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Second Order Derivative Estimates of the Solutions of a Class of Monge-Ampère Equations

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Abstract In this paper, we consider a class of Monge-Ampère equations in relative differential geometry. Given these equations with zero boundary values in a smooth strictly convex bounded domain, we obtain second order derivative estimates of the convex solutions.

Keywords Monge-Ampère equation; derivative estimate; relative differential geometry.

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1. Introduction

In equiaffine differential geometry and relative differential geometry, Li-Simon-Chen [1] and Wu-Zhao [2] considered the following equation

$$\begin{cases} \det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = S(x)(-u)^{-k} & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth strictly convex bounded domain in $\mathbb{R}^n$, k is a positive constant with $k > 1$, and $S(x) \in C^\infty(\Omega) \cap C^2(\bar{\Omega})$ with $S_n > 0$. In particular for $k = n + 2$ and $S(x) = \text{const}$, equation (1.1) is the well-known hyperbolic affine hypersphere equation.

Cheng-Yau [3] showed that (1.1) has a convex solution $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$, and the uniqueness follows from the maximum principal. Moreover, Lazer-McKenna [4] showed that the unique convex solution u satisfies

$$\frac{1}{C_0}\, d(x)^{\frac{n+1}{n+k}} \le -u(x) \le C_0\, d(x)^{\frac{n+1}{n+k}}, \tag{1.2}$$

where $d(x) := \text{dist}(x, \partial\Omega)$, and C_0 is a positive constant. In the following, we denote by $u_i, u_{ij}, u_{ijk}, \ldots$, the derivatives of u with respect to x, (u^{ij}) the inverse matrix of (u_{ij}). The main result of our paper is

Theorem 1.1 *The convex solution of (1.1) satisfies*

$$|u_{ij}| \le C\, d(x)^{\frac{n+1}{n+k}-2}, \quad 1 \le i, j \le n, \tag{1.3}$$

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where C is a constant depending only on Ω, n, k and $S(x)$.

For $k = n+2$ and $S(x) = \text{const}$, formula (1.3) can be seen in Loewner-Nirenberg [5] and Wu [6]. Second order estimates (1.3) can be used to describe the asymptotic behaviors of relative hypersurfaces of hyperbolic type, for details see [2] and [6]. On the other hand, (1.3) gives another proof of Lemma 3 in [2].

2. A barrier function

We consider the function

$$w = -c(R^2 - \sum_{1\le i\le n}(x_i - \bar{x}_i)^2)^\beta, \quad 0 < \beta < 1 \tag{2.1}$$

defined in the ball $\{(x_1, x_2, \dots, x_n) \mid \sum(x_i - \bar{x}_i)^2 \le R^2\}$. A direct calculation gives

$$w_i = 2c\beta(R^2 - (x_i - \bar{x}_i)^2)^{\beta-1}(x_i - \bar{x}_i), \tag{2.2}$$

$$\begin{aligned} w_{ij} &= 2c\beta(R^2 - (\sum x_i - \bar{x}_i)^2)^{\beta-1}[\delta_{ij} + 2(1-\beta)\frac{(x_i - \bar{x}_i)(x_j - \bar{x}_j)}{R^2 - \sum(x_i - \bar{x}_i)^2}] \\ &= 2\beta c^{\frac{1}{\beta}}(-w)^{\frac{\beta-1}{\beta}}[\delta_{ij} + 2(1-\beta)c^{\frac{1}{\beta}}(-w)^{-\frac{1}{\beta}}(x_i - \bar{x}_i)(x_j - \bar{x}_j)]. \end{aligned} \tag{2.3}$$

