Positive Solutions of Singular Second-Order Integral Boundary Value Problems

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Abstract By using the fixed point index in cone and the fixed theorem of cone expansion and compression, the existence of positive solutions to the singular second-order boundary value problem is considered.

Keywords   singular; integral boundary value problem; positive solution; cone.

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1. Introduction

It is known that many important phenomena such as heat conduction, chemical engineering, underground water flow and plasma physics can be represented by boundary value problems with integral boundary conditions for ordinary differential equations. As a result, they have been widely studied in the last few years [1–3, 5–7, 11, 12]. In [2], by applying fixed point index theory of strict contraction operators, Hao et al investigated the existence of multiple solutions for the following nth-order nonlocal boundary value problem (BVP) in Banach spaces.

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \ldots, x^{(n-2)}(t)) = \theta, & t \in (0, 1), \\ x^{(i)}(0) = \theta, & 0 \leq i \leq n-3, \\ ax^{(n-2)}(0) - bx^{(n-1)}(0) = \int_0^1 x^{(n-2)}(s)\mathrm{d}A(s), \\ cx^{(n-2)}(1) + dx^{(n-1)}(1) = \int_0^1 x^{(n-2)}(s)\mathrm{d}B(s), \end{cases} \tag{1.1}$$

where f may be singular at $t = 0$, $t = 1$. In [5], Yang established the existence of nontrivial solutions for the following Sturm-Liouville problem with integral boundary conditions.

$$\begin{cases} -(au')' + bu = g(t)f(t, u), & t \in (0, 1), \\ (\cos \gamma_0)u(0) - (\sin \gamma_0)u'(0) = \int_0^1 u(\tau)\mathrm{d}\alpha(\tau), \\ (\cos \gamma_1)u(0) + (\sin \gamma_1)u'(0) = \int_0^1 u(\tau)\mathrm{d}\beta(\tau), \end{cases} \tag{1.2}$$

where g may be singular at $t = 0$, $t = 1$. By using topological degree arguments and cone theory, Kang and Liu [8] established the theory that there exist at least two positive solutions for the

following boundary value problem

$$\begin{cases} u''(t) + f(t, u(t)) + g(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = 0, \ u(1) = 0 \end{cases} \tag{1.3}$$

where $f, g \in C((0, 1) \times (0, +\infty), [0, +\infty))$ may be singular at $t = 0$, $t = 1$, $u = 0$.

Motivated by the above works, the purpose of this paper is to discuss the existence of multiple solutions for the following BVP

$$\begin{cases} (p(t)(u'(t)))' = -a(t)f(t, u(t)), & t \in (0, 1), \\ \alpha u(0) - \beta p(0)u'(0) = \int_0^1 h(s)u(s)\mathrm{d}s, \\ \gamma u(1) + \delta p(1)u'(1) = \int_0^1 g(s)u(s)\mathrm{d}s, \end{cases} \tag{1.4}$$

where $p(t) \in C^1[0, 1]$ and $p(t) > 0$, $t \in [0, 1]$, $a(t) \in C([0, 1], [0, +\infty))$ does not vanish identically on any subinterval of $[0, 1]$, $f \in C((0, 1) \times (0, +\infty), [0, +\infty))$ may be singular at $t = 0$, $t = 1$, $u = 0$, $h(s), g(s) \in L(0, 1)$, are nonnegative, $\alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta \geq 0$ and $\rho = \alpha\gamma e + \alpha\delta + \beta\gamma > 0$ with $e = \int_0^1 \frac{1}{p(s)}\mathrm{d}s$. Some ideas of this paper are from [4] and [8]. The main results of these papers are based on the symmetry of Green function, but the associated Green function in our paper does not have the property. By employing the Krein-Rutmann theorem, we overcome the difficulty caused by the dissymmetric Green function. Moreover, by applying the fixed point index in cone and the fixed point theorem of cone expansion and compression, we obtain the existence of multiple positive solutions for BVP (1.4).

The paper is organized as follows. In Section 2, we introduce some notations, definitions, and lemmas. In Section 3, we present and prove our main results about the existence of positive solutions of BVP (1.4). In Section 4, we present two examples to illustrate the main results.

2. Some preliminaries and lemmas

In this section, we will give some preliminaries for obtaining the main results in the next section. For the purpose of convenience, we set

$$\psi(t) = \alpha \int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau + \beta, \quad \phi(t) = \gamma \int_t^1 \frac{1}{p(\tau)}\mathrm{d}\tau + \delta,$$

$$c_1 = \frac{1}{\rho} \int_0^1 \phi(t)h(t)\mathrm{d}t, \quad c_2 = \frac{1}{\rho} \int_0^1 \psi(t)h(t)\mathrm{d}t, \quad c_3 = \frac{1}{1 - c_1},$$

$$c_4 = \frac{1}{\rho} \int_0^1 \phi(t)g(t)\mathrm{d}t, \quad c_5 = \frac{1}{\rho} \int_0^1 \psi(t)g(t)\mathrm{d}t, \quad c_6 = \frac{1}{1 - c_5}$$

and list two assumptions to be used throughout this paper.

(I) $c_1, c_5 \in [0, 1)$.

(II) $c_2 c_3 c_4 c_6 \in [0, 1)$.

