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On $L(1,2)$-Edge-Labelings of Some Special Classes of Graphs

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Abstract For a graph G and two positive integers j and k, an m-$L(j,k)$-edge-labeling of G is an assignment on the edges to the set $\{0,\dots,m\}$, such that adjacent edges receive labels differing by at least j, and edges which are distance two apart receive labels differing by at least k. The $\lambda'_{j,k}$-number of G is the minimum m of an m-$L(j,k)$-edge-labeling admitted by G. In this article, we study the $L(1,2)$-edge-labeling for paths, cycles, complete graphs, complete multipartite graphs, infinite Δ-regular trees and wheels.

Keywords $L(j,k)$-edge-labeling; line graph; path; cycle; complete graph; complete multipartite graph; infinite Δ-regular tree; wheel.

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1. Introduction

In this paper, we consider undirected and simple graphs, and we use standard notations in graph theory [1]. Let G be a graph with non-empty edge set and j,k be two positive integers. An m-$L(j,k)$-labeling of G is a function which assigns each vertex of G with a label from the set $\{0,\dots,m\}$, such that the following two distance conditions are satisfied: $|f(u)-f(v)|\ge j$ if u and v are adjacent and $|f(u)-f(v)|\ge k$ if u and v are distance two apart. The $L(j,k)$-labeling number of a graph G, denoted by $\lambda_{j,k}(G)$, is the minimum m of an m-$L(j,k)$-labeling admitted by G.

The $L(j,k)$-labeling of graphs is motivated by the channel assignment problem introduced by Hale [11]. The $L(2,1)$-labeling was formulated and studied by Griggs and Yeh [10] in 1992. Since then $L(2,1)$-labeling and $L(j,k)$-labeling of graphs for $j\ge k$ have been studied extensively. Refer to surveys [2, 9, 16]. Most of the results on the $L(j,k)$-labeling dealt with the case $j\ge k$.

A variation of the channel assignment problem is the code assignment in computer networks [13]. The task is to assign integer "control codes" to a network of computer stations with distance restrictions. This is the same as $L(j,k)$-labelings such that $j\le k$ is allowed. In [13], Jin and Yeh studied the $L(j,k)$-labelings for $(j,k)\in\{(0,1),(1,1),(1,2)\}$. The authors gave a general upper bound for the $L(1,2)$-labeling number and obtained the $L(1,2)$-labeling numbers for several families of graphs. For example, they concluded the following results:

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Theorem 1.1 ([13]) *Suppose P_n is a path with $n \geq 2$ vertices, and C_n is a cycle of order $n \geq 3$. Then*

$$\lambda_{1,2}(P_n) = \begin{cases} 1, & \text{if } n = 2; \\ 2, & \text{if } n = 3; \\ 3, & \text{if } n \geq 4. \end{cases} \qquad \lambda_{1,2}(C_n) = \begin{cases} 2, & \text{if } n = 3; \\ 3, & \text{if } n = 0 \pmod 4; \\ 4, & \text{otherwise.} \end{cases}$$

In addition, Calamoneri, Pelc and Petreschi [3] investigated the $\lambda_{j,k}$-numbers of trees with $j \leq k$. Chen and Lin [4], Griggs and Jin [8], and Niu [15] also studied the $L(j,k)$-labelings for $j \leq k$.

In this paper, we study the edge version of $L(j,k)$-labeling, which is defined analogously to the above $L(j,k)$-labeling problem. Let G be a graph, whose line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are adjacent in G. Let e_1, e_2 be any two edges of G. The distance between e_1 and e_2, denoted by $d(e_1, e_2)$, is defined as the distance between the corresponding two vertices in $L(G)$. That is, two edges e_1 and e_2 are adjacent (at distance one) if they meet at a common vertex; and two edges e_1 and e_2 are distance two apart if they are nonadjacent but adjacent to a common edge in G. The degree of an edge e, denoted by $d(e)$, is the number of edges adjacent to e. The $L(j,k)$-edge-labeling number of G, denoted by $\lambda'_{j,k}(G)$, is the minimum m of an m-$L(j,k)$-edge-labeling admitted by G. We assume without loss of generality that the minimum label of an $L(j,k)$-edge-labeling is 0.

