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Majorization Properties for Certain New Classes of Multivalent Meromorphic Functions Defined by Sălăgean Operator

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Abstract In the present paper, we investigate the majorization property for certain new class of multivalent meromorphic analytic functions defined by Sălăgean operator. Moreover, we point out some new and interesting applications of our main result to the other classes of multivalent meromorphic functions.

Keywords analytic functions; multivalent functions; meromorphic functions; subordination; majorization property.

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1. Introduction

Let $\Sigma_{p,n}$ denote the class of functions of the form

$$f(z)=z^{-p}+\sum_{k=n}^{\infty}a_kz^k,\quad n\ge p,p\in\mathbb{N}=\{1,2,\ldots\},\tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $\Delta^*=\{z\in\mathbb{C}:0<|z|<1\}=\Delta\backslash\{0\}$, where Δ is the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$. Also, let $\Sigma_{1,1}=\Sigma$.

For functions $f_j\in\Sigma_{p,n}$ given by

$$f_j(z)=z^{-p}+\sum_{k=n}^{\infty}a_{k,j}z^k,\quad j=1,2,\tag{1.2}$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1*f_2)(z):=z^{-p}+\sum_{k=n}^{\infty}a_{k,1}a_{k,2}z^k=(f_2*f_1)(z),\quad z\in\Delta^*.$$

Let $f(z)$ and $g(z)$ be analytic in Δ. We say that the function $f(z)$ is subordinate to $g(z)$ if there exists a Schwarz function $\omega(z)$, analytic in Δ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\Delta)$, such that $f(z)=g(\omega(z)),z\in\Delta$ (see [1]). We denote this subordination by

$$f(z)\prec g(z),\quad z\in\Delta.$$

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Indeed it is known that (see for details [2, 3]; see also [4]):

$$f(z) \prec g(z) \ (z \in \Delta) \Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

Let $f(z)$ and $g(z)$ be analytic in Δ. We say that $f(z)$ is majorized by $g(z)$ in Δ (see [5]) and write

$$f(z) \ll g(z), \quad z \in \Delta \tag{1.3}$$

if there exists a function $\varphi(z)$, analytic in Δ, such that

$$|\varphi(z)| \leq 1 \text{ and } f(z) = \varphi(z)g(z), \quad z \in \Delta. \tag{1.4}$$

It may be noted here that (1.3) is closely related to the concept of qusi-subordination between analytic functions.

Let $f^{(q)}$ denote the qth-order ordinary differential operator for a function $f \in \Sigma_{p,n}$, that is,

$$f^{(q)} = (-1)^q \frac{(p+q-1)!}{(p-1)!} z^{-(p+q)} + \sum_{k=n}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q}, \tag{1.5}$$

where $p > q$; $p \in \mathbb{N}$; $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $z \in \Delta^*$.

The operator defined by (1.5) has been studied earlier by several researchers [6, 7].

We introduce the generalized Sălăgean operator $D^m f^{(q)}(z)$ as follows

$$D^m f^{(q)}(z) = (-1)^{q+m} \frac{(p+q-1)!(p+q)^m}{(p-1)!} z^{-(p+q)} + \sum_{k=n}^{\infty} \frac{k!(k-q)^m}{(k-q)!} a_k z^{k-q}, \tag{1.6}$$

where $n \geq p$, $p > q$; $p \in \mathbb{N}$; $m, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $z \in \Delta^*$.

In view of (1.6), it is clear that $D^0 f^{(0)}(z) = f(z)$, $D^1 f^{(0)}(z) = zf'(z)$ and $D^m f^{(0)}(z) = D^m f(z)$ is a known Sălăgean operator [8].

Using the operator $D^m f^{(q)}(z)$, we now introduce the following subclass of $\Sigma_{p,n}$:

Definition 1.1 *A function $f(z) \in \Sigma_{p,n}$ is said to be in the class $\Sigma_{p,q,n}^{j,l}[A, B; \alpha, \gamma]$ of p-valent meromorphic functions of complex order $\gamma \neq 0$ in Δ^* if and only if*

$$1 + \frac{1}{\gamma}\left\{\frac{D^j f^{(q)}(z)}{D^l f^{(q)}(z)} - (-1)^{j-l}(p+q)^{j-l}\right\} - \alpha\left|\frac{1}{\gamma}\left\{\frac{D^j f^{(q)}(z)}{D^l f^{(q)}(z)} - (-1)^{j-l}(p+q)^{j-l}\right\}\right| \prec \frac{1+Az}{1+Bz} \tag{1.7}$$

$$z \in \Delta^*; \ A, B \in \mathbb{C}, A \neq B, |B| \leq 1; j > l, l, q \in \mathbb{N}_0; \alpha \geq 0, \gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}.$$