Hence

$$\begin{aligned} \det(w_{ij}) &= 2^n\beta^n c^{\frac{n}{\beta}}(-w)^{\frac{(\beta-1)n}{\beta}}[1 + 2(1-\beta)c^{\frac{1}{\beta}}(-w)^{-\frac{1}{\beta}}\sum(x_i - \bar{x}_i)^2] \\ &= 2^n\beta^n c^{\frac{n}{\beta}}(-w)^{\frac{(\beta-1)n}{\beta}}[1 + 2(1-\beta)c^{\frac{1}{\beta}}(-w)^{-\frac{1}{\beta}}(R^2 - (-w)^{\frac{1}{\beta}}c^{-\frac{1}{\beta}})] \\ &= 2^n\beta^n c^{\frac{n}{\beta}}(-w)^{n-\frac{n+1}{\beta}}[(2\beta - 1)(-w)^{\frac{1}{\beta}} + 2R^2(1-\beta)c^{\frac{1}{\beta}}]. \end{aligned} \tag{2.4}$$

For $2\beta - 1 \ge 0$, we have

$$\begin{aligned} \det(w_{ij}) &\le 2^n\beta^n c^{\frac{n}{\beta}}(-w)^{n-\frac{n+1}{\beta}}[(2\beta - 1)R^2c^{\frac{1}{\beta}} + 2R^2(1-\beta)c^{\frac{1}{\beta}}] \\ &= 2^nR^2\beta^n c^{\frac{n+1}{\beta}}(-w)^{n-\frac{n+1}{\beta}} \\ &\le 2^nR^2c^{\frac{n+1}{\beta}}(-w)^{n-\frac{n+1}{\beta}}. \end{aligned} \tag{2.5}$$

For $2\beta - 1 \le 0$, we have

$$\begin{aligned} \det(w_{ij}) &\le 2^{n+1}R^2(1-\beta)\beta^n c^{\frac{n+1}{\beta}}(-w)^{n-\frac{n+1}{\beta}} \\ &\le 2R^2c^{\frac{n+1}{\beta}}(-w)^{n-\frac{n+1}{\beta}}. \end{aligned} \tag{2.6}$$

Let $S(x)$ be the function as in (1.1), and $k = \frac{n+1}{\beta} - n$. Now we choose the constant c as follows

$$c = (2^{-n}R^{-2} \cdot \min_{x\in\bar{\Omega}} S(x))^{\frac{1}{n+k}}. \tag{2.7}$$

Then from (2.5) and (2.6), we know

$$\det(w_{ij}) \le S(x)(-w)^{-k}. \tag{2.8}$$

Next we give a comparison result in [4].

Lemma 2.1 ([4]) *Let Ω be a bounded convex domain, and let $v_k \in C^2(\Omega) \cap C(\bar{\Omega})$ for $k = 1, 2$.*

Let $f(x,\xi)$ be defined for $x \in \Omega$ and ξ in some interval containing the ranges of v_1 and v_2 and assume that $f(x,\xi)$ is strictly increasing in ξ for all $x \in \Omega$. If

(1) the matrix $((v_1)_{ij})$ is positive definite in Ω,

(2) $\det((v_1)_{ij}) \geq f(x, v_1)$, $\forall x \in \Omega$,

(3) $\det((v_2)_{ij}) \leq f(x, v_2)$, $\forall x \in \Omega$,

(4) $v_1 \leq v_2$, $\forall x \in \partial\Omega$,

then $v_1 \leq v_2$, $\forall x \in \Omega$.

3. Second order derivative estimates

Proof We divide two steps to prove Theorem 1.1, and follow the calculations as in Loewner-Nirenberg [5] and Pogorelov [7].

Step 1. For any point $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \Omega$, set $d_0 = \mathrm{dist}(x^0, \partial\Omega)$. There is a number δ_0 such that for $\delta \leq \delta_0$ the domain Ω_δ consisting of the set of points in Ω whose distance to the boundary is at least δ has a smooth, strictly convex boundary.

Let $z = \sigma(x)$ be the equation of the tangent hyperplane to the graph $u(x)$ at the point $(x^0, u(x^0))$. We claim that there is a positive constant c_0 such that

$$\max_{\partial\Omega_\delta} \sigma(x) \leq -c_0 \cdot d_0^{\frac{n+1}{n+k}}, \quad \text{for } \delta \leq \min\{\delta_0, d_0/2\}. \tag{3.1}$$

To see this, let Q be the boundary point of Ω_δ where σ takes its maximum. For $\delta \leq \delta_0$, there is a fixed positive number R_0 independent of δ such that there is a closed disc D in $\bar{\Omega}_\delta$ of radius R_0 touching $\partial\Omega_\delta$ at Q. We may suppose that Q is the origin and that the inner normal to $\partial\Omega_\delta$ at Q has the direction $(0, 0, \dots, 1)$. Hence the disc D has center $(0, 0, \dots, R_0)$.