Definition 2.1 *A function $u(t) \in C[0, 1] \cap C^1(0, 1)$ with $p(t)u(t) \in C^1(0, 1)$ is said to be a positive solution of the BVP (1.4), if it satisfies BVP (1.4) and $u(t) > 0$ for $t \in (0, 1)$.*

To investigate BVP (1.4), we now consider the following linear BVP

$$\begin{cases} (p(t)u'(t))' = -y(t), & t \in (0,1), \\ \alpha u(0) - \beta p(0)u'(0) = \int_0^1 h(s)u(s)\mathrm{d}s, \\ \gamma u(1) + \delta p(1)u'(1) = \int_0^1 g(s)u(s)\mathrm{d}s. \end{cases} \tag{2.1}$$

For BVP (2.1), we can have the following lemmas immediately.

Lemma 2.1 *Assume that (I), (II) hold. Then for $y \in C[0,1]$ and $y \geq 0$, the BVP (2.1) has a unique solution $u(t) \geq 0$ for $t \in [0,1]$, such that*

$$u(t) = \int_0^1 H(t,s)y(s)\mathrm{d}s,$$

where $H(t,s)$ is the associated Green's function for (2.1), which can be expressed as

$$H(t,s) = G(t,s) + B(t)\int_0^1 G(\tau,s)h(\tau)\mathrm{d}\tau + C(t)\int_0^1 G(\tau,s)g(\tau)\mathrm{d}\tau$$

with

$$G(t,s) = \frac{1}{\rho}\begin{cases} \phi(t)\psi(s), & 0 \leq s \leq t \leq 1, \\ \phi(s)\psi(t), & 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$B(t) = \frac{1}{\rho(1 - c_2c_3c_4c_6)}[\phi(t)c_3 + \psi(t)c_3c_4c_6],$$

$$C(t) = \frac{1}{\rho(1 - c_2c_3c_4c_6)}[\psi(t)c_6 + \phi(t)c_2c_3c_6].$$

Proof The proof is similar to that of Lemma 2.1 in [7], so we omit it. □

Lemma 2.2 *The associated functions $G(t,s)$, $H(t,s)$ have the following properties.*

(a) *$G(t,s)$, $H(t,s)$ are continuous on $[0,1] \times [0,1]$ and $G(t,s) > 0$, $H(t,s) > 0$ for any $t,s \in (0,1)$.*

(b) *For any $t,s \in [0,1]$, $G(t,s) \leq G(s,s)$, $G(t,s) \geq v(t)G(s,s)$, where $v(t) = \min\{\frac{\psi(t)}{\alpha e + \beta}, \frac{\phi(t)}{\gamma e + \delta}\}$.*

(c) *For any $t,s \in [0,1]$, $H(t,s) \leq NG(s,s)$, where*

$$N = 1 + \frac{c_3(\gamma e + \delta) + c_3c_4c_6(\alpha e + \beta)}{\rho(1 - c_2c_3c_4c_6)}\int_0^1 h(s)\mathrm{d}s + \frac{c_6(\alpha e + \beta) + c_2c_3c_6(\gamma e + \delta)}{\rho(1 - c_2c_3c_4c_6)}\int_0^1 g(s)\mathrm{d}s.$$

(d) *For any $t,s \in [0,1]$, $H(t,s) \geq z(t)G(s,s)$, where*

$$z(t) = v(t) + B(t)\int_0^1 v(s)h(s)\mathrm{d}s + C(t)\int_0^1 v(s)g(s)\mathrm{d}s.$$

(e) *For any $t,s,\tau \in [0,1]$, $H(t,s) \geq k(t)H(\tau,s)$, where $k(t) = \frac{z(t)}{N}$, and $k(t)$ is continuous on $[0,1]$ and $k(t) > 0$, $t \in (0,1)$.*

Next we introduce a hypothesis:

(H$_0$) $0 < \int_0^1 G(s,s)a(s)f_{rR}(s)\mathrm{d}s < +\infty$, for any $0 < r \leq R$, where

$$f_{rR}(t) = \max\{f(t,u), u \in [k(t)r, R]\}, \quad t \in (0,1).$$

We suppose that (H_0) holds throughout the remainder of the paper. We will give examples of functions satisfying (H_0) in Section 4.

Let $E = C[0,1]$. Then E is a Banach space with a norm by $\max_{t \in [0,1]} |u(t)|$, $u \in E$. Define

$$P = \{u \in E : u(t) \geq k(t)\|u\|\}.$$

Then P is a cone in E and $\int_0^1 H(t,s)\mathrm{d}s \in P$. For $u \in P\backslash\{\theta\}$, define an operator A by

$$(Au)(t) = \int_0^1 H(t,s)a(s)f(s,u(s))\mathrm{d}s. \tag{2.2}$$

Since $u \in P\backslash\{\theta\}$, we have $\|u\| > 0$, and $k(t)\|u\| \leq u(t) \leq \|u\|$. From ($H_0$), we know A is well-defined.

Lemma 2.3 *If $u \in P\backslash\{\theta\}$, then we have $Au \in P$.*

Proof For any $u \in P\backslash\{\theta\}$, we have

$$(Au)(t) = \int_0^1 H(t,s)a(s)f(s,u(s))\mathrm{d}s \geq k(t)\int_0^1 H(\tau,s)a(s)f(s,u(s))\mathrm{d}s = k(t)(Au)(\tau).$$

Then, we have $Au \in P$. $\square$

Obviously, that u is a positive solution of BVP (1.4) is equivalent to that $Au = u$ in $P\backslash\{\theta\}$ has a fixed point.