The $L(j,k)$-edge-labeling is related to the $L(j,k)$-labeling. It is easy to see that $\lambda'_{j,k}(G) = \lambda_{j,k}(L(G))$, where $L(G)$ is the line graph of G. The $L(j,k)$-edge-labeling was first investigated by Georges and Mauro in [5], in which the authors determined the $\lambda'_{1,1}$-numbers and $\lambda'_{2,1}$-numbers for paths, cycles, complete graphs, Δ-regular trees for $\Delta \geq 2$, n-dimensional cubes for small n and wheels. In addition, the $L(j,k)$-edge-labeling was also studied in [4, 14]. The following theorem was proved in [4]:

Theorem 1.2 ([4]) *Let G be a simple or multiple graph and let Δ_L be the maximum degree of its line graph. Suppose $\Delta_L \geq 2$. Except the case that G is a 5-cycle and $j = k$, we have $\lambda'_{j,k}(G) \leq k\lfloor \Delta_L^2/2 \rfloor + j\Delta_L - 1$.*

The aim of this article is to investigate the $L(1,2)$-edge-labeling numbers of some graphs. These graphs have been well studied in the distance two labeling literature [6, 7, 10]. In Section 2, we determine the $\lambda'_{1,2}$-numbers for paths, cycles, complete graphs and complete multipartite graphs.

Let $T_\infty(\Delta)$ be an infinite tree with each vertex having degree Δ, which is called Δ-regular tree. Georges and Mauro [5] derived the $\lambda'_{2,1}$-numbers for $T_\infty(\Delta)$ with $\Delta \geq 3$.

Theorem 1.3 ([5]) *Suppose $T_\infty(\Delta)$ is an infinite Δ-regular tree. Then*

$$\lambda'_{2,1}(T_\infty(\Delta)) = \begin{cases} 2\Delta + 1, & \text{if } \Delta = 3 \text{ and } 4; \\ 2\Delta + 2, & \text{if } \Delta = 5; \\ 2\Delta + 3, & \text{if } \Delta \geq 6. \end{cases}$$

In Section 3, we show bounds of the $\lambda'_{1,2}$-number for $T_\infty(\Delta)$, and the bounds are sharp for $\Delta = 3$ and 4.

A wheel of order $n+1$, denoted by W_n, is a graph that contains a cycle of order n, and each vertex on the cycle is adjacent to a common vertex not on the cycle, called the hub of the wheel. The edges incident to the hub are called spokes. Georges and Mauro [5] determined the $\lambda'_{2,1}$-numbers for W_n. The authors gave the following conclusions:

Theorem 1.4 ([5]) *For $n \geq 3$,*

$$\lambda'_{2,1}(W_n)) = \begin{cases} 7, & \text{if } n = 3 \text{ and } 4; \\ 9, & \text{if } n = 5; \\ 2n-2, & \text{if } n \geq 6. \end{cases}$$

In Section 4, we get the $\lambda'_{1,2}$-numbers for W_n with $n \geq 3$.

2. Path, cycle, complete graph and complete multipartite graph

Recall the relationship between the $\lambda_{j,k}$-number and the $\lambda'_{j,k}$-number as indicated in Section 1, we can get the $\lambda'_{1,2}$-numbers for paths and cycles by Theorem 1.1.

Remark 2.1 *For $n \geq 2$,*

$$\lambda'_{1,2}(P_n) = \lambda_{1,2}(L(P_n)) = \lambda_{1,2}(P_{n-1}) = \begin{cases} 0, & \text{if } n = 2; \\ 1, & \text{if } n = 3; \\ 2, & \text{if } n = 4; \\ 3, & \text{if } n \geq 5. \end{cases}$$

Remark 2.2 *For $n \geq 3$,*

$$\lambda'_{1,2}(C_n) = \lambda_{1,2}(L(C_n)) = \lambda_{1,2}(C_n) = \begin{cases} 2, & \text{if } n = 3; \\ 3, & \text{if } n \equiv 0 \pmod 4; \\ 4, & \text{otherwise.} \end{cases}$$

We now turn to discuss the $L(1,2)$-edge-labeling numbers for complete graphs and complete multipartite graphs.

Lemma 2.3 *For a graph G with m edges, if $L(G)$ is hamiltonian, then $\lambda'_{1,2}(G) \leq m-1$.*

Proof In order to prove the result, it suffices to give an $(m-1)$-$L(1,2)$-edge-labeling for G. From the relationship between G and $L(G)$, we can assume that $E(G) = V(L(G)) = \{e_1, \ldots, e_m\}$. Since $L(G)$ is hamiltonian, there is a hamiltonian path in $L(G)$. Without loss of generality, let $e_1 \ldots e_m$ be a hamiltonian path in $L(G)$. We define a labeling function f as $f(e_i) = i-1$. It is easy to check that f is an $(m-1)$-$L(1,2)$-edge-labeling of G. Thus, Lemma 2.3 follows. □

Harary and Nash-Williams [12] discussed the hamiltonian properties of the line graph.

Lemma 2.4 ([12])

(1) If G is Eulerian, then $L(G)$ is hamiltonian.

(2) If G is hamiltonian, then $L(G)$ is hamiltonian.