Let w be the function

$$\begin{aligned} w &= -c\Big(R_0^2 - \sum_{1\leq i\leq n-1} x_i^2 - (x_n - R_0)^2\Big)^{\frac{n+1}{n+k}} \\ &= -c\Big(-\sum_{1\leq i\leq n} x_i^2 + 2R_0 x_n\Big)^{\frac{n+1}{n+k}}. \end{aligned} \tag{3.2}$$

From Section 2, we can choose the constant c such that

$$\begin{cases} \det(w_{ij}) \leq S(x)(-w)^{-k} & \text{in } D, \\ w = 0 \quad \text{on } \partial D. \end{cases}$$

By Lemma 2.1 we conclude $\sigma \leq u \leq w$. Since the maximum of σ on $\bar{D}$ occurs at Q, we see that the tangent hyperplane has the form $\sigma = u(x_0) + u_n(x^0)(x_n - x_n^0)$. Hence

$$\sigma(0, \dots, 0, x_n) \leq u(0, \dots, 0, x_n) \leq w(0, \dots, 0, x_n) = -c(2R_0 x_n - x_n^2)^{\frac{n+1}{n+k}}. \tag{3.3}$$

Since for some constant N

$$\sigma(0, \dots, 0, x_n) \leq -c(2R_0 x_n - x_n^2)^{\frac{n+1}{n+k}} \leq -N x_n^{\frac{n+1}{n+k}}, \quad 0 \leq x_n \leq R_0, \tag{3.4}$$

we may therefore assert that $\sigma(0, \dots, 0, x_n) \leq \tau(x_n)$, where $z = \tau(x_n)$ is the equation in the (x_n, z) plane of the tangent line from the point $(x_n^0, u(x^0))$ to the curve $z = -N x_n^{\frac{n+1}{n+k}}$, furthermore, this line touches the curve at a point $(t, -Nt^{\frac{n+1}{n+k}})$ with $t \le R_0$.

The slope of $\tau(x_n)$ is $-\frac{N(n+1)}{n+k}t^{\frac{1-k}{n+k}}$ and therefore

$$\sigma(Q) \le \tau(0) = -Nt^{\frac{n+1}{n+k}} + \frac{N(n+1)}{n+k}t^{\frac{1-k}{n+k}} \cdot t = -N\frac{k-1}{n+k}t^{\frac{n+1}{n+k}}. \tag{3.5}$$

We wish to find a lower bound for t. From the definition of $\tau(x_n)$, we have

$$-\frac{N(n+1)}{n+k}t^{\frac{1-k}{n+k}} = \frac{u(x^0) + Nt^{\frac{n+1}{n+k}}}{x_n^0 - t}, \tag{3.6}$$

or

$$N\frac{k-1}{n+k} \cdot t + u(x^0) \cdot t^{\frac{k-1}{n+k}} + N\frac{n+1}{n+k}x_n^0 = 0. \tag{3.7}$$

Put $s = t^{-1/(n+k)}$, then from (3.7) we get

$$N\frac{n+1}{n+k}x_n^0 \cdot s^{n+k} + u(x^0) \cdot s^{n+1} + N\frac{k-1}{n+k} = 0. \tag{3.8}$$

Consider a function

$$f(y) = N\frac{n+1}{n+k}x_n^0 \cdot y^{n+k} + u(x^0) \cdot y^{n+1} + N\frac{k-1}{n+k}, \tag{3.9}$$

and a point

$$s^* = [\frac{-(n+k)u(x^0)}{N(n+1)x_n^0}]^{\frac{1}{k-1}}. \tag{3.10}$$

Then $f(s^*) = N\frac{k-1}{n+k} > 0$, and the derivative of $f(y)$ is $f'(y) = N(n+1)x_n^0 \cdot y^{n+k-1} + (n+1)u(x^0) \cdot y^n$. Obviously for $y > s^*$, $f'(y) > 0$. It follows that $s \le s^*$, and hence

$$t = s^{-(n+k)} \ge (s^*)^{-(n+k)} = (\frac{-(n+k)u(x^0)}{N(n+1)x_n^0})^{\frac{n+k}{1-k}}. \tag{3.11}$$

