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The Pants Sum of High Distance Heegaard Splittings

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Abstract Let M_i be a compact orientable 3-manifold, and F_i be an incompressible surface on ∂M_i, $i = 1, 2$. Let $f : F_1 \to F_2$ be a homeomorphism, and $M = M_1 \cup_f M_2$. In this paper, under certain assumptions for the attaching surface F_i, we show that if both M_1 and M_2 have Heegaard splittings with distance at least $2(g(M_1)+g(M_2))+1$, then $g(M) = g(M_1)+g(M_2)$.

Keywords pants sum; Heegaard distance; Heegaard genus.

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1. Introduction

Let $M = M_1 \cup_F M_2$ be a decomposition of M into two compact connected orientable 3-manifolds along a connected incompressible separating surface F which is properly embedded in M. A central topic in 3-manifold theory is to study how $g(M)$ is related to $\chi(F)$, $g(M_1)$ and $g(M_2)$. Suppose F is a closed surface, and let $V_i \cup_{S_i} W_i$ be a Heegaard splitting of M_i for i=1,2. Then M has a natural Heegaard splitting $V \cup_S W$ which is called the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ (see [1]). From this point of view, $g(M) \leq g(M_1) + g(M_2) - g(F)$, and if the Heegaard genera of M_1 and M_2 are additive, then F must be a 2-sphere. Suppose F is a bounded surface, by the so-called disk version of Haken's lemma, the Heegaard genera of M_1 and M_2 are additive, and it has been shown that $g(M) \leq g(M_1) + g(M_2)$ always holds [2]. In recent years, many papers have given sufficient conditions on the additivity of Heegaard genera of 3-manifolds. For example, if F is an annulus, various results about if $g(M) = g(M_1) + g(M_2)$ holds or not have been given [2–4]. In general, if F is a bounded surface, $g(M)$ is determined by the Euler characteristic of F and the pattern of F on both ∂M_1 and ∂M_2.

The main results of this paper are the following:

Theorem 1.1 *Let $M = M_1 \cup_F M_2$, where M_i is an irreducible ∂-irreducible 3-manifold, and F is an incompressible pants on one component P_i of ∂M_i, $i = 1, 2$. If both M_1 and M_2 have Heegaard splittings with distance at least $2(g(M_1) + g(M_2)) + 1$, then $g(M) = g(M_1) + g(M_2)$.*

Remark 1.1 Theorem 1.1 above and Theorem 4 in [2] show that the Heegaard genera of 3-manifolds are additive under the annulus and pants sum of high distance Heegaard splitings, i.e.,

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the Heegaard genus of the surface sum of M_1 and M_2 has no relationship with the pattern of the attaching surfaces F on both P_1 and P_2.

Theorem 1.2 *Let $M = M_1 \cup_F M_2$, where M_i is an irreducible ∂-irreducible 3-manifold, and F is a bounded connected incompressible surface on one component P_i of ∂M_i, $i = 1, 2$. If both M_1 and M_2 have Heegaard splittings with distance at least $2(g(M_1) + g(M_2)) + 1$, and F is non-separating on at least one of P_1 and P_2 and $\chi(F) \geq 1 - g(P_i)$ for $i = 1, 2$, then $g(M) = g(M_1) + g(M_2)$.*

Theorem 1.3 *Let $M = M_1 \cup_F M_2$, where M_i is an irreducible ∂-irreducible 3-manifold, and F is a bounded connected incompressible surface on one component P_i of ∂M_i, $i = 1, 2$. If both M_1 and M_2 have Heegaard splittings with distance at least $2(g(M_1) + g(M_2)) + 1$, and F is complete separating on both P_1 and P_2 and $\chi(F) \geq 2 - n$, then $g(M) = g(M_1) + g(M_2)$, where n is the number of boundary components of F.*

Theorem 1.4 *Let $M = M_1 \cup_F M_2$, where M_i is an irreducible ∂-irreducible 3-manifold, and F is an incompressible pants on one component P_i of ∂M_i, $i = 1, 2$. If both M_1 and M_2 have Heegaard splittings with distance at least $2(g(M_1) + g(M_2)) + 1$, and F is non-separating on both P_1 and P_2, then the length of any minimal Heegaard splitting is three.*

2. Preliminaries

We are working in the PL category. All 3-manifolds M in this paper are assumed to be compact and orientable. Furthermore, we assume that ∂M contains no spherical component.