(3) $L(G)$ is hamiltonian if and only if there is a tour in G which includes at least one end-vertex of each edge of G.

Theorem 2.5 *If G is a complete graph or complete multipartite graph with $|E(G)| = m$, then $\lambda'_{1,2}(G) = m - 1$.*

Proof Since G is a complete graph or complete multipartite graph, the edges of G are pairwisely at most distance two apart. So, the labels assigned to the edges of G must be distinct. Thus, $\lambda'_{1,2}(G) \geq m - 1$. On the other hand, if G is a complete graph, then G is hamiltonian; if G is a complete multipartite graph, then it is not difficult to find a tour in G which includes at least one end-vertex of each edge of G. Therefore, the line graph of G is hamiltonian by Lemma 2.4. So, $\lambda'_{1,2}(G) \leq m - 1$ by Lemma 2.3. Thus, Theorem 2.5 holds. □

3. Infinite Δ-regular tree

For two nonnegative integers a and b with $a \leq b$, let $[a, b]$ denote the set $\{a, a+1, \ldots, b\}$. Two sets A and B are called 2-separated, if for every $x \in A$ and $y \in B$, it holds that $|x - y| \geq 2$. For a vertex u, let $N(u)$ denote the set of vertices adjacent to u.

For the infinite Δ-regular tree $T_\infty(\Delta)$, by Remark 2.1, we have $\lambda'_{1,2}(T_\infty(2)) = 3$. Hence, let $\Delta \geq 3$. The following result gives a lower bound for $\lambda'_{1,2}(T_\infty(\Delta))$:

Lemma 3.1 *For $\Delta \geq 3$, $\lambda'_{1,2}(T_\infty(\Delta)) \geq 2\Delta + 1$.*

Proof Suppose $\lambda'_{1,2}(T_\infty(\Delta)) \geq 2\Delta + 1$ is false. Then $\lambda'_{1,2}(T_\infty(\Delta)) \leq 2\Delta$. Let f be a 2Δ-$L(1,2)$-edge-labeling for $T_\infty(\Delta)$. Let uv be any edge in $T_\infty(\Delta)$. Since an infinite Δ-regular tree is edge-transitive, we may assume that $f(uv) = 0$.

Let $A = \{f(ut) | t \in N(u) \setminus \{v\}\}$ and $B = \{f(vt) | t \in N(v) \setminus \{u\}\}$. We have $|A| = |B| = \Delta - 1$ and $A \cap B = \varnothing$. Moreover, A and B are 2-separated and $A \cup B$ must be contained in $[1, 2\Delta]$. So, at least one of A and B consists of consecutive integers. Without loss of generality, assume A consists of consecutive integers. Then A and B can only be the following cases:

Case 1 $A = [1, \Delta - 1]$ and $B = [\Delta + 1, 2\Delta - 1]$.

Case 2 $A = [1, \Delta - 1]$ and $B = [\Delta + 2, 2\Delta]$.

Case 3 $A = [2, \Delta]$ and $B = [\Delta + 2, 2\Delta]$.

Case 4 $A = [1, \Delta - 1]$ and $B = [\Delta + 1, 2\Delta] \setminus \{\Delta + k\}$ for some $1 < k < \Delta$.

Case 5 $A = [k, \Delta + k - 2]$ and $B = [1, k - 2] \cup [\Delta + k, 2\Delta]$ for some $2 < k \leq \Delta$.

Case 6 $A = [\Delta + 2, 2\Delta]$ and $B = [1, k - 1] \cup [k + 1, \Delta]$ for some $1 < k < \Delta$.

In the following, in order to get contradictions in the above six cases, we distinguish them into $\Delta > 3$ and $\Delta = 3$. Assume $\Delta > 3$. Let x $(\neq v)$ be a vertex adjacent to u, and y $(\neq u)$ be a vertex adjacent to v. Let F_x (or F_y) be the set of forbidden labels for the $\Delta - 1$ edges incident to x (or y) except ux (or vy).

Case 1 Since $B = [\Delta+1, 2\Delta-1]$, we may assume that $f(vy) = \Delta+2$. Then $F_y = [\Delta, 2\Delta] \cup \{0, 1\}$ by the distance condition. Note that $|F_y| = \Delta + 3$. So, there are only $(2\Delta+1) - (\Delta+3) = \Delta - 2$ labels for those $\Delta - 1$ edges incident to y, a contradiction.

Case 2 Since $B = [\Delta+2, 2\Delta]$, we may assume that $f(vy) = \Delta+3$. Then $F_y = [\Delta+1, 2\Delta] \cup \{0, 1\}$ by the distance condition. That is, $\{f(yt)|t \in N(y) \setminus \{v\}\} = [2, \Delta]$. Let z $(\neq v)$ be a vertex adjacent to y. Assume $f(yz) = 2$. The forbidden labels for remaining $\Delta - 1$ edges incident to z are in $[2, \Delta + 4]$. Since $|[2, \Delta + 4]| = \Delta + 3$, there are only $\Delta - 2$ labels for these $\Delta - 1$ edges, a contradiction.