Since Ω_δ is convex, we get

$$x_n^0 \ge \operatorname{dist}(x^0, \partial\Omega_\delta) \ge d_0/2. \tag{3.12}$$

From (1.2), we get

$$-u(x^0) \le C_0 \cdot d_0^{\frac{n+1}{n+k}}. \tag{3.13}$$

Combining (3.5) and (3.11)-(3.13), we get

$$\begin{aligned}\sigma(Q) &\le -N\frac{k-1}{n+k}t^{\frac{n+1}{n+k}} \le -N\frac{k-1}{n+k}(\frac{-(n+k)u(x^0)}{N(n+1)x_n^0})^{\frac{n+1}{1-k}} \\ &\le -N\frac{k-1}{n+k}(\frac{n+k}{N(n+1)})^{\frac{n+1}{1-k}}(2C_0)^{\frac{n+1}{1-k}} \cdot d_0^{\frac{n+1}{n+k}},\end{aligned} \tag{3.14}$$

hence (3.1) is proved.

From (1.2) we see that

$$-u(x) \le C_0 \cdot \delta^{\frac{n+1}{n+k}}, \quad x \in \partial\Omega_\delta. \tag{3.15}$$

Note that N and C_0 are constants independent of δ, we now fix

$$\delta = \min\{\delta_0, \frac{1}{2}d_0, (\frac{c_0}{2C_0})^{\frac{n+k}{n+1}}d_0\}. \tag{3.16}$$

Then we have

$$-u(x) \leq \frac{c_0}{2} d_0^{\frac{n+1}{n+k}}, \quad x \in \partial\Omega_\delta. \tag{3.17}$$

It follows from (3.1) and (3.17) that

$$\alpha := \max_{\partial\Omega_\delta}(\sigma - u(x)) \leq -\frac{c_0}{2} d_0^{\frac{n+1}{n+k}}. \tag{3.18}$$

By formula (3.17) and the convexity of u, there exists a constant $c_1 > 0$ such that

$$|\mathrm{grad}\, u|, \ |\mathrm{grad}\,(\sigma - u)| \leq c_1 d_0^{\frac{1-k}{n+k}}, \quad \text{in } \Omega_\delta. \tag{3.19}$$

Step 2. Set

$$\gamma = u - \sigma + \alpha, \quad \text{in } \Omega_\delta, \tag{3.20}$$

then $\gamma \geq 0$ on $\partial\Omega_\delta$. With

$$\tau = \frac{2}{(c_1)^2} d_0^{\frac{2(k-1)}{k+n}}, \tag{3.21}$$

we consider in the region $\Omega_0 =$ the points in Ω_δ where $\gamma < 0$, the function

$$W = -\gamma[\exp(\tau u_r^2/2)]u_{rr}, \tag{3.22}$$

where the index r denotes differentiation in a fixed direction. In Ω_0, which contains x^0, the function W attains a maximum at some point O and some direction. By a unimodular transformation, we can make our choice of coordinate system in such a way that the direction is $(1, 0, \ldots, 0)$ and at the point O we have $u_{ij} = 0$ for $i \neq j$.

In Ω_0, the function $-\gamma[\exp(\tau u_1^2/2)]u_{11}$ takes a maximum at the point O. Take the logarithm and differentiate it twice with respect to x_i, then at the point O

$$\frac{\gamma_i}{\gamma} + \tau u_1 u_{1i} + \frac{u_{11i}}{u_{11}} = 0, \tag{3.23}$$

$$\frac{\gamma_{ii}}{\gamma} - \frac{\gamma_i^2}{\gamma^2} + \tau u_1 u_{1ii} + \tau u_{1i}^2 + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \leq 0. \tag{3.24}$$