Let F be either a properly embedded connected surface in a 3-manifold M or a connected sub-surface of ∂M. If there is an essential simple closed curve on F which bounds a disk in M or F is a 2–sphere which bounds a 3-ball in M, then we say F is compressible; otherwise, F is said to be incompressible. If F is an incompressible surface not ∂-parallel to ∂M, then F is said to be essential. If M contains an essential 2-sphere, then M is said to be reducible; otherwise, M is said to be irreducible.

A compression body V is a 3-manifold obtained by attaching 2–handles to $F \times I$, along a collection of pairwise disjoint simple closed curves on $F \times \{0\}$, then capping off resulting 2-sphere boundary components with 3-balls, where F is a connected closed surface. Let $\partial_+ V = F \times \{1\}$ and $\partial_- V = \partial V - \partial_+ V$. Note that if $\partial_- V = \emptyset$, then V is called a handlebody. In particular, if $V = F \times I$, then V is called a trivial compression body.

Let M be a 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $S = \partial_+ V = \partial_+ W$, then we say M has a Heegaard splitting, denoted by $M = V \cup_S W$; and S is called a Heegaard surface of M. Moreover, if the genus $g(S)$ of S is minimal among all the Heegaard surfaces of M, then $g(S)$ is called the genus of M, denoted by $g(M)$.

If there are essential disks $B \subset V$ and $D \subset W$ such that $\partial D = \partial B$ (resp., $\partial B \cap \partial D = \emptyset$), then $V \cup_S W$ is said to be reducible (resp., weakly reducible); otherwise, it is said to be irreducible (resp., strongly irreducible), see [5].

Let $M = V \cup_S W$ be a Heegaard splitting. Then $V \cup_S W$ has a thin position as

$$V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} \cdots \cup_{H_{n-1}} (V_n' \cup_{S_n'} W_n')$$

where $n \geq 2$, each component of $H_1, \ldots, H_{n-1}$ is an essential closed surface in M and $V_i' \cup_{S_i'} W_i'$ is a strongly irreducible Heegaard splitting for $1 \leq i \leq n$. We call n the length of the thin position [6].

Let $V \cup_S W$ be a Heegaard splitting of M. The distance between two essential simple closed curves α and β in S, denoted by $d(\alpha, \beta)$, is the smallest integer $n \geq 0$ so that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ in S such that α_i is disjoint from α_{i-1} for $1 \leq i \leq n$. The distance of the Heegaard splitting $V \cup_S W$ is $d(S) = \mathrm{Min}\{d(\alpha, \beta)\}$, where α bounds a disk in V and β bounds a disk in W. For more details, see [7].

Let M be a 3-manifold. If M is homeomorphic to $S \times I$, where S is a connected closed surface, then M is called a product I-bundle of closed surface S.

Let P be a connected closed surface, F be a bounded sub-surface on P. If $P - \mathrm{int}\, F$ is connected, then F is said to be non-separating on P; otherwise, F is said to be separating on P. If F is separating on P and each boundary component of F is also separating on P, then F is called complete separating on P; otherwise, F is called non-complete separating on P.

Let M_1 and M_2 be two 3-manifolds, P_i be one component of ∂M_i and F_i be a bounded connected incompressible sub-surface on P_i for $i = 1, 2$. Let $f : F_1 \to F_2$ be a homeomorphism. Then the manifold M obtained by gluing M_1 and M_2 along F_1 and F_2 via f is called the surface sum of M_1 and M_2 along the bounded surface F_1 and F_2, and we denote it by $M = M_1 \cup_f M_2$ or $M = M_1 \cup_F M_2$. Let $P_i \times [0, 1]$ be a regular neighborhood of P_i in M_i, denote $P_i = P_i \times \{0\}$, $P^i = P_i \times \{1\}$, $M^i = M_i - P_i \times [0, 1)$ for $i = 1, 2$, and $M^* = (P_1 \times I) \cup_F (P_2 \times I)$. Then $M = M^1 \cup_{P^1} M^* \cup_{P^2} M^2$, where M^* is called the surface sum of product I-bundle of closed surfaces P_1 and P_2 along F.