Case 3 Since $B = [\Delta + 2, 2\Delta]$, the argument is similar to Case 2.

Case 4 Since $B = [\Delta + 1, 2\Delta] \setminus \{\Delta + k\}$, we may assume that $f(vy) = 2\Delta$. Then $F_y = [\Delta, 2\Delta] \cup \{0, 1\}$ by the distance condition. Since $|F_y| = \Delta + 3$, there are only $\Delta - 2$ labels for those $\Delta - 1$ edges incident to y, a contradiction.

Case 5 Since $A = [k, \Delta + k - 2]$, we may assume that $f(ux) = k + 1$. Then $F_x = [k - 1, \Delta + k - 1] \cup \{0, 1\}$ by the distance condition. Note that $2 < k \leq \Delta$. We have $|F_x \cap [0, 2\Delta]| = \Delta + 3$. So, there are only $\Delta - 2$ labels for those $\Delta - 1$ edges incident to x, a contradiction.

Case 6 Since $A = [\Delta + 2, 2\Delta]$, the argument is similar to Case 2.

We now turn to $\Delta = 3$. Let f be a 6-$L(1,2)$-edge-labeling for $T_\infty(3)$. The above cases correspond to the following cases: (1) $A = \{1, 2\}$ and $B = \{4, 5\}, \{5, 6\}$ or $\{4, 6\}$; (2) $A = \{2, 3\}$ and $B = \{5, 6\}$; (3) $A = \{3, 4\}$ and $B = \{1, 6\}$; (4) $A = \{5, 6\}$ and $B = \{1, 3\}$.

Observation Suppose e_1e_2 is a path of length two in $T_\infty(3)$ with $f(e_1) = a$ and $f(e_2) = b$. Due to the distance condition, the set $\{a, b\}$ cannot be $\{1, 4\}, \{1, 5\}$ or $\{2, 5\}$.

For (1), the case of $A = \{1, 2\}$, we may assume that $f(ux) = 1$. Then $\{f(xt)|t \in N(x) \setminus \{u\}\} \cap \{4, 5\} = \varnothing$ by Observation. So, due to the distance condition, we have $\{f(xt)|t \in N(x) \setminus \{u\}\} \subset \{6\}$. It is a contradiction since $|\{f(xt)|t \in N(x) \setminus \{u\}\}| = 2$. With the similar argument, we can prove that $A \neq \{2, 3\}$ and $B \neq \{1, 3\}$, which are corresponding to (2) and (4), respectively. For (3), the case of $A = \{3, 4\}$, we may assume that $f(ux) = 4$. Then the labels of the remaining two edges incident to x are 5 and 6. Without loss of generality, let w be a vertex adjacent to x with $f(xw) = 5$. We have $\{f(wt)|t \in N(w) \setminus \{x\}\} \cap \{1, 2\} = \varnothing$ by Observation. So, due to the distance condition, it follows that $\{f(wt)|t \in N(w) \setminus \{x\}\} \subset \{0\}$. It is a contradiction by $|\{f(wt)|t \in N(w) \setminus \{x\}\}| = 2$. Therefore, the lemma holds for $\Delta = 3$. This completes the proof of Lemma 3.1. □

We now turn to the upper bound of $\lambda'_{1,2}$-number for $T_\infty(\Delta)$. If we fix an edge uv in $T_\infty(\Delta)$, then $T_\infty(\Delta) \setminus \{uv\}$ has two rooted trees T_u and T_v, rooted at u and v, respectively. For two vertices x and y in $T_\infty(\Delta)$, the distance between x and y, denoted by $d(x,y)$, is the length of the unique (x,y)-path in $T_\infty(\Delta)$. For any edge $e = xy$ in T_u, define $d(xy,u) = \min\{d(x,u), d(y,u)\}$. If $d(xy,u) = k$, then we call xy a k^{th}-generation edge descended from u. Similarly, we define the k^{th}-generation edge descended from v. For two adjacent edges e_1 and e_2 in T_u (or T_v), if e_1 is a k^{th}-generation edge and e_2 is a $(k+1)^{th}$-generation edge, then we say that e_1 is the father of e_2 and e_2 is a child of e_1.

Lemma 3.2 *For $\Delta \geq 3$, $\lambda'_{1,2}(T_\infty(\Delta)) \leq 2\Delta + 4$.*

Proof To prove the result, it suffices to produce a $(2\Delta+4)$-$L(1,2)$-edge-labeling f for $T_\infty(\Delta)$.