By specializing the parameter $p, q, n, j, l, A, B, \alpha, \gamma$, we obtain the following classes of multivalently meromorphic functions,

(1) $\Sigma_{p,q,n}^{j,l}[A, B; 0, \gamma] = \Sigma_{p,q,n}^{j,l}[A, B; \gamma]$

$$= \left\{f(z) \in \Sigma_{p,n} : 1 + \frac{1}{\gamma}\left\{\frac{D^j f^{(q)}(z)}{D^l f^{(q)}(z)} - (-1)^{j-l}(p+q)^{j-l}\right\} \prec \frac{1+Az}{1+Bz}\right\};$$

(2) $\Sigma_{p,0,n}^{1,0}[A, B; 0, 1] = S^*[p, A, B] = \left\{f(z) \in \Sigma_{p,n} : 1 + p + \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}\right\};$

(3) $\Sigma_{p,0,n}^{2,1}[A, B; 0, 1] = K[p, A, B] = \left\{f(z) \in \Sigma_{p,n} : 1 + p + \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}\right\};$

(4) $\Sigma_{p,0,n}^{1,0}[1-2\beta, -1; \alpha, \gamma] = \Sigma US_n(p, \alpha, \beta, \gamma)$

$= \Big\{f(z)\in\Sigma_{p,n}:\Re\Big\{1+\frac{1}{\gamma}(\frac{zf'(z)}{f(z)}+p)-\alpha|\frac{1}{\gamma}(\frac{zf'(z)}{f(z)}+p)|\Big\}>\beta\Big\}$

$(\alpha\geq 0, 0\leq\beta<1, \gamma\in\mathbb{C}^*)$;

(5) $\Sigma^{2,1}_{p,0,n}[1-2\beta,-1;\alpha,\gamma]=\Sigma UK_n(p,\alpha,\beta,\gamma)$

$= \Big\{f(z)\in\Sigma_{p,n}:\Re\Big\{1+\frac{1}{\gamma}(\frac{(zf'(z))'}{f'(z)}+p)-\alpha|\frac{1}{\gamma}(\frac{(zf'(z))'}{f'(z)}+p)|\Big\}>\beta\Big\}$

$(\alpha\geq 0, 0\leq\beta<1, \gamma\in\mathbb{C}^*)$.

Meromorphically multivalent functions have been extensively studied (for example) by many researchers such as Aouf [9, 10], Cho et al. [11], Liu and Srivastava [12], Mogra [13, 14], Srivastava et al. [15], Owa et al. [16], Raina and Srivastava [17], Kumar et al. [18] and El-Ashwah et al. [19].

Majorization problems for the normalized classes of starlike functions have been investigated by MacGregor [5] and Altinas et al. [20]. Recently, Goyal et al. [21] studied majorization properties for meromorphic classes. Further, Goyal and Goswami [22], Prajapat, Aouf [23], Goyal et al. [24], Goswami and Wang [25], Pranay Goswami and Aouf [26], Goswami et al. [27] and Li et al. [28] studied majorization properties for different classes. In the present paper, we investigate a majorization problem for the class $\Sigma^{j,l}_{p,q,n}[A,B;\alpha,\gamma]$. Further, we point out some new and interesting applications of our main result to the other classes of multivalent meromorphic functions.

In order to obtain our main theorem, we need the following lemma:

Lemma 1.2 ([29]) *Let $\varphi(z)$ be analytic in Δ satisfying $|\varphi(z)|<1$ for $z\in\Delta$. Then*

$$|\varphi'(z)|\leq\frac{1-|\varphi(z)|^2}{1-|z|^2},\quad z\in\Delta. \tag{1.8}$$

2. Majorization problem for the class $\Sigma^{j,l}_{p,q,n}[A,B;\alpha,\gamma]$

We begin by proving the following result.