Lemma 2.4 *For any $R_2 > R_1 > 0$, $A : \overline{P_{R_2}}\backslash P_{R_1} \longrightarrow P$ is completely continuous, where $P_r = \{u \in P, \|u\| < r\}$, $\overline{P_r} = \{u \in P, \|u\| \leq r\}$ $(r > 0)$.*

Proof For any $u \in \overline{P_{R_2}}\backslash P_{R_1}$, then $k(t)R_1 \leq u(t) \leq R_2$. From Lemma 2.2(c), we have

$$\|Au\| \leq N \int_0^1 G(s,s)a(s)f_{R_1 R_2}(s)\mathrm{d}s \triangleq M.$$

Thus $\|Au\| \leq M$, which implies that A is bounded on $\overline{P_{R_2}}\backslash P_{R_1}$.

Next, we prove that $\{(Au)(t), u \in V\}$ is equicontinuous on $[0,1]$, for all $V \subset \overline{P_{R_2}}\backslash P_{R_1}$. Notice that $(Au)(t)$ can be expressed as $(Au)(t) = (A_1 u)(t) + (A_2 u)(t)$, where $(A_1 u)(t), (A_2 u)(t)$ have the expressions

$$(A_1 u)(t) = \int_0^1 G(t,s)a(s)f(s,u(s))\mathrm{d}s,$$

and

$$(A_2 u)(t) = B(t)\int_0^1 \int_0^1 G(\tau,s)h(\tau)a(s)f(s,u(s))\mathrm{d}\tau\mathrm{d}s +$$
$$C(t)\int_0^1 \int_0^1 G(\tau,s)g(\tau)a(s)f(s,u(s))\mathrm{d}\tau\mathrm{d}s.$$

Then, to prove that $\{(Au)(t), u \in V\}$ is equicontinuous on $[0,1]$, we only need to show that $\{(A_1 u)(t), u \in V\}$ and $\{(A_2 u)(t), u \in V\}$ are equicontinuous on $[0,1]$.

First, we will show $\{(A_1 u)(t), u \in V\}$ is equicontinuous. The proof can be divided into four cases.

Case 1 $\beta\delta \neq 0$, then

$$\int_0^1 a(s)f_{R_1 R_2}(s)\mathrm{d}s < +\infty. \tag{2.3}$$

For any $t_1, t_2 \in [0, 1]$, $t_1 < t_2$,

$$\begin{aligned}
&|(A_1 u)(t_1) - (A_1 u)(t_2)| \\
&= \left| \int_0^1 G(t_1, s)a(s)f(s, u(s))\mathrm{d}s - \int_0^1 G(t_2, s)a(s)f(s, u(s))\mathrm{d}s \right| \\
&\leq \int_0^1 |G(t_1, s) - G(t_2, s)|a(s)f(s, u(s))\mathrm{d}s \\
&\leq \int_0^1 |G(t_1, s) - G(t_2, s)|a(s)f_{R_1 R_2}(s)\mathrm{d}s.
\end{aligned}$$

This, together with the uniform continuity of $G(t, s)$ on $[0, 1] \times [0, 1]$ and (2.3), guarantees that $\{(A_1 u)(t), u \in V\}$ is equicontinuous on $[0, 1]$.

Case 2 $\beta = 0$, $\delta \neq 0$, then

$$\int_0^1 \psi(s)a(s)f_{R_1 R_2}(s)\mathrm{d}s < +\infty. \tag{2.4}$$

We first show that $\lim_{t \to 0^+} A_1 u = 0$ uniformly with respect to $u \in V$.

Notice that

$$\begin{aligned}
(A_1 u)(t) &= \int_0^1 G(t, s)a(s)f(s, u(s))\mathrm{d}s \\
&= \frac{1}{\rho}\int_0^t \phi(t)\psi(s)a(s)f(s, u(s))\mathrm{d}s + \frac{1}{\rho}\int_t^1 \phi(s)\psi(t)a(s)f(s, u(s))\mathrm{d}s.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{\rho}\int_0^t \phi(t)\psi(s)a(s)f(s, u(s))\mathrm{d}s &\leq \frac{1}{\rho}\int_0^t \phi(s)\psi(s)a(s)f(s, u(s))\mathrm{d}s \\
&\leq \frac{1}{\rho}\int_0^t \phi(s)\psi(s)a(s)f_{R_1 R_2}(s)\mathrm{d}s,
\end{aligned}$$

and (2.4), we deduce that

$$\lim_{t \to 0^+} \frac{1}{\rho}\int_0^t \phi(t)\psi(s)a(s)f(s, u(s))\mathrm{d}s = 0 \tag{2.5}$$

uniformly with respect to $u \in V$.

