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Some Characterizations of Prüfer v-Multiplication Rings

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Abstract In this paper, we study Prüfer v-multiplication rings (PVMRs) and give some new characterizations of PVMRs. Moreover, we show that a Marot ring R is a PVMR if and only if every w-ideal of R is complete.

Keywords w-ideal; integrally closed; PVMR; complete.

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1. Introduction

Throughout this paper, R denotes a commutative ring with identity, $T(R)$ denotes its total quotient ring, and R^c denotes the integral closure of R in $T(R)$. R is called an integrally closed ring if $R = R^c$. An overring of R is a ring between R and $T(R)$. An element of R is regular if it is not a zero divisor. An ideal of R that contains a regular element is said to be a regular ideal.

Recall from [1] that an ideal J is called a Glaz-Vasconcelos ideal (GV-ideal), denoted by $J \in GV(R)$, if J is finitely generated and the natural homomorphism from R into $J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism. An R-module M is called a GV-torsionfree module if whenever $Jx = 0$, for some $J \in GV(R)$ and $x \in M$, then $x = 0$. A GV-torsionfree R-module M is called a w-module if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in GV(R)$, and the w-envelope of M is defined by $M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}$, where $E(M)$ is the injective envelope of M. M is a w-module if and only if $M_w = M$. M is said to be w-finite (or of finite type, when no confusion is likely) if $M_w = N_w$ for some finitely generated submodule N of M. A maximal w-ideal is an ideal that is maximal among the proper w-ideals. We denote by w-max(R) the set of maximal w-ideals of R. In this paper, we consider the case $w\text{-}\max(R) \neq \emptyset$. In fact, if $w\text{-}\max(R) = \emptyset$, and suppose that c is a regular element of R. Then $(c) = (c)_w = R$, and so c is a unit, therefore $R = T(R)$. Let $\mathfrak{F}(R)$ be the set of R-submodules of $T(R)$.

Recall that an integral domain R is called a Prüfer v-multiplication domain (PVMD) if each finitely generated ideal is w-invertible. In 1980, Huckaba and Papick [2], and Matsuda [3] extended this notion to rings with zero divisors by declaring that a ring R is said to be a Prüfer v-multiplication ring (PVMR) if each finitely generated regular ideal is w-invertible. For a monographic study on PVMDs, the reader may consult [4, 5].

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In [1], the theory of w-operations was developed for arbitrary commutative rings. The purpose of this paper is to characterize PVMRs by the w-operation.

2. PVMRs and completion of w-ideals

Let S be a multiplicatively closed subset of R. The large quotient ring of R with respect to S, denoted by $R_{[S]}$, is the set $R_{[S]} = \{z \in T(R) \mid sz \in R \text{ for some } s \in S\}$. If $A \in \mathfrak{F}(R)$, then $[A]R_{[S]} = \{z \in T(R) \mid sz \in A \text{ for some } s \in S\}$ is an $R_{[S]}$-submodule of $T(R)$. In particular, if I is an ideal of R, then $[I]R_{[S]}$ is an ideal of $R_{[S]}$. For a prime ideal $\mathfrak{p}$, we write $R_{[\mathfrak{p}]}$ in place of $R_{[R\backslash\mathfrak{p}]}$.

Proposition 2.1 *Let $A \in \mathfrak{F}(R)$. Then $[A]R_{[\mathfrak{p}]}$ is a w-module as an R-module for every prime w-ideal $\mathfrak{p}$ of R.*

Proof First, we prove that $[A]R_{[\mathfrak{p}]}$ is a GV-torsionfree R-module. Let $Jx = 0$ for some $J \in GV(R)$ and $x \in [A]R_{[\mathfrak{p}]}$. There exists a regular element $b \in R$ such that $bx \in R$. Thus $Jbx = 0$, and so $bx = 0$. Therefore, $x = 0$.