Let $X_0 = [0, \Delta+1]$ and $X_1 = [\Delta+3, 2\Delta+4]$. We note that X_0 and X_1 are 2-separated and $|X_0| = |X_1| = \Delta + 2$.

Fix an edge uv and label it by 0. Assign the $\Delta - 1$ edges incident to u expect uv by the distinct labels in $X_0 \setminus \{0\}$, and assign the $\Delta - 1$ incident to v expect uv by the distinct labels in X_1. Now assume that, for $0 \leq h \leq k$, the h^{th}-generation edges descended from u are labeled entirely from X_i, where $i \equiv h \pmod 2$, the h^{th}-generation edges descended from v are labeled entirely from X_j, where $j \equiv h+1 \pmod 2$. In the following, we prove that the $(k+1)^{th}$-generation edges descended from u or v can be labeled.

Without loss of generality, let e be a k^{th}-generation edge descended from u with label $f(e) \in X_i$, where $i \equiv k \pmod 2$, and let e' be the father of e with $f(e') \in X_j$, where $j \equiv k+1 \pmod 2$. We assign labels to the $\Delta - 1$ children of e from $W = X_j \setminus \{f(e')-1, f(e'), f(e')+1\}$. Since $|W| \geq \Delta - 1$, such a labeling can be achieved. Because the distance between any two $(k+1)^{th}$-generation edges descended from u with distinct parents is greater than two, all of the $(k+1)^{th}$-generation edges from u can be labeled in this manner.

A similar argument may be used to prove that the labeling works for T_v. Then $2\Delta + 4$ is an upper bound for $T_\infty(\Delta)$. So, Lemma 3.2 follows. □

In the following, we assign labels to the edges of $T_\infty(\Delta)$ in another manner and get another upper bound.

Lemma 3.3 *For $\Delta \geq 3$, $\lambda'_{1,2}(T_\infty(\Delta)) \leq 3\Delta - 2$.*

Proof In order to prove the result, it suffices to produce a $(3\Delta-2)$-$L(1,2)$-edge-labeling f for $T_\infty(\Delta)$.

Let $X_0 = [0, \Delta-2]$, $X_1 = [\Delta, 2\Delta-2]$ and $X_2 = [2\Delta, 3\Delta-2]$. We note that the sets X_i are pairwisely 2-separated and $|X_i| = \Delta - 1$ ($i \in \{0,1,2\}$).

Fix an edge uv and label it by $3\Delta - 2$. For the h^{th}-generation edges descended from u, which are in T_u, we assign them by labels in X_i, where $i \equiv h \pmod 3$. For the h^{th}-generation edges descended from v, which are in T_v, we assign them in the following manner: if $h \equiv 0 \pmod 3$, then we assign them by labels in X_1; if $h \equiv 1 \pmod 3$, then we assign them by labels in X_0; if $h \equiv 2 \pmod 3$, then we assign them by labels in X_2.

It is easy to check that the labeling f satisfies the constraints of distance, and f is a

$(3\Delta-2)$-$L(1,2)$-edge-labeling for $T_\infty(\Delta)$. Hence, Lemma 3.3 holds. □

By Lemmas 3.1, 3.2 and 3.3, we obtain:

Theorem 3.4 *For $\Delta \geq 3$,*

$$2\Delta+1 \leq \lambda'_{1,2}(T_\infty(\Delta)) \leq \begin{cases} 3\Delta-2, & \text{if } \Delta \leq 6; \\ 2\Delta+4, & \text{if } \Delta > 6. \end{cases}$$

It follows from Theorem 3.4 that $\lambda'_{1,2}(T_\infty(3)) = 7$ and $9 \leq \lambda'_{1,2}(T_\infty(4)) \leq 10$. Using a computer aided method to exhaust all possibilities, we can determine that there does not exist a 9-$L(1,2)$-edge-labeling for $T_\infty(4)$. Hence, we have:

Corollary 3.5 *For $\Delta = 3$ and 4, we have $\lambda'_{1,2}(T_\infty(3)) = 7$ and $\lambda'_{1,2}(T_\infty(4)) = 10$.*

4. The wheels

Recall the definition of a wheel W_n in Section 1. We denote the vertices of C_n outside of W_n by $u_0, u_1, \ldots, u_{n-1}$ and the edges $\{u_iu_{i+1}\}$ by e_i $(0 \leq i \leq n-1$ and $u_n = u_0)$. Additionally, we denote the hub by w and the spoke wu_i by s_i $(0 \leq i \leq n-1)$. For any m-$L(1,2)$-edge-labeling function f of W_n, let $A = \{f(e_i) | 0 \leq i \leq n-1\}$, $B = \{f(s_i) | 0 \leq i \leq n-1\}$, l_i and r be the cardinality of $\{e \in E(W_n) | f(e) = i\}$ and $\{i \in [0,m] | l_i \geq 2\}$, respectively. The following results give the $\lambda'_{1,2}$-numbers for W_n with $n \in \{3,4,5,6\}$.