On the other hand, for any given $\varepsilon > 0$, by (H_0) there exists $\xi > 0$ such that

$$\frac{1}{\rho}\int_t^\xi \phi(s)\psi(s)a(s)f_{R_1 R_2}(s)\mathrm{d}s < \varepsilon, \quad t \in (0, \xi). \tag{2.6}$$

In view of

$$\begin{aligned}
&\frac{1}{\rho}\int_t^1 \phi(s)\psi(t)a(s)f(s, u(s))\mathrm{d}s \\
&= \frac{1}{\rho}\int_t^\xi \phi(s)\psi(t)a(s)f(s, u(s))\mathrm{d}s + \frac{1}{\rho}\int_\xi^1 \phi(s)\psi(t)a(s)f(s, u(s))\mathrm{d}s
\end{aligned}$$

$$\leq \frac{1}{\rho} \int_t^\xi \phi(s)\psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s + \frac{1}{\rho}\frac{\psi(t)}{\psi(\xi)} \int_\xi^1 \phi(s)\psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s,$$

(2.4), (2.6) and $\lim_{t\to 0^+} \psi(t) = 0$, we have

$$\lim_{t\to 0^+} \frac{1}{\rho} \int_t^1 \phi(s)\psi(t)a(s)f(s,u(s))\mathrm{d}s = 0 \tag{2.7}$$

uniformly with respect to $u \in V$.

From (2.5) and (2.7) it follows that $\lim_{t\to 0^+} A_1 u = 0$ uniformly with respect to $u \in V$.

Now we are in position to show that for any $a \in (0, 1/2)$, $\{(A_1 u)(t), u \in V\}$ is equicontinuous on $[a, 1-a]$.

In fact, for any $t_1, t_2 \in [a, 1-a], t_1 < t_2$

$$|(A_1u)(t_1) - (A_1u)(t_2)|$$

$$= \left| \int_0^1 G(t_1,s)a(s)f(s,u(s))\mathrm{d}s - \int_0^1 G(t_2,s)a(s)f(s,u(s))\mathrm{d}s \right|$$

$$\leq \left| \int_0^a G(t_1,s)a(s)f(s,u(s))\mathrm{d}s - \int_0^a G(t_2,s)a(s)f(s,u(s))\mathrm{d}s \right| +$$

$$\left| \int_a^{1-a} G(t_1,s)a(s)f(s,u(s))\mathrm{d}s - \int_a^{1-a} G(t_2,s)a(s)f(s,u(s))\mathrm{d}s \right| +$$

$$\left| \int_{1-a}^1 G(t_1,s)a(s)f(s,u(s))\mathrm{d}s - \int_{1-a}^1 G(t_2,s)a(s)f(s,u(s))\mathrm{d}s \right|$$

$$\leq |\phi(t_1) - \phi(t_2)| \int_0^a \psi(s)a(s)f(s,u(s))\mathrm{d}s +$$

$$\left| \int_a^{1-a} G(t_1,s)a(s)f(s,u(s))\mathrm{d}s - \int_a^{1-a} G(t_2,s)a(s)f(s,u(s))\mathrm{d}s \right| +$$

$$|\psi(t_1) - \psi(t_2)| \int_{1-a}^1 \phi(s)a(s)f(s,u(s))\mathrm{d}s$$

$$\leq |\phi(t_1) - \phi(t_2)| \int_0^a \psi(s)a(s)f(s,u(s))\mathrm{d}s +$$

$$\left| \int_a^{1-a} G(t_1,s)a(s)f(s,u(s))\mathrm{d}s - \int_a^{1-a} G(t_2,s)a(s)f(s,u(s))\mathrm{d}s \right| +$$

$$|\psi(t_1) - \psi(t_2)| \frac{1}{\psi(1-a)} \int_{1-a}^1 \phi(s)\psi(s)a(s)f(s,u(s))\mathrm{d}s$$

$$\leq |\phi(t_1) - \phi(t_2)| \int_0^1 \psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s +$$

$$\int_a^{1-a} |G(t_1,s) - G(t_2,s)|a(s)f(s,u(s))\mathrm{d}s +$$

$$|\psi(t_1) - \psi(t_2)| \frac{1}{\phi(1-a)} \int_0^1 \phi(s)\psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s.$$

Let $m = \min_{t\in[a,1-a]} k(t)$. Then $u(t) \in [mR_1, R_2]$, so there exists $D > 0$, such that $\max_{t\in[a,1-a]} |a(t)f(t,u(t))| \leq D$, which implies

$$|(A_1u)(t_1) - (A_1u)(t_2)| \leq |\phi(t_1) - \phi(t_2)| \int_0^1 \psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s +$$

$$|\psi(t_1) - \psi(t_2)| \frac{1}{\phi(1-a)} \int_0^1 \phi(s)\psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s +$$

$$D \int_a^{1-a} |G(t_1, s) - G(t_2, s)|\mathrm{d}s.$$

By (2.4), it is easy to see that $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[a, 1-a]$.

Finally, for any given $\varepsilon > 0$, by (H_0), there exists $b > 0$, which satisfies

$$\frac{1}{\rho} \int_b^1 \phi(s)\psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s < \varepsilon. \tag{2.8}$$

Then, we will show that $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[b, 1]$.

