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Coleman Automorphisms of Finite Groups with a Unique Nontrivial Normal Subgroup

Zhengxing LI*, **Jinke HAI**, **Shuxian YANG**
College of Mathematics, Qingdao University, Shandong 266071, *P. R. China*

Abstract Let G be a finite group with a unique nontrivial normal subgroup. It is shown that every Coleman automorphism of G is an inner automorphism.

Keywords Coleman automorphism; minimal normal subgroup.

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1. Introduction

All groups considered in this paper are finite. Let G be a finite group and let σ be an automorphism of G. Recall that σ is said to be a Coleman automorphism if the restriction of σ to any Sylow subgroup of G equals the restriction of some inner automorphism of G. This concept was introduced by Hertweck and Kimmerle in [1]. Denote by $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ the automorphism group and the inner automorphism group of G respectively. Denote by $\mathrm{Aut}_{\mathrm{Col}}(G)$ the group formed by all Coleman automorphisms of G. It is clear that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}_{\mathrm{Col}}(G)$. Set $\mathrm{Out}_{\mathrm{Col}}(G) := \mathrm{Aut}_{\mathrm{Col}}(G)/\mathrm{Inn}(G)$. Recently, lots of results on Coleman automorphisms have appeared in the literature, see [2,3] for instance.

The aim of the present paper is to investigate Coleman automorphisms of finite groups having a unique nontrivial normal subgroup. The description of the structure of such groups had been obtained by Qinhai Zhang and Jianji Cao in [4]. By making use of the structure theorems therein, we can prove the following main result (Theorem 3.1).

Theorem A *Let G be a finite group having a unique nontrivial normal subgroup. Then every Coleman automorphism of G is inner, i.e.,* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

More generally, we have the following result (Theorem 3.2).

Theorem B *Let G be a finite group whose nontrivial normal subgroups have the same order. Then every Coleman automorphism of G is inner, i.e.,* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

2. Notation and preliminaries

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* Corresponding author
E-mail address: lzxlws@163.com (Zhengxing LI); haijinke2002@yahoo.com.cn (Jinke HAI); yangshuxian@qdu.edu.cn (Shuxian YANG)

In this section, we first fix some nation and then record some lemmas that will be used in the sequel. Let G be a finite group, $\sigma \in \mathrm{Aut}(G)$ and $H \leq G$. We write $\sigma|_H$ for the restriction of σ to H. If $H \trianglelefteq G$ and σ fixes H, then we write $\sigma|_{G/H}$ for the automorphism of the quotient G/H induced by σ in the natural way. Denote by $\mathrm{C}_G(H)$ and $\mathrm{N}_G(H)$ the centralizer and the normalizer of H in G, respectively. Denote by $\pi(G)$ the set of all primes dividing the order of G. Let $p \in \pi(G)$. Denote by $\mathrm{O}_p(G)$ and $\mathrm{O}_{p'}(G)$ the largest normal p-subgroup and p'-subgroup of G, respectively. C_p always denotes a cyclic group of order p. Denote by $\mathrm{F}(G)$ and $\mathrm{F}^*(G)$ the Fitting subgroup and the generalized Fitting subgroup of G, respectively. Denote by $\mathrm{Z}(G)$ the center of G. For a fixed $x \in G$, we write $\mathrm{conj}(x)$ for the inner automorphism of G induced by x via conjugacy, i.e., $g^{\mathrm{conj}(x)} = g^x$ for any $g \in G$. We refer to [5] for other notations, which are mostly standard.

Lemma 2.1 ([4, Theorem 1.2]) *Let G be a finite solvable group. If G has a unique nontrivial normal subgroup, then one of the following holds.*

(I) G is a cyclic p-group of order p^2;

(II) G is a semidirect product $G = P \rtimes Q$, where P is an elementary abelian p-group and Q is a cyclic group of order q, with p and q being distinct primes. Moreover, the action of Q on P is irreducible.

Lemma 2.2 ([4, Theorem 1.3]) *Let G be a finite non-solvable group. If G has a unique nontrivial normal subgroup K, then*

(I) K is solvable.

(i) If $K \leq \mathrm{Z}(G)$, then G is a covering group of a finite simple group and $\mathrm{Z}(G) \cong C_p$.