Theorem 4.1 *Suppose W_n is a wheel. Then*

$$\lambda'_{1,2}(W_n) = \begin{cases} 5, & \text{if } n = 3; \\ 7, & \text{if } n = 4; \\ 9, & \text{if } n = 5; \\ 11, & \text{if } n = 6. \end{cases}$$

Proof If $n = 3$, then $W_3 = K_4$. The result follows by Theorem 2.5.

It is easy to see that W_4 and W_5 have edge diameter two and $|E(W_4)| = 8$ and $|E(W_5)| = 10$. So, $\lambda'_{1,2}(W_4) \geq 7$ and $\lambda'_{1,2}(W_5) \geq 9$. In Figure 1, we provide a 7- and 9-$L(1,2)$-edge-labelings for W_4 and W_5, respectively. Hence, Theorem holds for $n = 4$ and 5.

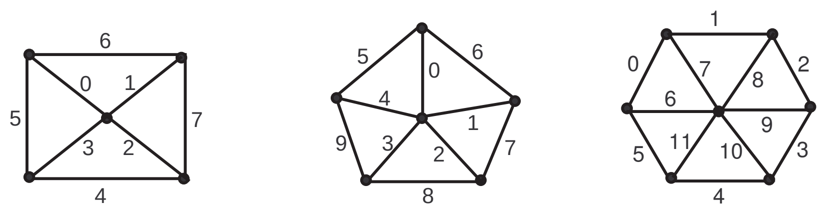

Figure 1 Optimal $L(1,2)$-edge-labelings for W_4, W_5 and W_6, respectively

When $n = 6$, we provide a 11-$L(1,2)$-edge-labeling for W_6 in Figure 1. Then $\lambda'_{1,2}(W_6) \leq 11$. On the other hand, suppose $\lambda'_{1,2}(W_6) \leq 10$. Let f be a 10-$L(1,2)$-edge-labeling for W_6. Due to the distance condition, we have $l_i \leq 2$ and $r \leq 3$. Moreover, if $l_i = 2$, then $l_{i-1} = l_{i+1} = 0$. So,

$\sum_{i=0}^{10} l_i \le 11$, contradicting $|E(W_6)| = 12$. Therefore, $\lambda'_{1,2}(W_6) = 11$. This completes Theorem 4.1. □

We now consider $n \ge 7$. In the following, we first investigate the properties of the labels on C_n.

Lemma 4.2 *Suppose f is an $L(1,2)$-edge-labeling for W_n with $n \ge 7$. Then the labeling function f satisfies the following properties:*

(1) $l_i \le \lfloor \frac{n}{3} \rfloor$.

(2) If the label k is assigned to two different edges on C_n, then $k \pm 1 \notin B$.

(3) If $|A| = 4$, then $r \ge 3$; if $|A| = 5$ and $n \ge 10$, then $r \ge 3$.

Proof (1) Since any two spokes are adjacent and the distance between any spoke and any edge on C_n is at most two, the label used on spokes can appear only once. Therefore, if $l_i \ge 2$, then the label i must be assigned to the edges on C_n. So, $l_i \le \lfloor \frac{n}{3} \rfloor$.

(2) Suppose two edges on C_n have label k, then these two edges must be nonadjacent, thus any spoke cannot be adjacent to both of these two edges. Therefore, labels $k+1$ and $k-1$ cannot be used on spokes. That is $k \pm 1 \notin B$.

(3) Suppose $r \le 2$. If $|A| = 4$, then $\sum_{i \in A} l_i \le 2 \cdot \lfloor \frac{n}{3} \rfloor + 2 < n$ for $n > 6$. If $|A| = 5$, then $\sum_{i \in A} l_i \le 2 \cdot \lfloor \frac{n}{3} \rfloor + 3 < n$ for $n > 9$. These contradict $\sum_{i \in A} l_i = n$. Hence, (3) follows. □

Next, we give the lower bounds of $\lambda'_{1,2}$-number for W_n.

Lemma 4.3 *For $n \ge 7$, $\lambda'_{1,2}(W_n) \ge n + 4$.*

Proof Suppose to the contrary, $\lambda'_{1,2}(W_n) \le n + 3$. Let f be an $(n+3)$-$L(1,2)$-edge-labeling for W_n. Recall the description of A and B, we have $A \cap B = \varnothing$ and $|B| = n$ by the distance condition. So, $|A| \le 4$ as there are $n+4$ labels. Obviously, $|A| \ge 3$. In the following, we consider the following two cases:

Case 1 $|A| = 3$.