Let $Jx \subseteq [A]R_{[\mathfrak{p}]}$ for some $J \in GV(R)$ and $x \in E(R)$. Then there exists an element $s_1 \in R\backslash\mathfrak{p}$ such that $Js_1x \subseteq A$. Note that $J \not\subseteq \mathfrak{p}$. So there exists an element $s_2 \in J$, but $s_2 \notin \mathfrak{p}$ such that $s_1s_2x \in A$. Thus $x \in [A]R_{[\mathfrak{p}]}$. $\square$

Note that for every prime ideal $\mathfrak{p}$ of R, $R_{[\mathfrak{p}]}$ is a w-module as an R-module. In fact, let $Jx \subseteq R_{[\mathfrak{p}]}$ for some $J \in GV(R)$ and $x \in E(R)$. There exists an element $s \in R \setminus \mathfrak{p}$ such that $Jsx \subseteq R$. Thus $sx \in R$ since R is a w-module. It follows that $x \in R_{[\mathfrak{p}]}$.

Corollary 2.1 *Let $A \in \mathfrak{F}(R)$. Then $[A_w]R_{[\mathfrak{p}]} = [A]R_{[\mathfrak{p}]}$ for every prime w-ideal $\mathfrak{p}$ of R.*

Proof For $x \in [A_w]R_{[\mathfrak{p}]}$, there exists an element $s \in R\backslash\mathfrak{p}$ such that $sx \in A_w$. Then there exists $J \in GV(R)$ such that $Jsx \subseteq A$. Thus $Jx \subseteq [A]R_{[\mathfrak{p}]}$, and so $x \in [A]R_{[\mathfrak{p}]}$, as required. $\square$

Proposition 2.2 *If $A \in \mathfrak{F}(R)$, then $A_w = \bigcap_{\mathfrak{m}\in w\text{-}\max(R)} A_wR_{[\mathfrak{m}]} = \bigcap_{\mathfrak{m}\in w\text{-}\max(R)}[A_w]R_{[\mathfrak{m}]}$.*

Proof Clearly, $A_w \subseteq \bigcap_{\mathfrak{m}\in w\text{-}\max(R)} A_wR_{[\mathfrak{m}]} \subseteq \bigcap_{\mathfrak{m}\in w\text{-}\max(R)}[A_w]R_{[\mathfrak{m}]}$.

Let $x \in \bigcap_{\mathfrak{m}\in w\text{-}\max(R)}[A_w]R_{[\mathfrak{m}]}$. Then $I = (A_w :_R x) = \{r \in R \mid rx \in A_w\}$ is a w-ideal of R. For each $\mathfrak{m} \in w\text{-}\max(R)$, there exists $s \in R\backslash\mathfrak{m}$ such that $sx \in A_w$. Thus $I \not\subseteq \mathfrak{m}$, and so $I = R$. Therefore, $x \in A_w$. $\square$

Corollary 2.2 *Let R be a ring. Then $R = \bigcap_{\mathfrak{m}\in w\text{-}\max(R)} R_{[\mathfrak{m}]}$.*

Lemma 2.1 *The following are equivalent for a ring R:*

(1) R is integrally closed.

(2) If a is a regular element of R and I is a finitely generated regular ideal of R such that $I^2 = Ia$, then $I_w = (a)$.

(3) If a is a regular element of R and I is a finitely generated regular ideal of R such that $I^2 = Ia$, then $I_w \subseteq (a)$.

Proof (1) $\Rightarrow$ (2). By [6, Proposition 24.1], $I = (a)$. Thus $I_w = (a)$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). It follows from [6, Proposition 24.1] since $I \subseteq I_w \subseteq (a)$. $\square$

Lemma 2.2 *Let R be an integrally closed ring. Suppose that $a, b \in R$ with a regular. If there exists $n > 1$ such that $a^{n-1}b \in (a^n, b^n)_w$, then (a, b) is w-invertible.*

Proof We prove this by an induction on n. Assume that $n = 2$. Then there exists $J = (c_1, c_2, \ldots, c_m) \in GV(R)$ such that $Jab \subseteq (a^2, b^2)$. For each $1 \leqslant i \leqslant m$, $c_i ab = x_i a^2 + y_i b^2$, where $x_i, y_i \in R$. Multiplying this equation by $\frac{y_i}{a^2}$, we have $(\frac{y_i b}{a})^2 - c_i(\frac{y_i b}{a}) + x_i y_i = 0$. Thus $\frac{y_i b}{a}$ is integral over R, and is therefore in R. Write $z_i = \frac{y_i b}{a}$. Note that

$$c_i a \in (a, b)(y_i, c_i - z_i) = (ay_i, az_i, a(c_i - z_i), ax_i) \subseteq (a).$$

Set $I = (y_1, c_1 - z_1) + \cdots + (y_m, c_m - z_m)$. Then $Ja \subseteq (a, b)I \subseteq (a)$. Hence $J \subseteq (a, b)Ia^{-1} \subseteq R$, and therefore, $((a, b)Ia^{-1})_w = R$.