Multiplying (3.24) by $u^{ii}u_{11}$ and summing over i, with the aid of (3.23), we obtain (Summation convention is used)

$$u^{ii}u_{11ii} - u^{ii}\frac{u_{11i}^2}{u_{11}} - \sum_{i>1} u^{ii}\frac{u_{11i}^2}{u_{11}} + \frac{n}{\gamma}u_{11} - \frac{\gamma_1^2}{\gamma^2} + \tau u_{11}^2 + \tau u_1 u_{11} u^{ii} u_{ii1} \leq 0. \tag{3.25}$$

We now differentiate (1.1) twice with respect to x_1

$$u^{ij}u_{ij1} = -k\frac{u_1}{u} + \frac{S_1}{S}. \tag{3.26}$$

$$u_1^{ij}u_{ij1} + u^{ij}u_{ij11} = -k\frac{u_{11}}{u} + k\frac{u_1^2}{u^2} + \frac{S_{11}}{S} - \frac{S_1^2}{S^2}. \tag{3.27}$$

From (3.27), we get

$$\begin{aligned} u^{ii}u_{ii11} &= -k\frac{u_{11}}{u} + k\frac{u_1^2}{u^2} + \frac{S_{11}}{S} - \frac{S_1^2}{S^2} - u_1^{ij}u_{ij1} \\ &= -k\frac{u_{11}}{u} + k\frac{u_1^2}{u^2} + \frac{S_{11}}{S} - \frac{S_1^2}{S^2} + u^{ii}u^{jj}u_{ij1}^2. \end{aligned} \tag{3.28}$$

Note that

$$u^{ii}u^{jj}u_{ij1}^2 - u^{ii}\frac{u_{11i}^2}{u_{11}} - \sum_{i>1} u^{ii}\frac{u_{11i}^2}{u_{11}} \geq 0. \tag{3.29}$$

On the other hand, by (3.26) we have

$$\tau u_1 u_{11} u^{ii} u_{ii1} = -k\tau\frac{u_1^2}{u}u_{11} + \tau u_1 u_{11}\frac{S_1}{S} \geq \tau u_1 u_{11}\frac{S_1}{S}. \tag{3.30}$$

Combining (3.25) and (3.28)–(3.30), we get

$$\tau u_{11}^2 + (\frac{n}{\gamma} + \tau\frac{S_1}{S}u_1)u_{11} + \frac{S_{11}}{S} - \frac{S_1^2}{S^2} - \frac{\gamma_1^2}{\gamma^2} \leq 0. \tag{3.31}$$

Multiplying inequality (3.31) by $\gamma^2 \exp{(\tau u_1^2)}$, we obtain

$$\tau \cdot W^2 - (n + \tau\gamma u_1\frac{S_1}{S})e^{\frac{\tau}{2}u_1^2} \cdot W + (\gamma^2(\frac{S_{11}}{S} - \frac{S_1^2}{S^2}) - \gamma_1^2)e^{\tau u_1^2} \leq 0. \tag{3.32}$$

Applying (3.19) and (3.21), there exist constants c_2, c_3 and c_4 such that

$$c_2 d_0^{\frac{2(k-1)}{n+k}} W^2 - c_3 W - c_4 d_0^{\frac{2(1-k)}{n+k}} \leq 0. \tag{3.33}$$

It follows that

$$W(O) \leq c_5 \cdot d_0^{\frac{2(1-k)}{n+k}}, \quad \text{for constant } c_5 > 0. \tag{3.34}$$

Hence at the point x^0, we have

$$-\gamma[\exp{(\tau u_r^2/2)}]u_{rr} \leq c_5 \cdot d_0^{\frac{2(1-k)}{n+k}}. \tag{3.35}$$

Since

$$-\gamma(x^0) = -\alpha \geq \frac{c_0}{2}d_0^{\frac{n+1}{n+k}}, \tag{3.36}$$

we can find a positive constant c_6 so that

$$|u_{ii}(x^0)| \leq c_6\ d_0^{\frac{n+1}{n+k}-2}, \quad 1 \leq i \leq n. \tag{3.37}$$

By (3.37) and the convexity of u, we obtain (1.3).

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