Suppose $A = \{a, b, c\}$ with $a < b < c$. Then the labels a, b and c must be assigned to the edges on C_n in a cyclic order. Due to the distance condition, $b \ne a + 1$ and $c \ne b + 1$. So, the labels in $F = \{a-1, a, a+1, b, b+1, c, c+1\}$ are distinct. By (2) of Lemma 4.2, $F \cap B = \varnothing$. Then $|B| \le n - 1$, since there are $n + 4$ labels and $|F \cap [0, n+3]| \ge 5$, a contradiction.

Case 2 $|A| = 4$.

Suppose $A = \{a, b, c, d\}$. This implies that $l_k \ge 1$ for all $k \in [0, n+3]$. By (3) of Lemma 4.2, there exist at least three labels in A, say a, b and c, such that $l_i \ge 2$ $(i = a, b, c)$. If $l_d \ge 2$, then at least one of labels in $[0, n+3]$ cannot be assigned to the spokes by (2) of Lemma 4.2, a contradiction. Hence, $l_d = 1$. Then the labels in the set $F = \{a-1, a, a+1, b-1, b, b+1, c-1, c, c+1, d\}$ are forbidden for the spokes. That is, $F \cap [0, n+3] = \{a, b, c, d\}$. So, the labels a, b and c are consecutive integers. Then $A = \{0, 1, 2, 3\}$ and $d = 3$ (i.e., $l_3 = 1$), or $A = \{n, n+1, n+2, n+3\}$ and $d = n$ (i.e., $l_n = 1$). By symmetry, we only consider the case of

$A = \{0,1,2,3\}$. Since $l_3 = 1$ and $n \geq 6$, it is impossible to label the other $n-1$ edges on C_n by $0,1$ and 2. Therefore, $|A| \neq 4$.

By the above two cases, $|A|$ cannot be 3 or 4. It is a contradiction and the assumption is false. Thus, Lemma 4.3 holds. □

Lemma 4.4 *For $n \equiv 2 \pmod 4$ and $n \geq 10$, $\lambda'_{1,2}(W_n) \geq n+5$.*

Proof Suppose to the contrary, $\lambda'_{1,2}(W_n) \leq n+4$. Let f be an $(n+4)$-$L(1,2)$-edge-labeling for W_n. Since there are $n+5$ labels, we have $3 \leq |A| \leq 5$. With similar arguments in Case 1 of Lemma 4.3, we can prove $|A| \neq 3$. So, we consider the following two cases:

Case 1 $|A| = 4$.

By (3) of Lemma 4.2, we have $r \geq 3$. Since f is an $(n+4)$-$L(1,2)$-labeling function, there are at most 5 labels forbidden to the spokes. So, by (2) of Lemma 4.2, A can be the following possibilities: (1) $A = \{0,1,2,3\}$ or $\{n+1, n+2, n+3, n+4\}$; (2) $A = \{0,1,2,m\}$ for some $m \in [4, n+4]$ and $l_m = 1$; (3) $|A| = \{m, n+2, n+3, n+4\}$ for some $m \in [0,n]$ and $l_m = 1$. For (1), it contradicts $\lambda'_{1,2}(C_n) = 4$ when $n \equiv 2 \pmod 4$ by Theorem 2.2. For (2), since $l_m = 1$ and $n \geq 10$, it is impossible to label the other $n-1$ edges by $0,1$ and 2. By the symmetry of labels, A cannot be the pattern of (3). Hence, $|A| \neq 4$.

Case 2 $|A| = 5$.

By (3) of Lemma 4.2, we also have $r \geq 3$. Suppose $A = \{a,b,c,d,e\}$ and $l_i \geq 2$ for $i \in \{a,b,c\}$. Without loss of generality, assume $a < b < c$. Then $a+1 = b$ or $b+1 = c$. Otherwise, the labels in $F = \{a-1, a, a+1, b-1, b, b+1, c-1, c, c+1\}$ are forbidden for the spokes. It follows that $|F \cap [0, n+4]| \geq 6$, a contradiction. Then we may assume that $a+1 = b$. By (2) of Lemma 4.2, we know $r \neq 5$. So, $r = 3$ or $r = 4$.

If $r = 3$, then $l_d = l_e = 1$. Without loss of generality, assume $f(e_0) = d$ and $f(e_j) = e$. Consider the path $P = e_1e_2\ldots e_{j-1}$ and $P' = e_{j+1}e_{j+2}\ldots e_{n-1}$. Let l and l' be the length of P and P', respectively. Since $n \geq 10$, at least one of l and l' is larger than 3. Assume $l \geq 4$. So, it is impossible to label the edges on P by the labels $a, a+1$ and c. Hence, $r \neq 3$.

