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On Vasantha Kandasamy's Problem

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Abstract In this paper, we solve the problem proposed by Vasantha Kandasamy about the torsion-free non-abelian groups.

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1. Introduction

In 1995, Vasantha Kandasamy [1] introduced the definition of the essential group which is defined as follows:

Definition 1.1 ([1]) *Let G be a group and H any subgroup of group G. If for every subgroup K of G we have $H \cap K = \{e\}$, then we say H is an essential subgroup of G. A group G is called an essential group if every subgroup H of G is an essential subgroup.*

Definition 1.2 ([1]) *Let G be a group, and H a normal subgroup of G. If for every normal subgroup K of G we have $H \cap K = \{e\}$, then H is said to be a strongly essential subgroup of G. A group G is called a strongly essential group if every normal subgroup H of G is a strongly essential subgroup.*

Definition 1.3 ([1]) *Let G be a group, and H a normal subgroup of G. If for every subgroup K of G we have $H \cap K = \{e\}$, then we say H is a weakly essential subgroup of G. A group G is called a weakly essential group if any normal subgroup H of G is a weakly essential subgroup.*

In [1], Vasantha Kandasamy discussed the essentiality of the symmetric group S_n and proved the following results:

(i) $S_n (n > 3)$ is not an essential group;

(ii) $S_n (n > 3)$ is not a weakly essential group;

(iii) $S_n (n \neq 4)$ is a strongly essential group.

In particular, S_3 is an essential group, weakly essential and strongly essential group.

Example 1.1 (i) If n is a prime, then D_{2n} is an essential group, weakly essential group and strongly essential group;

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(ii) If n is not a prime, then D_{2n} is not an essential group, weakly essential group and strongly essential group.

Proof Let $G = D_{2n} = \langle a, b \mid a^n = 1, b^2 = 1, b^{-1}ab = a^{-1}\rangle$.

(i) If $n = p$ is a prime, then D_{2p} has only one subgroup Z_p of order p and p subgroups of order 2, and Z_p is the unique nontrivial normal subgroup of D_{2p}. It follows that D_{2p} is an essential group, weakly essential group and strongly essential group.

(ii) If $n = uv$ is not a prime, where u and v are positive proper divisors of n. Let $H = \langle a^v\rangle$. Then by the fact that H is a characteristic subgroup of $\langle a\rangle$ and $\langle a\rangle \trianglelefteq G$, we have $H \trianglelefteq G$. From $H \cap \langle a\rangle = H \neq 1$, we know that G is neither a strongly essential group, nor essential or weakly essential. $\square$

Vasantha Kandasamy also proposed the following Problem:

Problem 1.1 Is every torsion-free non-abelian group:

(i) an essential group?

(ii) a weakly essential group?

(iii) a strongly essential group?

Recall that a group is said to be torsion-free if apart from the identity all its elements have infinite order [2]. In this paper, we will answer the above problem on the torsion-free non-abelian group.

2. On Vasantha Kandasamy's problem

In this section, we will answer the problem proposed by Vasantha Kandasamy.

The answer of the problem is negative, so we will give a counter-example here. One important kind of non-abelian group is the linear group, such as $\mathrm{SL}(n, \mathbb{C})$. We consider the $n \times n$ upper triangular matrices:

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ 0 & 1 & a_{23} & \dots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ 0 & 0 & 1 & \dots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{n-2,n-1} & a_{n-2,n} \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

In $\mathrm{SL}(n, \mathbb{C})$, all upper triangular matrices and the identity matrix E form a group G under the matrix multiplication. It is clear that G is non-abelian. We first prove that G is a torsion-free group.

Proposition 2.1 G is a torsion-free group.

Proof We assume that there exists a non-trivial element L_1 of finite order. Since the eigenvalue of every matrix in G is 1, the diagonal elements of the Jordan classical form J_1 of L_1 are all 1, and there exists P such that $P^{-1}L_1P = J_1$. Because the order of L_1 is finite, say m, we have $L_1^m=E$, and $J_1^m=(P^{-1}L_1P)^m=P^{-1}L_1^mP = E$. But J_1 satisfies $J_1^m = E$ if and only if $J_1 = E$. It follows that $L_1 = E$, which contradicts the fact that L_1 is a non-trivial element of G. $\square$

Since the essential group is related to the subgroup and normal subgroup, we need to discuss the subgroups and normal subgroups of the torsion-free non-abelian group G which we introduced above.

Consider the set $H_{\mathbb{C}}$ of $n \times n$ matrices P with the following form:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & a \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad a \in \mathbb{C}.$$

It is obvious that the product of any two elements in $H_{\mathbb{C}}$ is also contained in $H_{\mathbb{C}}$, and for $P \in H_{\mathbb{C}}$, $P^{-1} \in H_{\mathbb{C}}$, so that $H_{\mathbb{C}} \leq G$.

Proposition 2.2 *$H_{\mathbb{C}}$ is a subgroup of $Z(G)$.*

Proof Let $P \in H_{\mathbb{C}}$ and $L \in G$ be defined as follows:

$$P = \begin{pmatrix} 1 & 0 & \dots & 0 & a \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = E + \begin{pmatrix} 0 & 0 & \dots & 0 & a \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} = E + P_1,$$

$$L = \begin{pmatrix} 1 & l_{12} & \dots & l_{1,n-1} & l_{1,n} \\ 0 & 1 & \dots & l_{2,n-1} & l_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & l_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = E + \begin{pmatrix} 0 & l_{12} & \dots & l_{1,n-1} & l_{1,n} \\ 0 & 0 & \dots & l_{2,n-1} & l_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & l_{n-1,n} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} = E + L_1.$$

We have

$$PL = (E + P_1)(E + L_1) = E + P_1 + L_1 + P_1L_1,$$

where

$$P_1L_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & a \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & l_{12} & \dots & l_{1,n-1} & l_{1,n} \\ 0 & 0 & \dots & l_{2,n-1} & l_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & l_{n-1,n} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} = 0 = L_1P_1.$$

It follows that

$$LP = (E + L_1)(E + P_1) = E + P_1 + L_1 = PL.$$

Therefore, $H_{\mathbb{C}} \leq Z(G)$. $\Box$

Now let $H_{\mathbb{R}}$ be the set of the following matrices

$$\begin{pmatrix} 1 & 0 & \dots & 0 & x \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

where $x \in \mathbb{R}$. Clearly, $H_{\mathbb{R}} \leq H_{\mathbb{C}}$, and $H_{\mathbb{R}} \trianglelefteq G$, $H_{\mathbb{C}} \trianglelefteq G$. Since $H_{\mathbb{R}} \cap H_{\mathbb{C}} = H_{\mathbb{R}} \neq \{e\}$, we see that $H_{\mathbb{R}}$ is not the strongly essential subgroup of G. Hence G is not a strongly essential group, so it is neither an essential group, nor a weakly essential group.

Hence, we have constructed a counter-example, and give the negative answer to Problem 1.1 proposed by Vasantha Kandasamy. This yields

Theorem 2.1 Not any torsion-free non-abelian group is

(1) an essential group;

(2) a weakly essential group;

(3) a strongly essential group.

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