Suppose that this lemma is true for the integer $n - 1$. Let $a^{n-1}b \in (a^n, b^n)_w$. Then $c_i a^{n-1} b = x_i a^n + y_i b^n$, where $J = (c_1, c_2, \ldots, c_m) \in GV(R)$ and $x_i, y_i \in R$ $(1 \leqslant i \leqslant m)$. We also have $\frac{y_i b}{a}$ is integral over R, and therefore is in R. Write $z_i = \frac{y_i b}{a}$. Then $c_i a^{n-1} b = x_i a^n + a z_i b^{n-1}$. Thus $c_i a^{n-2} b = x_i a^{n-1} + z_i b^{n-1} \in (a^{n-1}, b^{n-1})$. Therefore $Ja^{n-2}b \subseteq (a^{n-1}, b^{n-1})$. It follows that $a^{n-2}b \subseteq (a^{n-1}, b^{n-1})_w$. The induction hypothesis then shows that (a, b) is w-invertible. $\square$

Recall that a valuation is a map ν from a ring T onto a totally ordered Abelian group $(G, +)$ and a symbol ∞ with $g < \infty$, $g + \infty = \infty + \infty = \infty$ for all $g \in G$, such that for all $x, y \in T$:

(1) $\nu(xy) = \nu(x) + \nu(y)$.

(2) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

(3) $\nu(1) = 0$ and $\nu(0) = \infty$.

Lemma 2.3 [7, Proposition 1] *Let R be a subring of the ring T, and let $\mathfrak{p}$ be a prime ideal of R. Then the following are equivalent:*

(1) If B is a subring of T containing R and if Q is an ideal of B such that $Q \bigcap R = \mathfrak{p}$, then $B = R$.

(2) If $x \in T \backslash R$, then there exists $x' \in \mathfrak{p}$ with $xx' \in R \backslash \mathfrak{p}$.

(3) There is a valuation on T such that $R = \{x \in T \mid \nu(x) \geq 0\}$ and $\mathfrak{p} = \{x \in T \mid \nu(x) > 0\}$.

If the above conditions hold, then $(R, \mathfrak{p})$ is called a valuation pair of T and R is said to be a valuation ring of T. When $T = T(R)$, we simply say that $(R, \mathfrak{p})$ is a valuation pair.

Theorem 2.1 *The following are equivalent for a ring R:*

(1) R is a PVMR.

(2) Every regular ideal generated by two elements is w-invertible.

(3) Every w-finite regular ideal is w-invertible.

(4) $(R_{[\mathfrak{m}]}, [\mathfrak{m}]R_{[\mathfrak{m}]})$ is a valuation pair for every maximal w-ideal $\mathfrak{m}$.

(5) If A, B and C are ideals of R with A finitely generated and regular, then $AB = AC$ implies that $B_w = C_w$.

(6) R is integrally closed and there exists a positive integer $n > 1$ such that $((a, b)^n)_w = (a^n, b^n)_w$ for all $a, b \in R$ with a regular.

(7) R is integrally closed and there exists a positive integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)_w$ for all $a, b \in R$ with a regular.

Proof (1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (4). We appeal to Lemma 2.3. For $\mathfrak{m} \in w\text{-}\max(R)$, let $x \in T(R)\backslash R_{[\mathfrak{m}]}$. Then there exists a regular element $b \in R$ such that $bx \in R$. Thus $((b, bx)I)_w = R$ for some $I \in \mathfrak{F}(R)$. For any $a \in I$, $ab \in \mathfrak{m}$. Otherwise, $abx \in R$ implies $x \in R_{[\mathfrak{m}]}$, a contradiction. Since $((b, bx)I)_w \not\subseteq \mathfrak{m}$, there exists $a \in I$ such that $abx \in R\backslash\mathfrak{m}$, as required.