Theorem 2.1 *Let the function $f\in\Sigma_{p,n}$ and suppose that $g\in\Sigma^{j,l}_{p,q,n}[A,B;\alpha,\gamma]$. If $D^jf^{(q)}(z)$ is majorized by $D^lg^{(q)}(z)$ in Δ^* and*

$$\Big[\frac{|A-B||\gamma|}{|1-\alpha|}+(p+q)^{j-l}|B|\Big]\delta\leq(p+q)^{j-l},$$

then

$$|D^{j+1}f^{(q)}(z)|\leq|D^{l+1}g^{(q)}(z)|,\quad |z|\leq r_0, \tag{2.1}$$

where $r_0=r_0(p,q,\alpha,\gamma,j,l,A,B)$ is the smallest positive root of the equation

$$\Big[\frac{|A-B||\gamma|}{|1-\alpha|}+(p+q)^{j-l}|B|\Big]r^3-\big[(p+q)^{j-l}+2|B|\big]r^2- \\ \Big[\frac{|A-B||\gamma|}{|1-\alpha|}+(p+q)^{j-l}|B|+2\Big]r+(p+q)^{j-l}=0 \tag{2.2}$$

$z\in\Delta^*; A,B\in\mathbb{C}, A\neq B, |B|\leq 1; j>l; p,j\in\mathbb{N}; l,q\in\mathbb{N}_0; 0\leq\alpha\neq 1, \gamma\in\mathbb{C}^*; 0\leq\delta\leq r_0.$

Proof Suppose that $g \in \Sigma_{p,q,n}^{j,l}[A,B;\alpha,\gamma]$. Then, making use of the fact that

$$\varpi - \alpha|\varpi - 1| \prec \frac{1+Az}{1+Bz} \Leftrightarrow \varpi(1-\alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1+Az}{1+Bz}, \quad \phi \in \mathbb{R}$$

and letting

$$\varpi = 1 + \frac{1}{\gamma}\Big\{\frac{D^j g^{(q)}(z)}{D^l g^{(q)}(z)} - (-1)^{j-l}(p+q)^{j-l}\Big\}$$

in (1.7), we obtain

$$\Big\{1 + \frac{1}{\gamma}\Big[\frac{D^j g^{(q)}(z)}{D^l g^{(q)}(z)} - (-1)^{j-l}(p+q)^{j-l}\Big]\Big\}(1-\alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1+Az}{1+Bz},$$

or, equivalently,

$$1 + \frac{1}{\gamma}\Big\{\frac{D^j g^{(q)}(z)}{D^l g^{(q)}(z)} - (-1)^{j-l}(p+q)^{j-l}\Big\} \prec \frac{1 + [\frac{A-\alpha B e^{-i\phi}}{1-\alpha e^{-i\phi}}]z}{1+Bz} \tag{2.3}$$

which holds true for all $z \in \Delta^*$.

We find from (2.3) that

$$1 + \frac{1}{\gamma}\Big\{\frac{D^j g^{(q)}(z)}{D^l g^{(q)}(z)} - (-1)^{j-l}(p+q)^{j-l}\Big\} = \frac{1 + [\frac{A-\alpha B e^{-i\phi}}{1-\alpha e^{-i\phi}}]\omega(z)}{1+B\omega(z)}, \tag{2.4}$$

where $\omega(z) = c_1 z + c_2 z^2 + \cdots$, $\omega \in \mathcal{P}$, $\mathcal{P}$ denotes the well-known class of the bounded analytic functions in Δ and satisfies the conditions

$$\omega(0) = 0 \text{ and } |\omega(z)| \le |z|, \quad z \in \Delta.$$

From (2.4), we get

$$\frac{D^j g^{(q)}(z)}{D^l g^{(q)}(z)} = \frac{(-1)^{j-l}(p+q)^{j-l} + \Big[\frac{(A-B)\gamma}{1-\alpha e^{-i\phi}} + (-1)^{j-l}(p+q)^{j-l}B\Big]\omega(z)}{1+B\omega(z)}. \tag{2.5}$$

By virtue of (2.5), we get

$$\begin{aligned} |D^l g^{(q)}(z)| \le& \frac{1+|B||z|}{(p+q)^{j-l} - |\frac{(A-B)\gamma}{1-\alpha e^{-i\phi}} + (-1)^{j-l}(p+q)^{j-l}B||z|}|D^j g^{(q)}(z)| \\ \le& \frac{1+|B||z|}{(p+q)^{j-l} - [\frac{|A-B||\gamma|}{|1-\alpha|} + (p+q)^{j-l}|B|]|z|}|D^j g^{(q)}(z)|. \end{aligned} \tag{2.6}$$