(ii) If $K \nleq \mathrm{Z}(G)$, then $G/K \cong D$ is a finite non-abelian simple group. D acts irreducibly on K.

(II) K is non-solvable.

(i) Assume that T is a non-abelian simple group. Then $K \cong T$ if and only if G is almost simple group and $G/K \cong C_p$.

(ii) If $K \cong T^n$, where T is a non-abelian simple group and $n > 1$, then $G/K \cong D$, where D is a simple group.

Lemma 2.3 ([4, Theorem 1.1]) *Let G be a finite group. Then all nontrivial normal subgroups of G have the same order if and only if G is one of the following:*

(1) G is a simple group;

(2) G has a unique nontrivial normal subgroup;

(3) $G \cong T \times T$, where T is a finite simple group;

(4) $G \cong A_8 \times L_3(4)$;

(5) $G \cong B_n(q) \times C_n(q)$, where $n \geq 3$ and q is odd.

Lemma 2.4 ([1, Proposition 1]) *Let G be a finite group. Then $\pi(\mathrm{Aut}_{\mathrm{Col}}(G)) \subseteq \pi(G)$.*

Lemma 2.5 ([1, Corollary 3]) *Let $N \trianglelefteq G$ and let p be a prime which does not divide the order* *of G/N. Then the following hold.*

(i) If $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, then $\sigma|_N \in \mathrm{Aut}_{\mathrm{Col}}(N)$;

(ii) If $\mathrm{Out}_{\mathrm{Col}}(N)$ is a p'-group, then so is $\mathrm{Out}_{\mathrm{Col}}(G)$.

Lemma 2.6 ([1, Corollary 16]) *Let G be a quasinilpotent group. Then $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.*

Recall that a finite group G is said to be p-constrained group if $\mathrm{C}_{\bar{G}}(\mathrm{O}_p(\bar{G})) \leq \mathrm{O}_p(\bar{G})$, where $\bar{G} := G/\mathrm{O}_{p'}(G)$. Recall that an automorphism σ of G is said to be a p-central automorphism of G if the restriction of σ to a Sylow p-subgroup of G is trivial.

Lemma 2.7 ([6, Corollary 2.4]) *Let G be a p-constrained group with $\mathrm{O}_{p'}(G) = 1$ for some prime p. Then p-central automorphisms of G are inner automorphisms, given by conjugation with element from $\mathrm{Z}(\mathrm{O}_p(G))$.*

Lemma 2.8 ([1, Theorem 14]) *Let G be a simple group. Then there is a prime $p \in \pi(G)$ such that p-central automorphisms of G are inner automorphisms.*

Lemma 2.9 ([7, Lemma 2]) *Let p be a prime, and φ be a p-power order automorphism of a finite group G. Suppose that there is a normal subgroup N of G such that φ fixes all elements of N, and that φ induces the identity on the quotient group G/N. Then φ induces the identity on $G/\mathrm{O}_p(\mathrm{Z}(N))$. Further, if φ fixes element-wise a Sylow p-subgroup of G , then φ is an inner automorphism of G.*

Lemma 2.10 ([1, Lemma 19]) *Assume that no chief factor of $G/\mathrm{F}^*(G)$ is isomorphic to C_p, and let $\sigma \in \mathrm{Aut}(G)$ be of p-power order. If σ induces the identity on G/N for some $N \trianglelefteq G$ with $N^\sigma = N$ and Q is a Sylow subgroup of N with $\sigma|_Q = \mathrm{conj}(x)|_Q$ for some $x \in G$, then there is $g \in \mathrm{O}_p(G)N$ such that $\sigma|_Q = \mathrm{conj}(g)|_Q$.*

3. Proofs of Theorems A and B

In this section, we present proofs for Theorems A and B. For convenience, we record Theorem A here as

Theorem 3.1 *Let G be a finite group having a unique nontrivial normal subgroup. Then every Coleman automorphism of G is an inner automorphism, i.e., $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.*

Proof The proof of Theorem 3.1 splits into two cases according to the solvability of G.

Case 1 G is solvable.

According to Lemma 2.1, the proof of this case splits into two subcases.