(4) $\Rightarrow$ (1). Let I be a finitely generated regular ideal of R. Assume that $I \subseteq \mathfrak{m}$, where $\mathfrak{m} \in w\text{-}\max(R)$. Since the prime at infinity of $(R_{[\mathfrak{m}]}, [\mathfrak{m}]R_{[\mathfrak{m}]})$ is not regular, it does not contain I. Hence there exists an element $t \in T(R)$ such that tI is in $R_{[\mathfrak{m}]}$ but is not contained in $[\mathfrak{m}]R_{[\mathfrak{m}]}$. Thus there exists an element $s \in R\backslash\mathfrak{m}$ such that $stI \subseteq R\backslash\mathfrak{m}$, and therefore, $II^{-1} \not\subseteq \mathfrak{m}$.

(1) $\Leftrightarrow$ (3) is clear.

(1) $\Rightarrow$ (5). Since $(AA^{-1}B)_w = (AA^{-1}C)_w$, $B_w = C_w$.

(5) $\Rightarrow$ (6). By Lemma 2.1, R is integrally closed. Since $(a, b)^3 = (a^3, a^2b, ab^2, b^3) = (a, b)(a^2, b^2)$, $((a, b)^2)_w = (a^2, b^2)_w$ for all $a, b \in R$ with a regular.

(6) $\Rightarrow$ (7). Trivial.

(7) $\Rightarrow$ (2). By Lemma 2.2. $\square$

A ring R is said to be a Marot ring if each regular ideal of R is generated by its set of regular elements.

Remark 2.1 Let R be a Marot ring. Assume that (5), (6) and (7) are respectively replaced by the following

(5$'$) If A, B and C are finitely generated regular ideals of R, then $AB = AC$ implies that $B_w = C_w$.

(6$'$) R is integrally closed and there exists a positive integer $n > 1$ such that $((a, b)^n)_w = (a^n, b^n)_w$ for all $a, b \in R$ with a, b regular.

(7$'$) R is integrally closed and there exists a positive integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)_w$ for all $a, b \in R$ with a, b regular.

Then (1), (2), (3), (4), (5$'$), (6$'$) and (7$'$) are equivalent. We only need to show that (7$'$) $\Rightarrow$ (2). In fact, let I be a regular ideal generated by two elements. Then I admits a finite system of regular elements as generators by [8, Theorem 7.1]. Note that [6, Proposition 22.2] holds when replacing the phase "invertible" with "w-invertible". Therefore, I is w-invertible.

Lemma 2.4 *Let R be a Marot ring, and $a, b \in R$ with a and b regular. If n is an integer greater than 1 such that $(a^n, b^n)_w = \bigcap_{\lambda\in\Gamma} I_\lambda$, where $\{I_\lambda \mid \lambda \in \Gamma\}$ is a set of ideals of R such that $I_\lambda = I_\lambda V_\lambda \bigcap R$ for some valuation overring V_λ of R, then*

$$(a^n, b^n)_w = (a^n, a^{n-1}b, \ldots, ab^{n-1}, b^n)_w = ((a, b)^n)_w.$$

Proof If i, j are positive integers with their sum n, then

$$(a^i b^j)^n = (a^n)^i (b^n)^j \in (a^n, b^n)^n \subseteq ((a^n, b^n)_w)^n \subseteq (I_\lambda)^n$$