Next, since $D^j f^{(q)}(z)$ is majorized by $D^l g^{(q)}(z)$ in the punctured unit disk Δ^*, from (1.4), we have

$$D^j f^{(q)}(z) = \varphi(z) D^l g^{(q)}(z).$$

Differentiating the above equality with respect to z and multiplying by z, we get

$$D^{j+1} f^{(q)}(z) = z\varphi'(z) D^l g^{(q)}(z) + \varphi(z) D^{l+1} g^{(q)}(z). \tag{2.7}$$

Thus, by Lemma 1.2, the Schwarz function $\varphi(z)$ satisfies the inequality in (1.8) and using (2.6) in (2.7), we get

$$|D^{j+1} f^{(q)}(z)| \le \Big[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \cdot \frac{(1+|B||z|)|z|}{[(p+q)^{j-l} - (\frac{|A-B||\gamma|}{|1-\alpha|} + (p+q)^{j-l}|B|)|z|]}\Big].$$

$$|D^{l+1}g^{(q)}(z)|, \tag{2.8}$$

which upon setting

$$D^j f^{(q)}(z) = \varphi(z)D^l g^{(q)}(z),\ |z| = r \ \text{ and } \ |\varphi(z)| = \rho,\quad 0 \le \rho \le 1$$

leads us to the inequality

$$|D^{j+1}f^{(q)}(z)| \le \Big[\frac{\Psi(\rho, r)}{(1-r^2)[(p+q)^{j-l} - (\frac{|A-B||\gamma|}{|1-\alpha|} + (p+q)^{j-l}|B|)r]}\Big]|D^{l+1}g^{(q)}(z)|,$$

where

$$\begin{aligned}\Psi(\rho, r) =& - r(1+|B|r)\rho^2 + (1-r^2)\Big[(p+q)^{j-l} - \Big(\frac{|A-B||\gamma|}{|1-\alpha|} + (p+q)^{j-l}|B|\Big)r\Big]\rho + \\ & r(1+|B|r)\end{aligned} \tag{2.9}$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(p, q, \alpha, \gamma, j, l, A, B)$, where r_0 is the smallest positive root of Eq. (2.2). Furthermore, if $0 \le \delta \le r_0(p, q, \alpha, \gamma, j, l, A, B)$, then the function $\Psi(\rho, \delta)$ defined by

$$\begin{aligned}\Psi(\rho, \delta) =& - \delta(1+|B|\delta)\rho^2 + (1-\delta^2)\Big[(p+q)^{j-l} - \Big(\frac{|A-B||\gamma|}{|1-\alpha|} + (p+q)^{j-l}|B|\Big)\delta\Big]\rho + \\ & (1+|B|\delta)\delta\end{aligned} \tag{2.10}$$

is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\Psi(\rho, \delta) \le \Psi(1, \delta) = (1-\delta^2)\Big[(p+q)^{j-l} - \Big(\frac{|A-B||\gamma|}{|1-\alpha|} + (p+q)^{j-l}|B|\Big)\delta\Big], \tag{2.11}$$

$$0 \le \rho \le 1; 0 \le \delta \le r_0(p, q, \alpha, \gamma, j, l, A, B).$$

Hence, upon setting $\rho = 1$, in (2.10), we conclude that (2.1) of Theorem 2.1 holds true for $|z| \le r_0 = r_0(p, q, \alpha, \gamma, j, l, A, B)$, which completes the proof of Theorem 2.1. □

Setting $\alpha = 0$ in Theorem 2.1, we get the following result.

Corollary 2.2 *Let the function $f \in \Sigma_{p,n}$ and suppose that $g \in \Sigma_{p,q,n}^{j,l}[A, B; \gamma]$. If $D^j f^{(q)}(z)$ is majorized by $D^l g^{(q)}(z)$ in Δ^* and $(p+q)^{j-l} \ge [|A-B||\gamma| + (p+q)^{j-l}|B|]\delta$, then*

$$|D^{j+1}f^{(q)}(z)| \le |D^{l+1}g^{(q)}(z)|,\quad |z| \le r_0,$$

where $r_0 = r_0(p, q, \gamma, j, l, A, B)$ is the smallest positive root of the equation

$$\begin{aligned}&[|A-B||\gamma| + (p+q)^{j-l}|B|]r^3 - [(p+q)^{j-l} + 2|B|]r^2 - \\ &[|A-B||\gamma| + (p+q)^{j-l}|B| + 2]r + (p+q)^{j-l} = 0\end{aligned}$$

$$z \in \Delta^*; A, B \in \mathbb{C}, A \ne B, |B| \le 1; j > l; p, j \in \mathbb{N}; l, q \in \mathbb{N}_0; \gamma \in \mathbb{C}^*; 0 \le \delta \le r_0.$$

Putting $q = 0$, $j = 1$, $l = 0$ and $\gamma = 1$ in Corollary 2.2, we obtain the following result.