If $r = 4$, then we may assume that $l_d \geq 2$ and $l_e = 1$. By (2) of Lemma 4.2, since $l_k \geq 1$ for each $k \in [0, n+4]$, A must be $\{0,1,2,3,4\}$ and $l_4 = 1$, or $\{n, n+1, n+2, n+3, n+4\}$ and $l_n = 1$. By symmetry, we only consider the case of $A = \{0,1,2,3,4\}$. As $l_4 = 1$, the labels $0,1,2$ and 3 must appear on the other $n-1$ edges of C_n in a cyclic order. Since $n \equiv 2 \pmod 4$ and $n \geq 10$, this is impossible. Hence, $r \neq 4$.

By the above cases, $|A|$ cannot be $3, 4$ or 5. It is a contradiction and the assumption is false. Lemma 4.4 follows. □

In the following, we get the $\lambda'_{1,2}$-numbers for W_n with $n \geq 7$:

Lemma 4.5 *For $n \geq 7$,*

$$\lambda'_{1,2}(W_n) = \begin{cases} n+5, & \text{if } n \equiv 2 \pmod 4; \\ n+4, & \text{otherwise.} \end{cases}$$

Proof By Lemma 4.3 and 4.4, it suffices to produce $L(1,2)$-edge-labelings for W_n with different values of n.

Case 1 $n \equiv 0 \pmod 4$.

The labeling function f on $E(W_n)$ to the set $[0, n+4]$ is defined as:

$$f(s_i) = 5 + i,\ f(e_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 4; \\ 1, & \text{if } i \equiv 1 \pmod 4; \\ 2, & \text{if } i \equiv 2 \pmod 4; \\ 3, & \text{if } i \equiv 3 \pmod 4, \end{cases} \text{ for all } 0 \leq i \leq n-1.$$

It is straightforward to check that f is an $(n+4)$-$L(1,2)$-edge-labeling. Thus $\lambda'_{1,2}(W_n) = n+4$ when $n \equiv 0 \pmod 4$.

Case 2 $n \equiv 1 \pmod 4$.

The labeling function f on $E(W_n)$ to the set $[0, n+4]$ is defined as:

$f(s_i) = 5 + i$ for $0 \leq i \leq n-1$;

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 4; \\ 1, & \text{if } i \equiv 1 \pmod 4; \\ 2, & \text{if } i \equiv 2 \pmod 4; \\ 3, & \text{if } i \equiv 3 \pmod 4, \end{cases} \text{ for } 0 \leq i \leq n-2; \text{ and } f(e_{n-1}) = 4.$$

It is straightforward to check that f is an $(n+4)$-$L(1,2)$-edge-labeling. Thus $\lambda'_{1,2}(W_n) = n+4$ when $n \equiv 1 \pmod 4$.

Case 3 $n \equiv 2 \pmod 4$.

By Remark 2.2, $\lambda'_{1,2}(C_n) = 4$ when $n \equiv 2 \pmod 4$. So, we can define $f(e_i)$ with the labels in $[0,4]$. And then we define $f(s_i) \in [6, n+5]$ for all $i \in [0, n-1]$. It is obvious that f is an $(n+5)$-$L(1,2)$-edge-labeling for W_n. Thus, $\lambda'_{1,2}(W_n) = n+5$ when $n \equiv 2 \pmod 4$.

Case 4 $n \equiv 3 \pmod 4$.

The labeling function f on $E(W_n)$ to the set $[0, n+4]$ is defined as:

$f(s_{n-1}) = 5$, $f(s_i) = 6 + i$ for $0 \leq i \leq n-2$;

$$f(e_i) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 4; \\ 3, & \text{if } i \equiv 1 \pmod 4; \\ 0, & \text{if } i \equiv 2 \pmod 4; \\ 1, & \text{if } i \equiv 3 \pmod 4, \end{cases} \text{ for } 0 \leq i \leq n-1 \text{ and } i \neq n-2; \text{and } f(e_{n-2}) = 4.$$

It is straightforward to check that f is an $(n+4)$-$L(1,2)$-edge-labeling. Thus $\lambda'_{1,2}(W_n) = n+4$ when $n \equiv 3 \pmod 4$.

By the cases above, the proof is completed. □

By Lemmas 4.1 and 4.5, we obtain:

Theorem 4.6 *For $n \geq 3$,*

$$\lambda'_{1,2}(W_n) = \begin{cases} 5, & \text{if } n = 3; \\ 7, & \text{if } n = 4; \\ n+5, & \text{if } n \equiv 2 \pmod 4; \\ n+4, & \text{otherwise.} \end{cases}$$

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