3$, then

$$\mathcal{S}^{\lambda}_{\mu,p}(a, b, c_2)(\phi) \subset \mathcal{S}^{\lambda}_{\mu,p}(a, b, c_1)(\phi).$$

Theorem 3.4 *(i) Let $-p < \lambda_2 \leq \lambda_1,\ \mu \geq 0$ and $\phi \in M$ with (3.1) holding. If $\lambda_1 \geq 2 - p$ or $\lambda_1 + \lambda_2 \geq 3 - 2p$, then*

$$\mathcal{K}^{\lambda_1}_{\mu,p}(a, b, c)(\phi) \subset \mathcal{K}^{\lambda_2}_{\mu,p}(a, b, c)(\phi). \tag{3.8}$$

(ii) Let $0 < a_2 \leq a_1,\ \lambda > -p,\ \mu \geq 0$ and $\phi \in M$ with (3.1) holding. If $a_1 \geq 2$ or $a_1 + a_2 \geq 3$, then

$$\mathcal{K}^{\lambda}_{\mu,p}(a_2, b, c)(\phi) \subset \mathcal{K}^{\lambda}_{\mu,p}(a_1, b, c)(\phi).$$

Proof We first prove the part (i). Let $f \in \mathcal{K}^{\lambda_1}_{\mu,p}(a, b, c)(\phi)$. Then from (1.8) and (3.2), we have

$$\begin{aligned} f \in \mathcal{K}^{\lambda_1}_{\mu,p}(a, b, c)(\phi) &\Longleftrightarrow \frac{zf'}{p} \in \mathcal{S}^{\lambda_1}_{\mu,p}(a, b, c)(\phi) \\ &\Longrightarrow \frac{zf'}{p} \in \mathcal{S}^{\lambda_2}_{\mu,p}(a, b, c)(\phi) \\ &\Longleftrightarrow f \in \mathcal{K}^{\lambda_2}_{\mu,p}(a, b, c)(\phi). \end{aligned}$$

Therefore, the assertion (3.8) of Theorem 3.4 holds true. Similarly, we can prove that the part (ii) also holds true. □

Theorem 3.5 *(i) Let $0 < b_2 \leq b_1,\ \lambda > -p,\ \mu \geq 0$ and $\phi \in M$ with (3.1) holding. If $b_1 \geq 2$ or $b_1 + b_2 \geq 3$, then*

$$\mathcal{K}^{\lambda}_{\mu,p}(a, b_2, c)(\phi) \subset \mathcal{K}^{\lambda}_{\mu,p}(a, b_1, c)(\phi).$$

(ii) Let $0 < c_1 \leq c_2,\ \lambda > -p,\ \mu \geq 0$ and $\phi \in M$ with (3.1) holding. If $c_2 \geq 2$ or $c_1 + c_2 \geq 3$, then

$$\mathcal{K}^{\lambda}_{\mu,p}(a, b, c_2)(\phi) \subset \mathcal{K}^{\lambda}_{\mu,p}(a, b, c_1)(\phi).$$

Proof Applying the same techniques as in the proof of Theorem 3.4, and using (1.8) in conjunction with Theorem 3.3, we obtain the results asserted by Theorem 3.5. □

Corollary 3.1 *Let $p \in \mathbb{N}$ and*

$$\mathrm{Re}\left(\frac{1 + Az}{1 + Bz}\right) > \frac{p-1}{p}, \quad -1 \leq B < A \leq 1;\ z \in \mathbb{U}.$$

If λ_i, a_i, b_i, and c_i $(i = 1, 2)$ satisfy the following conditions:

(1) $-p < \lambda_2 \leq \lambda_1$ *and* $\lambda_1 \geq \min\{2 - p, 3 - 2p - \lambda_2\}$,