Subcase 1.1 G is a cyclic p-group of order p^2.

Since G is a p-group, the assertion holds trivially.

Subcase 1.2 G is a semidirect product of an elementary abelian p-group P and a cyclic group Q of order q with p, q distinct primes.

Since p does not divide $|Q|$, it follows from Lemmas 2.4 and 2.5 that $\mathrm{Out}_{\mathrm{Col}}(G)$ is a q-group. Thus to confirm the assertion in this case, it suffices to show that $\sigma \in \mathrm{Inn}(G)$ for any Coleman automorphism σ of q-power order. Since $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, it follows from the definition of Coleman automorphism that there exists some $g \in G$ such that $\sigma|_P = \mathrm{conj}(g)|_P$. Without loss of generality, we may assume that

$$\sigma|_P = \mathrm{id}|_P. \tag{3.1}$$

Note that σ acts naturally on the set $\mathrm{Syl}_q(G)$ of all Sylow q-subgroup of G. Since the order of σ and the cardinality of $\mathrm{Syl}_q(G)$ are coprime, it follows that σ fixes a Sylow q-subgroup of G. Without loss of generality, we may assume that σ fixes Q, i.e.,

$$Q^\sigma = Q. \tag{3.2}$$

Since $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, there exists some $x \in G$ such that

$$\sigma|_Q = \mathrm{conj}(x)|_Q. \tag{3.3}$$

Without loss of generality, we may assume that x is a q-element. Then, by (3.2) and (3.3), one obtains $x \in \mathrm{N}_G(Q)$ and hence $x \in Q$. But note that Q is cyclic, so we must have

$$\sigma|_Q = \mathrm{conj}(x)|_Q = \mathrm{id}|_Q. \tag{3.4}$$

Since $G = P \rtimes Q$, it follows from (3.1) and (3.4) that $\sigma = \mathrm{id}$.

Case 2 G is non-solvable.

Let K be the unique nontrivial normal subgroup of G. According to Lemma 2.2, the proof of this case splits into two subcases:

Subcase 2.1 K is solvable.

Firstly, we consider the case in which $K \leq \mathrm{Z}(G)$. By Lemma 2.2, $K = \mathrm{Z}(G) \cong C_p$ and G is a covering group of a simple group, i.e., $G/\mathrm{Z}(G)$ is isomorphic to a non-abelian simple group. Note that $\mathrm{F}^*(G) \geq K = \mathrm{Z}(G)$, so we must have $\mathrm{F}^*(G) = G$. In effect, assume that $\mathrm{F}^*(G) < G$, then $\mathrm{F}^*(G) = K = \mathrm{Z}(G)$, which implies that $\mathrm{C}_G(\mathrm{F}^*(G)) = \mathrm{C}_G(\mathrm{Z}(G)) = G > \mathrm{F}^*(G)$. However, this cannot occur since $\mathrm{F}^*(G)$ is self-centralized for any finite group G. Hence we must have $\mathrm{F}^*(G) = G$. This shows that G is a quasinilpotent group and thus by Lemma 2.6 the assertion holds.

Secondly, we consider the case in which $K \not\leq \mathrm{Z}(G)$. On the one hand, it is clear that $\mathrm{C}_G(K)$ is proper normal subgroup of G. On the other hand, since K is solvable, it follows that K is ableian and hence $\mathrm{C}_G(K)$ is a nontrivial proper normal subgroup of G containing K. Consequently, we must have $\mathrm{C}_G(K) = K$ since by hypothesis K is the unique nontrivial normal subgroup. In addition, note that $K = \mathrm{O}_p(G)$ for some prime $p \in \pi(G)$, so $\mathrm{O}_{p'}(G) = 1$. This shows that G is a p-constrained group and hence by Lemma 2.7 every p-central automorphism of G is inner. In particular, $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Subcase 2.2 K is non-solvable.