for each $\lambda \in \Gamma$. Thus $a^i b^j \in I_\lambda$. If not, then $a^i b^j \notin I_\lambda V_\lambda$. Set $c_{ij} = a^i b^j$. Moreover, if we show that $c_{ij}^n \notin (I_\lambda V_\lambda)^n$, then $c_{ij}^n \notin (I_\lambda)^n$, which is a contradiction. Therefore, without loss of generality we may assume that $R = V_\lambda$. Note that I_λ is generated by its set of regular elements. For each regular generator $x \in I_\lambda$, $\frac{c_{ij}}{x} \notin R$, and so $\frac{x}{c_{ij}} \in R$. Thus $I_\lambda \subset (c_{ij})$. Then $(I_\lambda)^n \subseteq I_\lambda (c_{ij})^{n-1} \subset (c_{ij})^n$. Therefore, $c_{ij}^n \notin (I_\lambda)^n$. It follows that $a^i b^j \in (a^n, b^n)_w = \bigcap_{\lambda\in\Gamma} I_\lambda$. Therefore, $(a^n, b^n) \subseteq (a^n, a^{n-1}b, \ldots, ab^{n-1}, b^n) = (a, b)^n \subseteq (a^n, b^n)_w$, as required. □

For $A \in \mathfrak{F}(R)$, define the completion of A by $A' = \bigcap AV_\lambda$, where $\{V_\lambda\}$ is the set of valuation overrings of R. A is said to be complete if $A = A'$. Gilmer [6] showed that a ring R is a Prüfer domain if and only if each ideal of R is complete. The completion of a flat ideal over an integrally closed domain was studied by Sally and Vasconcelos [9] and they showed that a flat ideal over a GCD domain, a Krull domain or an integrally closed coherent domain is complete. The completeness of flat ideals was also discussed by Glaz and Vasconcelos [10]. In [11], Wang discussed the completion of w-ideals and indicated that every w-ideal of a PVMD is complete, which generalized the work of Sally and Vasconcelos since flat ideals are w-ideals and GCD domains, Krull domains and integrally closed coherent domains are PVMDs. For the definition of completeness of an ideal over an integral domain, the reader may consult Gilmer [6].

For rings with zero divisors, it was shown that a Marot ring R is a Prüfer ring if and only if every ideal of R is complete [8, Theorem 21.3]. In the final theorem, we take a step forward in this direction and establish that a Marot ring R is a PVMR if and only if every w-ideal of R is complete. Here we have

Theorem 2.2 *Let R be a Marot ring. Then the following are equivalent:*

(1) R is a PVMR.

(2) For any nonzero ideal I of R, $I' \subseteq I_w$.

(3) Every w-ideal of R is complete.

(4) Every finite type w-ideal of R is complete.

(5) For every finite type w-ideal A of R, $A = \bigcap_{\lambda\in\Gamma} B_\lambda$, where $\{B_\lambda \mid \lambda \in \Gamma\}$ is a set of ideals of R such that $B_\lambda = B_\lambda V_\lambda \bigcap R$ for some valuation overring V_λ of R.

Proof (1) ⇒ (2). By Proposition 3, we have $I_w = \bigcap_{\mathfrak{m}\in w\text{-}\max(R)} I_w R_{[\mathfrak{m}]}$. By Theorem 2.1, $R_{[\mathfrak{m}]}$ is a valuation ring for every $\mathfrak{m} \in w\text{-}\max(R)$. Thus $I' \subseteq I_w$.

(2) ⇒ (3) ⇒ (4) ⇒ (5) are clear.

(5) ⇒ (1). We first show that R is an integrally closed ring. If not, then there exists $\frac{x}{y} \in R^c$, but $\frac{x}{y} \notin R$, where $x, y \in R$ with y regular. By [8, Theorems 7.7 and 9.1], we have

$$(y) = \bigcap_{\lambda\in\Gamma} (yV_\lambda \bigcap R) = y(\bigcap_{\lambda\in\Gamma} V_\lambda) \bigcap R = yR^c \bigcap R.$$

Then $x = y(\frac{x}{y}) \in (y)$, a contradiction.

By Remark 2.1, to complete the proof, we need to show that for any regular elements a, b of R, there exists an integer n for which $((a,b)^n)_w = (a^n, b^n)_w$. Since $(a^n, b^n)_w$ is a finite type w-ideal for any positive integer $n > 1$, $(a^n, b^n)_w = \bigcap_{\lambda\in\Gamma} I_\lambda$, where $\{I_\lambda \mid \lambda \in \Gamma\}$ is a set of ideals of R such that $I_\lambda = I_\lambda V_\lambda \bigcap R$ for some valuation overring V_λ of R. By Lemma 2.4, the result follows. □

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