(2) $0 < a_2 \leq a_1$ *and* $a_1 \geq \min\{2, 3 - a_2\}$,

(3) $0 < b_2 \leq b_1$ *and* $b_1 \geq \min\{2, 3 - b_2\}$,

(4) $0 < c_1 \le c_2$ *and* $c_2 \ge \min\{2, 3 - c_1\}$,

then for $\mu \ge 0$,

$$\mathcal{S}^{\lambda_1}_{\mu,p}(a_2, b_2, c_2; A, B) \subset \mathcal{S}^{\lambda_2}_{\mu,p}(a_2, b_2, c_2; A, B) \subset \mathcal{S}^{\lambda_2}_{\mu,p}(a_1, b_2, c_2; A, B) \subset \mathcal{S}^{\lambda_2}_{\mu,p}(a_1, b_1, c_2; A, B) \subset \mathcal{S}^{\lambda_2}_{\mu,p}(a_1, b_1, c_1; A, B) \tag{3.9}$$

and

$$\mathcal{K}^{\lambda_1}_{\mu,p}(a_2, b_2, c_2; A, B) \subset \mathcal{K}^{\lambda_2}_{\mu,p}(a_2, b_2, c_2; A, B) \subset \mathcal{K}^{\lambda_2}_{\mu,p}(a_1, b_2, c_2; A, B) \subset \mathcal{K}^{\lambda_2}_{\mu,p}(a_1, b_1, c_2; A, B) \subset \mathcal{K}^{\lambda_2}_{\mu,p}(a_1, b_1, c_1; A, B). \tag{3.10}$$

Proof Taking $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, we have $\phi \in M$. Thus, by applying Theorems 3.1–3.3, we obtain (3.9), and using Theorems 3.4 and 3.5, we get (3.10). □

To prove next theorems, we will use the following lemma.

Lemma 3.1 *Let* $p \in \mathbb{N}$ *and* $\phi \in M$ *with (3.1) holding. If* $f \in \mathcal{K}$ *and* $q \in \mathcal{S}^*_p(\phi)$, *then* $(z^{p-1}f) * q \in \mathcal{S}^*_p(\phi)$.

Proof If $q \in \mathcal{S}^*_p(\phi)$, then, from the definition of the class $\mathcal{S}^*_p(\phi)$, we know that

$$zq'(z) = p\phi(\omega(z))q(z),$$

where ω is a Schwarz function. Thus,

$$\frac{z[(z^{p-1}f(z)) * q(z)]'}{p[(z^{p-1}f(z)) * q(z)]} = \frac{(z^{p-1}f(z)) * zq'(z)}{p[(z^{p-1}f(z)) * q(z)]} = \frac{z^{p-1}f(z) * p\phi(\omega(z))q(z)}{p[z^{p-1}f(z) * q(z)]} = \frac{f(z) * \phi(\omega(z))z^{1-p}q(z)}{f(z) * z^{1-p}q(z)}. \tag{3.11}$$

By using similar method to those in the proof of Theorem 3.1, we deduce that (3.11) is subordinate to ϕ in $\mathbb{U}$, and hence $(z^{p-1}f) * q \in \mathcal{S}^*_p(\phi)$. □

Lemma 4 in [14] is a special case of the above Lemma 3.1.

Theorem 3.6 *Let* $p \in \mathbb{N}$, $-p < \lambda_2 \le \lambda_1$, $\mu \ge 0$ *and* $\phi, \psi \in M$, *and let* ϕ, ψ *satisfy (3.1). If* $\lambda_1 \ge 2 - p$ *or* $\lambda_1 + \lambda_2 \ge 3 - 2p$, *then*

$$\mathcal{C}^{\lambda_1}_{\mu,p}(a, b, c)(\phi, \psi) \subset \mathcal{C}^{\lambda_2}_{\mu,p}(a, b, c)(\phi, \psi).$$

Proof Let $f \in \mathcal{C}^{\lambda_1}_{\mu,p}(a, b, c)(\phi, \psi)$. Then there exists a function $q_1 \in \mathcal{S}^*_p(\phi)$ such that

$$\frac{z[\mathcal{I}^{\lambda_1}_{\mu,p}(a, b, c)f(z)]'}{pq_1(z)} \prec \psi(z), \quad z \in \mathbb{U},$$

which implies that

$$z[\mathcal{I}^{\lambda_1}_{\mu,p}(a, b, c)f(z)]' = pq_1(z)\psi[\omega(z)],$$

where ω is a Schwarz function.