Lemma 2.1 ([2]) *Let M be the surface sum of two irreducible, ∂-irreducible 3-manifolds M_1 and M_2 along a bounded connected surface F, and let ∂_i be the component of ∂M_i containing F. If both M_1 and M_2 have Heegaard splittings with distance at least $2(g(M_1) + g(M_2)) + 1$, then any minimal Heegaard splitting of M is the amalgamation of Heegaard splittings of M^1, M^2 and M^* along P^1 and P^2.*

Let $M = M_1 \cup_F M_2$ be the surface sum of two irreducible, ∂-irreducible 3-manifolds M_1 and M_2 along a bounded connected surface F. Let $M_i = V_i \cup_{S_i} W_i$ be a Heegaard splitting of M_i such that $F \subset P_i \subset \partial_- W_i$ and $P_i \times I$ is disjoint from S_i $(i = 1, 2)$. Now let γ_i be a vertical arc in W_i such that the endpoints $e_1(\gamma_i) \subset \partial_+ W_i$ and $e_2(\gamma_1) = e_2(\gamma_2) \subset intF$. Let $N(\gamma_1 \cup \gamma_2)$ be a regular neighborhood of $\gamma_1 \cup \gamma_2$ in $W_1 \cup W_2$. Let $V = V_1 \cup N(\gamma_1 \cup \gamma_2) \cup V_2$, and let W be the closure of $(W_1 \cup W_2) - N(\gamma_1 \cup \gamma_2)$.

The following Lemma 2.2 indicates that $V \cup_S W$ is a Heegaard splitting of M. We call it the surface sum of Heegaard splittings $M_1 = V_1 \cup_{S_1} W_1$ and $M_2 = V_2 \cup_{S_2} W_2$ along F.

Lemma 2.2 ([2]) *$V \cup_S W$ is a Heegaard splitting of M, where $S = \partial_+ V = \partial_+ W$.*

Remark 2.1 By Lemma 2.2, $g(M) \leq g(M_1) + g(M_2)$.

Lemma 2.3 ([8]) *Let M be a Haken 3-manifold containing an orientable incompressible surface of genus g. Then any Heegaard splitting of M has distance at most $2g$.*

Lemma 2.4 ([9]) *Let $M^0 = (P_1 \times I) \cup_F (P_2 \times I)$, where P_i $(i = 1, 2)$ is a connected orientable closed surface with genus at least two, and F is an incompressible pants on both $P_1 \times \{0\}$ and $P_2 \times \{0\}$. If F is non-separating on $P_1 \times \{0\}$ (resp., $P_2 \times \{0\}$). Then*

(1) M^0 contains no essential closed surface, if F is non-separating on $P_2 \times \{0\}$ (resp., $P_1 \times \{0\}$).

(2) M^0 contains six types of essential closed surfaces up to isotopy, if F is complete separating on $P_2 \times \{0\}$ (resp., $P_1 \times \{0\}$).

(3) M^0 contains two types of essential closed surfaces up to isotopy, if F is non-complete separating on $P_2 \times \{0\}$ (resp., $P_1 \times \{0\}$).

Definitions and terms which are not defined here are standard [10, 11].

3. Heegaard genus of the surface sum of product I-bundle of closed surfaces

Proposition 3.1 *Let $M^* = (P_1 \times I) \cup_F (P_2 \times I)$, where P_i $(i = 1, 2)$ is a connected closed surface, and F is a bounded connected incompressible surface on both $P_1 \times \{0\}$ and $P_2 \times \{0\}$. If F is non-separating on at least one of $P_1 \times \{0\}$ and $P_2 \times \{0\}$, and $\chi(F) \geq 1 - g(P_i)$ for $i = 1, 2$, then $g(M^*) = g(P_1) + g(P_2)$.*

Proof Let $P_i \times \{0\} = P_i$, and $P_i \times \{1\} = P^i$, $i = 1, 2$. Since F is non-separating on at least one of P_1 and P_2, M^* contains three boundary components P^1, P^2 and $P^* = (P_1 - \operatorname{int} F) \cup (P_2 - \operatorname{int} F)$.

Let $M^* = V \cup_S W$ be a minimal Heegaard splitting of M^*. Then $g(S) = g(M^*)$. Since any Heegaard splitting of M^* has to have at least two of P^1, P^2 and P^* on one side, it follows

$$g(S) \geq g(P^1) + g(P^2) \text{ or } g(S) \geq g(P^1) + g(P^*) \text{ or } g(S) \geq g(P^2) + g(P^*).$$