In fact, for any $t_1, t_2 \in [b, 1], t_1 < t_2,$

$$|(A_1u)(t_1) - (A_1u)(t_2)|$$

$$= \left| \int_0^1 G(t_1, s)a(s)f(s, u(s))\mathrm{d}s - \int_0^1 G(t_2, s)a(s)f(s, u(s))\mathrm{d}s \right|$$

$$= \left| \frac{1}{\rho} \int_0^{t_1} \phi(t_1)\psi(s)a(s)f(s, u(s))\mathrm{d}s + \frac{1}{\rho} \int_{t_1}^1 \phi(s)\psi(t_1)a(s)f(s, u(s))\mathrm{d}s - \right.$$

$$\left. \frac{1}{\rho} \int_0^{t_2} \phi(t_2)\psi(s)a(s)f(s, u(s))\mathrm{d}s - \frac{1}{\rho} \int_{t_2}^1 \phi(s)\psi(t_2)a(s)f(s, u(s))\mathrm{d}s \right|$$

$$\leq \left| \frac{1}{\rho} \int_0^{t_1} \phi(t_1)\psi(s)a(s)f(s, u(s))\mathrm{d}s - \frac{1}{\rho} \int_0^{t_2} \phi(t_2)\psi(s)a(s)f(s, u(s))\mathrm{d}s \right| +$$

$$\frac{1}{\rho} \int_{t_1}^1 \phi(s)\psi(t_1)a(s)f(s, u(s))\mathrm{d}s + \frac{1}{\rho} \int_{t_2}^1 \phi(s)\psi(t_2)a(s)f(s, u(s))\mathrm{d}s$$

$$\leq \frac{1}{\rho}|\phi(t_1) - \phi(t_2)| \int_0^{t_1} \psi(s)a(s)f(s, u(s))\mathrm{d}s + \frac{\phi(t_2)}{\rho} \int_{t_1}^{t_2} \psi(s)a(s)f(s, u(s))\mathrm{d}s +$$

$$\frac{1}{\rho} \int_{t_1}^1 \phi(s)\psi(t_1)a(s)f(s, u(s))\mathrm{d}s + \frac{1}{\rho} \int_{t_2}^1 \phi(s)\psi(t_2)a(s)f(s, u(s))\mathrm{d}s$$

$$\leq \frac{1}{\rho}|\phi(t_1) - \phi(t_2)| \int_0^{t_1} \psi(s)a(s)f(s, u(s))\mathrm{d}s + \frac{\phi(t_2)}{\rho} \int_{t_1}^{t_2} \psi(s)a(s)f(s, u(s))\mathrm{d}s +$$

$$\frac{1}{\rho} \int_{t_1}^1 \phi(s)\psi(s)a(s)f(s, u(s))\mathrm{d}s + \frac{1}{\rho} \int_{t_2}^1 \phi(s)\psi(s)a(s)f(s, u(s))\mathrm{d}s$$

$$\leq \frac{1}{\rho}|\phi(t_1) - \phi(t_2)| \int_0^1 \psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s + \frac{\phi(t_2)}{\rho} \int_{t_1}^{t_2} \psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s +$$

$$\frac{1}{\rho} \int_b^1 \phi(s)\psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s + \frac{1}{\rho} \int_b^1 \phi(s)\psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s.$$

This, together with (2.4), (2.8) and uniform continuity of $\phi(t)$ on $[0, 1]$, guarantees that $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[b, 1]$. Therefore, $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[0, 1]$.

Case 3 $\beta \neq 0, \delta = 0$, then $\int_0^1 \phi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s < +\infty.$

The proof of Case 3 is similar to that in Case 2. We just list the main steps: Firstly, we prove $\lim_{t \to 1^-} A_1u = 0$ uniformly with respect to $u \in V$. Secondly, for any $a \in (0, 1/2)$, $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[a, 1-a]$. Thirdly, there exists $b > 0$ small enough, such

that $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[0, b]$. Therefore, $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[0, 1]$.

Case 4 $\beta = 0$, $\delta = 0$, then $\int_0^1 \phi(s)\psi(s)a(s)f_{R_1R_2}(s)\mathrm{d}s < +\infty$.

The case can be proved by justifying $\lim_{t \to 0^+} A_1u = 0$ and $\lim_{t \to 1^-} A_1u = 0$ uniformly with respect to $u \in V$, which then implies that for any $a \in (0, 1/2)$, $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[a, 1-a]$.

Through above discussions, we know that $\{(A_1u)(t), u \in V\}$ is equicontinuous on $[0, 1]$. The equicontinuity of $\{(A_2u)(t), u \in V\}$ on $[0, 1]$ can be proved easily. Therefore, $\{(Au)(t), u \in V\}$ is equicontinuous on $[0, 1]$.

In addition, according to the Lebesgue dominated convergence theorem and

$$\int_0^1 G(s, s)a(s)f_{R_1R_2}(s)\mathrm{d}s < +\infty,$$

we can easily get the continuity of A. Thus $A : \overline{P_{R_2}} \backslash P_{R_1} \longrightarrow P$ is completely continuous. The proof is completed. $\square$

Lemma 2.5 (Krein-Rutmann [9]) *Let E be a real Banach space, E^* the dual space, P a total cone in E and P^* the dual cone of P. Let $L : E \longrightarrow E$ be a positive, completely continuous, linear operator, $r(L)$ the spectral radius of L and L^* the dual operator of L. If there exist $\psi \in E\backslash(-P)$ and a positive constant c such that $cL(\psi) \geq \psi$, then the spectral radius $r(L) \neq 0$ and there are $p \in P \backslash \{\theta\}$ and $w \in P^* \backslash \{\theta\}$ such that $Lp = r(L)p$ and $L^*w = r(L)w$.*

Define $K = \{u \in E, u(t) \geq 0, t \in [0, 1]\}$, then K is a cone in E. Let the dual space of E and the dual cone of K be denoted by E^* and K^*, respectively. They are represented by

$E^* = \{v: v$ is right continuous on $[0,1)$ and is of bounded variation on $[0, 1]$ with $v(0) = 0\}$

$K^* = \{u \in E^*, v$ is nondecreasing on $[0, 1]\}$.