From Lemma 3.1, we easily find that

$$q_2(z) = \phi_p(\lambda_2 + p, \lambda_1 + p)(z) * q_1(z) \in \mathcal{S}^*_p(\phi).$$

Then, by using the same method of the proof of Theorem 3.1, we have

$$\frac{z[\mathcal{I}_{\mu,p}^{\lambda_2}(a,b,c)f(z)]'}{pq_2(z)} = \frac{\phi_p(\lambda_2+p,\lambda_1+p)(z)*z[\mathcal{I}_{\mu,p}^{\lambda_1}(a,b,c)f(z)]'}{p\phi_p(\lambda_2+p,\lambda_1+p)(z)*q_1(z)}$$
$$= \frac{\phi_p(\lambda_2+p,\lambda_1+p)(z)*pq_1(z)\psi[\omega(z)]}{p\phi_p(\lambda_2+p,\lambda_1+p)(z)*q_1(z)}$$
$$= \frac{z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)(z)*z^{1-p}q_1(z)\psi[\omega(z)]}{z^{1-p}\phi_p(\lambda_2+p,\lambda_1+p)(z)*z^{1-p}q_1(z)} \prec \psi(z) \quad (z\in\mathbb{U}).$$

Therefore we have $f\in\mathcal{C}_{\mu,p}^{\lambda_2}(a,b,c)(\phi,\psi)$. □

Finally, by using arguments similar to those in the proof of Theorem 3.6, we easily derive the following results. Here, we choose to omit the details involved.

Theorem 3.7 *Let $0<a_2\le a_1$, $\lambda>-p$, $\mu\ge 0$ and $\phi,\psi\in M$, and let ϕ,ψ satisfy (3.1). If $a_1\ge 2$ or $a_1+a_2\ge 3$, then*

$$\mathcal{C}_{\mu,p}^{\lambda}(a_2,b,c)(\phi,\psi)\subset\mathcal{C}_{\mu,p}^{\lambda}(a_1,b,c)(\phi,\psi).$$

Theorem 3.8 *(i) Let $0<b_2\le b_1$, $\lambda>-p$, $\mu\ge 0$ and $\phi,\psi\in M$, and let ϕ,ψ satisfy (3.1). If $b_1\ge 2$ or $b_1+b_2\ge 3$, then*

$$\mathcal{C}_{\mu,p}^{\lambda}(a,b_2,c)(\phi,\psi)\subset\mathcal{C}_{\mu,p}^{\lambda}(a,b_1,c)(\phi,\psi).$$

(ii) Let $0<c_1\le c_2$, $\lambda>-p$, $\mu\ge 0$ and $\phi,\psi\in M$, and let ϕ,ψ satisfy (3.1). If $c_2\ge 2$ or $c_1+c_2\ge 3$, then

$$\mathcal{C}_{\mu,p}^{\lambda}(a,b,c_2)(\phi,\psi)\subset\mathcal{C}_{\mu,p}^{\lambda}(a,b,c_1)(\phi,\psi).$$

Upon setting

$$\phi(z)=\psi(z)=\frac{1+Az}{1+Bz},\quad -1\le B<A\le 1;\ z\in\mathbb{U}$$

in Theorems 3.6–3.8, we get the following result.

Corollary 3.2 *Under the conditions of Corollary 3.1, we have*

$$\mathcal{C}_{\mu,p}^{\lambda_1}(a_2,b_2,c_2;A,B)\subset\mathcal{C}_{\mu,p}^{\lambda_2}(a_2,b_2,c_2;A,B)\subset\mathcal{C}_{\mu,p}^{\lambda_2}(a_1,b_2,c_2;A,B)$$
$$\subset\mathcal{C}_{\mu,p}^{\lambda_2}(a_1,b_1,c_2;A,B)\subset\mathcal{C}_{\mu,p}^{\lambda_2}(a_1,b_1,c_1;A,B).$$

Remark 3.1 (i) Putting $p=1$ and $\lambda=\lambda_2=\lambda_1-1$ $(\lambda\ge 0)$ in Theorems 3.1 and 3.6, respectively, we have the results obtained by Srivastava et al. [16, Theorems 1 and 4, respectively].

(ii) Taking $p=1$ and $a=a_2=a_1-1$ $(a\ge 1)$ in Theorems 3.2 and 3.7, respectively, we get the results obtained by Srivastava et al. [16, Theorems 2 and 5, respectively].

(iii) Setting $p=1$, $\lambda=\lambda_2=\lambda_1-1$ $(\lambda\ge 0)$ and $a=a_2=a_1-1$ $(a\ge 1)$ in the assertions (i) and (ii) of Theorems 3.4, respectively, we obtain the results obtained by Srivastava et al. [16, Corollary 3].

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