Since $\chi(P^*) = \chi(P_1) + \chi(P_2) - 2\chi(F)$, we have $g(P^*) = g(P_1) + g(P_2) + \chi(F) - 1$. Note that $\chi(F) \geq 1 - g(P_i)$ for $i = 1, 2$, thus

$$g(P^*) = g(P_1) + g(P_2) + \chi(F) - 1 \geq g(P_1),$$

$$g(P^*) = g(P_1) + g(P_2) + \chi(F) - 1 \geq g(P_2).$$

By the above assumption, $g(P_i) = g(P^i)$ for $i = 1, 2$. Hence $g(S) \geq g(P_1) + g(P_2)$. Clearly, $g(P_i \times I) = g(P_i)$, $i = 1, 2$. By Lemma 2.2, $g(S) \leq g(P_1) + g(P_2)$. Therefore, $g(M^*) = g(P_1) + g(P_2)$. □

Proposition 3.2 *Let $M^* = (P_1 \times I) \cup_F (P_2 \times I)$, where P_i $(i = 1, 2)$ is a connected orientable closed surface, and F is a bounded connected incompressible surface on both $P_1 \times \{0\}$ and*

$P_2 \times \{0\}$. If F is complete separating on both $P_1 \times \{0\}$ and $P_2 \times \{0\}$, and $\chi(F) \geq 2-n$, then $g(M^*) = g(P_1) + g(P_2)$, where n is the number of boundary components of F.

Proof Let $P_i \times \{0\} = P_i$, and $P_i \times \{1\} = P^i$, $i = 1, 2$. Since F is complete separating on both P_1 and P_2, without loss of generality, we assume $P_1 - \operatorname{int} F = P_1^1 \cup P_1^2 \cup \cdots \cup P_1^n$, $P_2 - \operatorname{int} F = P_2^1 \cup P_2^2 \cup \cdots \cup P_2^n$, and $\partial P_i^j = \partial P_{3-i}^j$ for $i = 1, 2, 1 \leq j \leq n$, where n is the number of boundary components of F. Thus M^* contains $n+2$ boundary components P^1, P^2, $P_1^1 \cup P_2^1$, $P_1^2 \cup P_2^2, \ldots, P_1^n \cup P_2^n$. Let $P_i^j \cup P_{3-i}^j = P_*^j$ for $i = 1, 2, 1 \leq j \leq n$.

Let $M^* = V \cup_S W$ be a minimal Heegaard splitting of M^*. Then $g(S) = g(M^*)$. Since $P_i \times I$ is a trivial compression body, $g(P_i \times I) = g(P_i)$, $i = 1, 2$. By Lemma 2.2, $g(S) \leq g(P_1) + g(P_2)$, then P^1, P^2, $P_*^1, P_*^2, \ldots, P_*^n$ cannot lie in one side of S; otherwise $g(S) > g(P_1) + g(P_2)$, a contradiction. Furthermore, if P^1 and P^2 lie in one side of S, then no other boundary component of M^* lies in the same side of S.

Since each side of S contains at least one boundary component of M^*, we have

$$2g(S) \geq g(P^1) + g(P^2) + g(P_*^1) + g(P_*^2) + \cdots + g(P_*^n).$$

Note that

$$\chi(P_*^1) + \chi(P_*^2) + \cdots + \chi(P_*^n) = \chi(P_1) + \chi(P_2) - 2\chi(F),$$

and $g(P_i) = g(P^i)$ for $i = 1, 2$, then $g(S) \geq (g(P_1) + g(P_2))$. Therefore, $g(M^*) = g(P_1) + g(P_2)$. □

Proposition 3.3 *Let $M^* = (P_1 \times I) \cup_F (P_2 \times I)$, where P_i $(i = 1, 2)$ is a connected orientable closed surface, and F is an incompressible pants on both $P_1 \times \{0\}$ and $P_2 \times \{0\}$. Then $g(M^*) = g(P_1) + g(P_2)$.*

Proof Let $P_i \times \{0\} = P_i$, and $P_i \times \{1\} = P^i$, $i = 1, 2$. Since F is an incompressible pants on both P_1 and P_2, then $g(P_i) \geq 2$ for $i = 1, 2$.

By Propositions 3.1 and 3.2, if F is non-separating on at least one of P_1 and P_2 or complete separating on both P_1 and P_2, $g(M^*) = g(P_1) + g(P_2)$.

Now there are two cases.

Case 1 F is non-complete separating on one of P_1 and P_2, while complete separating on the other.