Firstly, we consider the case in which K is a non-abelian simple group and $G/K \cong C_p$. Let $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$. We have to show that $\sigma \in \mathrm{Inn}(G)$. By Lemmas 2.4 and 2.5, we may assume that σ is of p-power order. By Lemma 2.8, there exists some $q \in \pi(K)$ such that q-central automorphisms of K are inner. Thus, modifying σ by some inner automorphism, we may assume that $\sigma|_K \in \mathrm{Inn}(K)$. So there exists some $k \in K$ such that

$$\sigma|_K = \mathrm{conj}(k)|_K. \tag{3.5}$$

Without loss of generality, we may assume that

$$\sigma|_K = \mathrm{id}|_K. \tag{3.6}$$

Note that G/K is a cyclic group of order p, so we have

$$\sigma|_{G/K} = \mathrm{id}|_{G/K}. \tag{3.7}$$

Consequently, by Lemma 2.9, $\sigma|_{G/\mathrm{O}_p(\mathrm{Z}(K))} = \mathrm{id}|_{G/\mathrm{O}_p(\mathrm{Z}(K))}$, which implies that $\sigma = \mathrm{id}$ since $\mathrm{O}_p(\mathrm{Z}(K)) = 1$.

Secondly, we consider the case in which K is a direct product of n copies of a non-abelian simple group and $G/K \cong C_p$, where $n > 1$. The proof of this case is similar to that of the previous case in which K is a non-abelian simple group and $G/K \cong C_p$, so we omit it.

Finally, we consider the case in which K is a direct product of n copies of a non-abelian simple group and G/K is isomorphic to a non-abelian simple group, where $n > 1$. Let $p \in \pi(G)$ and let $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$ be an arbitrary Coleman automorphism of p-power order. Note that K is the unique minimal normal subgroup of G, so $\mathrm{F}(G) = 1$ and thus $\mathrm{F}^*(G) = K$. Since by hypothesis $G/\mathrm{F}^*(G) = G/K$ is a non-abelian simple group, it follows that $G/\mathrm{F}^*(G)$ has no chief factor isomorphic to C_p. Note that $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$ implies that $\sigma|_{G/K} \in \mathrm{Aut}_{\mathrm{Col}}(G/K)$. But G/K is simple, so by Lemma 2.8 we have $\sigma|_{G/K} \in \mathrm{Inn}(G/K)$. Without loss of generality, we may assume that

$$\sigma|_{G/K} = \mathrm{id}|_{G/K}. \tag{3.8}$$

Let Q be a Sylow subgroup of K. Then $\sigma|_Q = \mathrm{conj}(x)|_Q$ for some $x \in G$ since $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$. Consequently, by Lemma 2.10, there exists some $g \in \mathrm{O}_p(G)K = K$ with $\sigma|_Q = \mathrm{conj}(g)|_Q$. As Q is arbitrary Sylow subgroup of K, we have $\sigma|_K \in \mathrm{Aut}_{\mathrm{Col}}(K)$. Remember that K is a direct product of n copies of a non-abelian simple group, so we have $\mathrm{Aut}_{\mathrm{Col}}(K) = \mathrm{Inn}(K)$. It follows that there exists some $k \in K$ such that $\sigma|_K = \mathrm{conj}(k)|_K$. Replacing σ with $\sigma\mathrm{conj}(k^{-1})$, we may assume that

$$\sigma|_K = \mathrm{id}|_K. \tag{3.9}$$

Again, by Lemma 2.9, $\sigma|_{\mathrm{O}_p(\mathrm{Z}(K))} = \mathrm{id}|_{\mathrm{O}_p(\mathrm{Z}(K))}$, which implies that $\sigma = \mathrm{id}$. This completes the proof of Theorem 3.1. □

More generally, we have the following result (Theorem B).

Theorem 3.2 *Let G be a finite group whose nontrivial normal subgroups have the same order. Then every Coleman automorphism of G is inner, i.e.,* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Proof Since by hypothesis G is a finite group whose nontrivial normal subgroups have the same order, it follows from Lemma 2.3 that G is a simple group, a direct product of two simple groups or G has a unique nontrivial normal subgroup. If it is the first or second case, then by Lemma 2.8 $\mathrm{Out}_{\mathrm{Col}}(G) = 1$. If it is the third case, then the assertion follows immediately from Theorem 3.1. Therefore, in any case, we have $\mathrm{Out}_{\mathrm{Col}}(G) = 1$. We are done. □

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