Corollary 2.3 *Let the function $f \in \Sigma_{p,n}$ and suppose that $g \in S^*[p, A, B]$. If $Df(z)$ is majorized by $g(z)$ in Δ^* and $[|A-B| + p|B|]\delta \le p$, then*

$$|D^2 f(z)| \le |Dg(z)|,\quad |z| \le r_0,$$

where $r_0 = r_0(p, A, B)$ is the smallest positive root of the equation

$$[|A-B|+p|B|]r^3-[p+2|B|]r^2-[|A-B|+p|B|+2]r+p=0,$$

$$z \in \Delta^*; A, B \in \mathbb{C}, A \neq B, |B| \leq 1; p \in \mathbb{N}; 0 \leq \delta \leq r_0.$$

Also, putting $q = 0, j = 2, l = 1$ and $\gamma = 1$ in Corollary 2.2, we obtain the following result.

Corollary 2.4 *Let the function $f \in \Sigma_{p,n}$ and suppose that $g \in K[p, A, B]$. If $D^2 f(z)$ is majorized by $Dg(z)$ in Δ^* and $[|A-B|+p|B|]\delta \leq p$, then*

$$|D^3 f(z)| \leq |D^2 g(z)|, \quad |z| \leq r_0,$$

where $r_0 = r_0(p, A, B)$ is the smallest positive root of the equation

$$[|A-B|+p|B|]r^3-[p+2|B|]r^2-[|A-B|+p|B|+2]r+p=0,$$

$$z \in \Delta^*; A, B \in \mathbb{C}, A \neq B, |B| \leq 1; p \in \mathbb{N}; 0 \leq \delta \leq r_0.$$

The following two results can also be obtained from Theorem 2.1:

Corollary 2.5 *Let the function $f \in \Sigma_{p,n}$ and suppose that $g \in \Sigma US_n(p, \alpha, \beta, \gamma)$. If $Df(z)$ is majorized by $g(z)$ in Δ^* and*

$$\Big[\frac{2(1-\beta)|\gamma|}{|1-\alpha|}+p\Big]\delta \leq p,$$

then

$$|D^2 f(z)| \leq |Dg(z)|, \quad |z| \leq r_0,$$

where $r_0 = r_0(p, \alpha, \beta, \gamma)$ is the smallest positive root of the equation

$$\Big[\frac{2(1-\beta)|\gamma|}{|1-\alpha|}+p\Big]r^3-[p+2]r^2-\Big[\frac{2(1-\beta)|\gamma|}{|1-\alpha|}+p+2\Big]r+p=0,$$

$$z \in \Delta^*; p \in \mathbb{N}; 0 \leq \alpha \neq 1; 0 \leq \beta < 1; \gamma \in \mathbb{C}^*; 0 \leq \delta \leq r_0.$$

Corollary 2.6 *Let the function $f \in \Sigma_{p,n}$ and suppose that $g \in \Sigma UK_n(p, \alpha, \beta, \gamma)$. If $D^2 f(z)$ is majorized by $Dg(z)$ in Δ^* and*

$$\Big[\frac{2(1-\beta)|\gamma|}{|1-\alpha|}+p\Big]\delta \leq p,$$

then

$$|D^3 f(z)| \leq |D^2 g(z)|, \quad |z| \leq r_0,$$

where $r_0 = r_0(p, \alpha, \beta, \gamma)$ is the smallest positive root of the equation

$$\Big[\frac{2(1-\beta)|\gamma|}{|1-\alpha|}+p\Big]r^3-[p+2]r^2-\Big[\frac{2(1-\beta)|\gamma|}{|1-\alpha|}+p+2\Big]r+p=0,$$

$$z \in \Delta^*; p \in \mathbb{N}; 0 \leq \alpha \neq 1; 0 \leq \beta < 1; \gamma \in \mathbb{C}^*; 0 \leq \delta \leq r_0.$$

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