In this case, without loss of generality, we suppose F is non-complete separating on P_1, while complete separating on P_2. Let $P_1 - \operatorname{int} F = P_1^1 \cup P_1^2$, $P_2 - \operatorname{int} F = P_2^1 \cup P_2^2 \cup P_2^3$, $\partial P_1^1 = \partial P_2^1 \cup \partial P_2^2$, where each boundary component of P_1^1 is non-separating on P_1. By the above assumptions, M^* contains four boundary components P^1, P^2, $P^* = P_1^1 \cup P_2^1 \cup P_2^2$ and $P^{**} = P_1^2 \cup P_2^3$.

Let $M^* = V \cup_S W$ be a minimal Heegaard splitting of M^*. Then $g(S) = g(M^*)$. Thus as above argument, if P^1 and P^2 lie in one side of S, then no other boundary component of M^* lies in the same side of S. No mater how the Heegaard surface S separates the boundary components of M^*, by the definition of compression body, $2g(S) \geq g(P^1) + g(P^2) + g(P^*) + g(P^{**})$ always holds. Note that $\chi(P^*)+\chi(P^{**})=\chi(P_1)+\chi(P_2)-2\chi(F)$, and $\chi(F)=-1$. Thus

$$g(P^*)+g(P^{**})=g(P^1)+g(P^2)-1,$$

therefore, $g(S)\geq g(P_1)+g(P_2)-1/2$.

Since the Heegaard genus is an integer, $g(S)\geq g(P_1)+g(P_2)$.

Since $P_i\times I$ is a trivial compression body, $g(P_i\times I)=g(P_i)$, $i=1,2$. By Lemma 2.2, $g(S)\leq g(P_1)+g(P_2)$. Therefore $g(M^*)=g(P_1)+g(P_2)$.

Case 2 F is non-separating on both P_1 and P_2.

Let $P_1-\operatorname{int}F=P_1^1\cup P_1^2$ and $P_2-\operatorname{int}F=P_2^1\cup P_2^2$, where each boundary component of P_1^1 is non-separating on P_1, and each boundary component of P_2^1 is non-separating on P_2.

Now there are two subcases.

Subcase 2.1 $\partial P_1^1=\partial P_2^1$.

In this case, M^* contains four boundary components. By the proof of Case 1, $g(M^*)=g(P_1)+g(P_2)$.

Subcase 2.2 $\partial P_1^1\neq\partial P_2^1$.

If $\partial P_1^1\neq\partial P_2^1$, M^* contains three boundary components. By the proof of Proposition 3.1, $g(M^*)=g(P_1)+g(P_2)$.

By the above argument, Proposition 3.3 holds. □

Remark 3.1 Proposition 3.3 cannot be generalized to an n-punctured 2-sphere for $n\geq 4$. Although $g(M^*)=g(P_1)+g(P_2)$ holds if F is a four punctured 2-sphere and it is complete separating on both P_1 and P_2.

4. The proofs of the main results

Proof of Theorems 1.1, 1.2 and 1.3 Let $M=M^1\cup_{P^1}M^*\cup_{P^2}M^2$, where $M^*=(P_1\times I)\cup_F(P_2\times I)$, and $P_i\times\{1\}=P^i$, $i=1,2$. If F is a pants, by Proposition 3.3, $g(M^*)=g(P_1)+g(P_2)$. By Lemma 2.1, $g(M)=g(M_1)+g(M_2)+g(M^*)-g(P_1)+g(P_2)$, therefore, $g(M)=g(M_1)+g(M_2)$. Hence Theorem 1.1 holds. By Propositions 3.1 and 3.2, and by a similar argument as above, Theorems 1.2 and 1.3 hold. □

Proof of Theorem 1.4 Let $M=M^1\cup_{P^1}M^*\cup_{P^2}M^2$, where $M^*=(P_1\times I)\cup_F(P_2\times I)$, and $P_i\times\{1\}=P^i$, $i=1,2$. By Lemma 2.1, any minimal Heegaard splitting of M is the amalgamation of Heegaard splittings of M^1, M^2 and M^* along P^1 and P^2. It is easy to see that both P^1 and P^2 are essential closed surfaces in M.

By the assumption of Theorem 1.4, M^i $(i=1,2)$ has a Heegaard splitting $V_i\cup_{S_i}W_i$ with distance at least $2(g(M_1)+g(M_2))+1$. By Lemma 2.3, M^i contains no essential closed surface. Then by Lemma 2.4, the length of any minimal Heegaard splitting of M is three if F is non-separating on both P_1 and P_2. This completes the proof of Theorem 1.4. □

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