Moreover, the bounded linear functional on E can be represented in the Riemann-Stieltjes integral

$$\langle v, u \rangle = \int_0^1 u(t)\mathrm{d}v(t), \quad u \in E, \ v \in E^*.$$

Define

$$(Lu)(t) = \int_0^1 H(t, s)a(s)u(s)\mathrm{d}s, \quad u \in E.$$

It is easy to see that $L : E \longrightarrow E$ is a completely continuous, linear operator, satisfying $L(K) \subset K$. That is, L is a positive, completely continuous, linear operator. It is easy to know that

$$(L^*v)(s) = \int_0^s \int_0^1 H(t, \tau)a(\tau)\mathrm{d}v(t)\mathrm{d}\tau, \quad v \in E^*,$$

where $L^* : K^* \longrightarrow K^*$ is the dual operator of L.

Lemma 2.6 *The spectral radius of L is positive and there exists $q \in K \backslash \{\theta\}$ such that $r(L)q(s) = \int_0^1 H(t, s)a(s)q(t)\mathrm{d}t$ and $\int_0^1 q(t)\mathrm{d}t = 1$. Furthermore there exists $\omega > 0$, such that $\int_0^1 u(t)q(t)\mathrm{d}t \geq \omega\|u\|$, for $u \in P$.*

Proof From Lemma 2.2(a) we know that there exists $[t_1, t_2] \subset (0, 1)$ such that $H(t, s) > 0$, for

$t, s \in [t_1, t_2]$. Take $u(t) \in E$, such that $u(t) \geq 0$, for $t \in [0, 1]$, and $u(t_3) > 0$, and $u(t) = 0$ for $t \notin [t_1, t_2]$. Then for $t \in [t_1, t_2]$, we have

$$(Lu)(t) = \int_0^1 H(t, s)a(s)u(s)\mathrm{d}s = \int_{t_1}^{t_2} H(t, s)a(s)u(s)\mathrm{d}s > 0.$$

So there exists a constant $c > 0$ such that $c(Lu)(t) \geq u(t)$ for all $t \in [0, 1]$. From Lemma 2.5, we know that the spectral radius $r(L) \neq 0$ and there are $p \in K \setminus \{\theta\}$ and $w \in K^* \setminus \{\theta\}$ such that $Lp = r(L)p$, $L^*w = r(L)w$ with $w(1) = 1$. Let $q(t) = w'(t)$. Then $q \in K \setminus \{\theta\}$, $r(L)q(s) = \int_0^1 H(t, s)a(s)q(t)\mathrm{d}t$ and $\int_0^1 q(t)\mathrm{d}t = 1$.

Notice that for $u \in P$, we have $u(t) \geq k(t)\|u\|$, so $\int_0^1 u(t)q(t)\mathrm{d}t \geq \int_0^1 k(t)q(t)\mathrm{d}t\|u\|$. Let $\omega = \int_0^1 k(t)q(t)\mathrm{d}t$. We have $\omega > 0$ and $\int_0^1 u(t)q(t)\mathrm{d}t \geq \omega\|u\|$. $\square$

Lemma 2.7 ([10]) *Suppose that E is a real Banach space and $P \subset E$ is a cone. Let $A : \overline{P_{R_3}} \setminus P_{R_1} \longrightarrow P$ be a completely continuous operator with $R_1 < R_2 < R_3$. Suppose the following three conditions hold*

    (i) $\|Ax\| \geq \|x\|, \forall x \in \partial P_{R_1}$;

    (ii) $\|Ax\| < \|x\|, \forall x \in \partial P_{R_2}$;

    (iii) $\|Ax\| \geq \|x\|, \forall x \in \partial P_{R_3}$,

where $\partial P_r = \{u \in P, \|u\| = r\}$. Then A has at least two fixed points in $\overline{P_{R_2}} \setminus P_{R_1} \cup \overline{P_{R_3}} \setminus P_{R_2}$.

Lemma 2.8 ([10]) *Let E be a real Banach space, and $P \subset E$ be a cone. Suppose that $A : \overline{P_r} \longrightarrow P$ is a completely continuous operator. If $A\varphi \neq \mu\varphi, \forall \varphi \in \partial P_r, \mu \geq 1$, then the fixed point index $i(A, P_r, P) = 1$.*

Lemma 2.9 ([10]) *Let E be a real Banach space, $P \subset E$ be a cone. Suppose that $A : \overline{P_r} \longrightarrow P$ is a completely continuous operator. If there exists $\varphi_0 \in P \setminus \{\theta\}$ such that $\varphi - A\varphi \neq \mu\varphi_0, \forall \varphi \in \partial P_r, \mu \geq 0$, then the fixed point index $i(A, P_r, P) = 0$.*

3. The main results

In this section, we shall discuss that BVP (1.4) has at last two positive solutions under some conditions. For convenience, we list the following conditions

(H_1) There exist $b(t) \in C([0, 1], [0, +\infty))$, which does not vanish identically on any subinterval of $[0, 1]$, and $\varepsilon_0 > 0$, such that $f(t, u) \geq (\lambda_1 + \varepsilon_0)u - b(t)$, for $t \in (0, 1)$, where $\lambda_1 r(L) = 1$.

(H_2) There exist $r_1 > 0$ and $\varepsilon_1 > 0$, such that $f(t, u) \geq (\lambda_1 + \varepsilon_1)u$, for $0 \leq u \leq r_1, t \in (0, 1)$.

(H_3) There exists $R_0 > 0$, such that $\int_0^1 G(s, s)a(s)f_{R_0 R_0}(s)\mathrm{d}s < R_0/N$.

(H_4) There exist $[c, d] \subset [0, 1]$ and $l > 0$, such that $f(t, u) \geq nu$, for $t \in [c, d], u \in [l, +\infty)$, where $n = \frac{1}{\int_c^d H(\frac{1}{2}, s)a(s)k(s)\mathrm{d}s}$.

(H_5) There exist $r_2 > 0$ and $\varphi(t) \in L(0, 1)$ such that $f(t, u) \geq \varphi(t)$ for $t \in (0, 1), u \in (0, r_2]$, and $\int_0^1 G(s, s)a(s)\varphi(s)\mathrm{d}s > 0$.

In terms of these conditions, we can show the main result about BVP (1.4) in the next theorem.

Theorem 3.1 *If (I), (II) (H_0)–(H_3) are satisfied, then BVP (1.4) has at least two positive solutions.*

Proof Let $\beta = r(L) \int_0^1 q(t)b(t)\mathrm{d}t$, $R > \max\{\frac{\beta}{\varepsilon_0 \omega r(L)}, R_0\}$. Then, for $\varphi \in \partial P_R$ and $\mu \geq 0$, we have $\varphi - A\varphi \neq \mu p_0$, where $p_0 = \int_0^1 H(t,s)\mathrm{d}s$. Otherwise, there exist $\varphi_0 \in \partial P_R$, and $\mu_0 \geq 0$, such that $\varphi_0 - A\varphi_0 = \mu_0 p_0$, thus, by H_1, we have

$$\varphi_0 = A\varphi_0 + \mu_0 p_0 \geq A\varphi_0 = \int_0^1 H(t,s)a(s)f(s,\varphi_0(s))\mathrm{d}s$$

$$\geq (\lambda_1 + \varepsilon_0) \int_0^1 H(t,s)a(s)\varphi_0(s)\mathrm{d}s - \int_0^1 H(t,s)a(s)b(s)\mathrm{d}s,$$

which implies

$$\int_0^1 q(t)\varphi_0(t)\mathrm{d}t \geq (\lambda_1 + \varepsilon_0) \int_0^1 q(t) \int_0^1 H(t,s)a(s)\varphi_0(s)\mathrm{d}s\mathrm{d}t - \int_0^1 q(t) \int_0^1 H(t,s)a(s)b(s)\mathrm{d}s\mathrm{d}t.$$

$$= (\lambda_1 + \varepsilon_0) \int_0^1 \varphi_0(s) \int_0^1 H(t,s)a(s)q(t)\mathrm{d}t\mathrm{d}s - \int_0^1 b(s) \int_0^1 H(t,s)a(s)q(t)\mathrm{d}t\mathrm{d}s.$$

$$= (\lambda_1 + \varepsilon_0)r(L) \int_0^1 \varphi_0(s)q(s)\mathrm{d}s - \beta.$$

$$= (1 + \varepsilon_0 r(L)) \int_0^1 \varphi_0(s)q(s)\mathrm{d}s - \beta.$$

So, $\beta \geq \varepsilon_0 r(L)) \int_0^1 \varphi_0(s)q(s)\mathrm{d}s \geq \varepsilon_0 r(L))\omega\|\varphi_0\| = \varepsilon_0 r(L))\omega R$, which is a contradiction with the assumption of R. Then we have $i(A, P_R, P) = 0$.

Let $r < \min\{R_0, r_1\}$. Then for $\varphi \in \partial P_r$, and $\tau \geq 0$, we have $\varphi - A\varphi \neq \tau p_0$. Otherwise, there exist $\varphi_0 \in \partial P_r$, and $\tau_0 \geq 0$, such that $\varphi_0 - A\varphi_0 = \tau_0 p_0$, thus, by ($H_2$), we have

$$\varphi_0 = A\varphi_0 + \tau_0 p_0 \geq A\varphi_0 = \int_0^1 H(t,s)a(s)f(s,\varphi_0(s))\mathrm{d}s$$

$$\geq (\lambda_1 + \varepsilon_1) \int_0^1 H(t,s)a(s)\varphi_0(s)\mathrm{d}s,$$

which implies

$$\int_0^1 q(t)\varphi_0(t)\mathrm{d}t \geq (\lambda_1 + \varepsilon_1) \int_0^1 q(t) \int_0^1 H(t,s)a(s)\varphi_0(s)\mathrm{d}s\mathrm{d}t,$$

$$= (\lambda_1 + \varepsilon_1)r(L) \int_0^1 \varphi_0(t)q(t)\mathrm{d}t,$$

$$= \int_0^1 \varphi_0(t)q(t)\mathrm{d}t + \varepsilon_1 r(L) \int_0^1 \varphi_0(t)q(t)\mathrm{d}t.$$

So, $0 \geq \int_0^1 \varphi_0(t)q(t)\mathrm{d}t > 0$, which is a contradiction. Then we have $i(A, P_r, P) = 0$.

For $u \in \partial P_{R_0}$, we have $Au \neq \mu u$, $\mu \geq 1$. Otherwise, there exist $u_0 \in \partial P_{R_0}$, and $\mu_0 \geq 1$, such that $Au_0 = \mu_0 u_0$. By (H_3), we have

$$\mu_0 u_0 = Au_0 = \int_0^1 H(t,s)a(s)f(s,u_0(s))\mathrm{d}s$$

$$\leq N \int_0^1 G(s,s)a(s)f_{R_0R_0}(s)\mathrm{d}s < R_0.$$

Thus $\mu_0 R_0 < R_0$, which is a contradiction, so we have $i(A, P_{R_0}, P) = 1$.

So, $i(A, P_{R_0}\backslash\overline{P_r}, P) = 1$ and $i(A, P_R\backslash\overline{P_{R_0}}, P) = -1$. Therefore, BVP (1.4) has at least two positive solutions. $\square$

It is interesting that, from different conditions, we can show the same result as in Theorem 3.1, which is summarized in the following theorem.

Theorem 3.2 *If (I), (II), (H_0), (H_3)–(H_5) are satisfied, then BVP (1.4) has at least two positive solutions.*

Proof Let $k_0 = \min_{t\in[c,d]} k(t)$. Then $u(t) \geq k_0\|u\|$. Suppose $\|u\| = R_1$ large enough, such that $u(t) \geq k_0\|u\| = k_0 R_1 \geq l$, so, by ($H_4$), we have

$$\|Au\| \geq Au(1/2) = \int_0^1 H(1/2,s)a(s)f(s,u(s))\mathrm{d}s$$

$$\geq \int_c^d H(1/2,s)a(s)f(s,u(s))\mathrm{d}s \geq n\int_c^d H(1/2,s)a(s)u(s)\mathrm{d}s$$

$$\geq n\int_c^d H(1/2,s)a(s)k(s)\mathrm{d}s\|u\| = \|u\|.$$

Let $\|u\| = r$ small enough such that $r < \min\{\int_0^1 H(1/2,s)a(s)\varphi(s)\mathrm{d}s, r_2, R_0\}$. Then by ($H_5$),

$$\|Au\| \geq Au(1/2) = \int_0^1 H(1/2,s)a(s)f(s,u(s))\mathrm{d}s$$

$$\geq \int_0^1 H(1/2,s)a(s)\varphi(s)\mathrm{d}s \geq \|u\|.$$

For $u \in \partial P_{R_0}$, by (H_3), we have

$$\|Au\| \leq N\int_0^1 G(s,s)a(s)f_{R_0R_0}(s)\mathrm{d}s < R_0 = \|u\|.$$

To sum up, by Lemma 2.7, our conclusion follows.

Corollary 3.1 *If (I), (II), (H_0), (H_1), (H_3), (H_5) are satisfied, then BVP (1.4) has at least two positive solutions.*

Corollary 3.2 *If (I), (II), (H_0), (H_2), (H_3), (H_4) are satisfied, then BVP(1.4) has at least two positive solutions.*

Remark Under the assumptions of (I), (II), if (H_0), (H_1), (H_3) or (H_0), (H_2), (H_3) or (H_0), (H_3), (H_4) or (H_0), (H_3), (H_5) are satisfied, then BVP (1.4) has at least one positive solution.

Two examples are presented in Section 4 to illustrate how our main results can be used in practice.

4. Examples

Example 4.1 Consider the singular integral boundary value problem

$$
\begin{cases}
u''(t) = -\frac{1}{9\sqrt{t(1-t)}}\left(\frac{1}{\sqrt{u(t)}} + (u(t))^2\right), & t \in (0, 1), \\
u(0) = \int_0^1 \frac{1}{2}u(s)\mathrm{d}s, \quad u(1) = \int_0^1 \frac{1}{2}u(s)\mathrm{d}s.
\end{cases}
\tag{4.1}
$$

Then BVP (4.1) has at least two positive solutions.

Proof BVP(4.1) can be regarded as a BVP of the form (1.4), where $p(t) = a(t) = 1$, $f(t, u) = \frac{1}{9\sqrt{t(1-t)}}\left(\frac{1}{\sqrt{u}} + u^2\right)$, $h(s) = g(s) = \frac{1}{2}$, $\alpha = \gamma = 1$, $\beta = \delta = 0$ and $e = 1$, $\rho = 1$. Let $R_0 = 1$. We can easily find that the conditions of Theorem (3.1) are satisfied, so it has at least two positive solutions. $\square$

Example 4.2 Consider the singular integral boundary value problem

$$
\begin{cases}
u''(t) = -\frac{1}{3\sqrt{t(1-t)}}\left(\frac{1}{u(t)} + u(t)\right), & t \in (0, 1), \\
u(0) = \int_0^1 \frac{1}{2}u(s)\mathrm{d}s, \quad u(1) = \int_0^1 \frac{1}{2}u(s)\mathrm{d}s.
\end{cases}
\tag{4.2}
$$

Then BVP (4.2) has at least two positive solutions.

Proof BVP(4.2) can be regarded as a BVP of the form (1.4), where $p(t) = a(t) = 1$, $f(t, u) = \frac{1}{3\sqrt{t(1-t)}}\left(\frac{1}{u} + u\right)$, $h(s) = g(s) = \frac{1}{2}$, $\alpha = \gamma = 1$, $\beta = \delta = 0$ and $e = 1$, $\rho = 1$. Let $R_0 = 3$, $\varphi(t) = \frac{1}{\sqrt{t(1-t)}}$. We can easily find that the conditions of Theorem (3.2) are satisfied, so it has at least two positive solutions. $\square$

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