Second Order Derivative Estimates of the Solutions of a Class of Monge-Ampère Equations

Yadong WU$^{1,*}$, Hepeng LI$^2$

1. College of Mathematics and Information Science, Jiangxi Normal University, Jiangxi 330022, P. R. China;
2. College of Mathematics and Finance, Sichuan University of Arts and Science, Sichuan 635000, P. R. China

Abstract In this paper, we consider a class of Monge-Ampère equations in relative differential geometry. Given these equations with zero boundary values in a smooth strictly convex bounded domain, we obtain second order derivative estimates of the convex solutions.

Keywords Monge-Ampère equation; derivative estimate; relative differential geometry.

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1. Introduction

In equiaffine differential geometry and relative differential geometry, Li-Simon-Chen [1] and Wu-Zhao [2] considered the following equation

\[
\begin{cases}
\det \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right) = S(x)(-u)^{-k} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \) is a smooth strictly convex bounded domain in \( \mathbb{R}^n \), \( k \) is a positive constant with \( k > 1 \), and \( S(x) \in C^\infty(\Omega) \cap C^2(\bar{\Omega}) \) with \( S_n > 0 \). In particular for \( k = n + 2 \) and \( S(x) = \text{const} \), equation (1.1) is the well-known hyperbolic affine hypersphere equation.

Cheng-Yau [3] showed that (1.1) has a convex solution \( u \in C^\infty(\Omega) \cap C^0(\bar{\Omega}) \), and the uniqueness follows from the maximum principal. Moreover, Lazer-McKenna [4] showed that the unique convex solution \( u \) satisfies

\[
\frac{1}{C_0} d(x)^{n+1} \leq -u(x) \leq C_0 d(x)^{n+1},
\]

(1.2)

where \( d(x) := \text{dist}(x, \partial \Omega) \), and \( C_0 \) is a positive constant. In the following, we denote by \( u_i, u_{ij}, u_{ijk}, \ldots \), the derivatives of \( u \) with respect to \( x \), \((u^{ij})\) the inverse matrix of \((u_{ij})\). The main result of our paper is

**Theorem 1.1** The convex solution of (1.1) satisfies

\[
|u_{ij}| \leq C d(x)^{n+1} \frac{n+1}{n+k} - 2, \quad 1 \leq i, j \leq n,
\]

(1.3)

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* Corresponding author

E-mail address: wydmath@gmail.com (Yadong WU); lihepeng19840163.com (Hepeng LI)
where $C$ is a constant depending only on $\Omega$, $n$, $k$ and $S(x)$.

For $k = n+2$ and $S(x) = \text{const}$, formula (1.3) can be seen in Loewner-Nirenberg [5] and Wu [6]. Second order estimates (1.3) can be used to describe the asymptotic behaviors of relative hypersurfaces of hyperbolic type, for details see [2] and [6]. On the other hand, (1.3) gives another proof of Lemma 3 in [2].

2. A barrier function

We consider the function

$$w = -c(R^2 - \sum_{1 \leq i \leq n} (x_i - \bar{x}_i)^2)^\beta, \hspace{1cm} 0 < \beta < 1$$

(2.1)

defined in the ball $\{(x_1, x_2, \ldots, x_n) \mid \sum (x_i - \bar{x}_i)^2 \leq R^2\}$. A direct calculation gives

$$w_i = 2c\beta(R^2 - (x_i - \bar{x}_i)^2)^{\beta-1}(x_i - \bar{x}_i),$$

(2.2)

$$w_{ij} = 2c\beta(R^2 - \sum (x_i - \bar{x}_i)^2)^{\beta-1}[\delta_{ij} + 2(1 - \beta)(x_i - \bar{x}_i)(x_j - \bar{x}_j)]R^2 - \sum (x_i - \bar{x}_i)^2]$$

$$= 2c\beta (-w)^{\frac{n+1}{n}}[\delta_{ij} + 2(1 - \beta)\bar{c}(-w)^{-\frac{1}{n}}(x_i - \bar{x}_i)(x_j - \bar{x}_j)].$$

(2.3)

Hence

$$\det(w_{ij}) = 2^n \beta^n c^n \bar{c}(-w)^{\frac{n+1}{n}}[1 + 2(1 - \beta)\bar{c}(-w)^{-\frac{1}{n}}\sum (x_i - \bar{x}_i)^2]$$

$$= 2^n \beta^n c^n \bar{c}(-w)^{\frac{n+1}{n}}[1 + 2(1 - \beta)\bar{c}(-w)^{-\frac{1}{n}}(R^2 - (-w)^{\frac{1}{n}}c^{-\frac{1}{n}})]$$

$$= 2^n \beta^n c^n \bar{c}(-w)^{n-\frac{n+1}{n}}[(2\beta - 1)(-w)^{\frac{1}{n}} + 2R^2(1 - \beta)c^{\frac{1}{n}}]$$

(2.4)

For $2\beta - 1 \geq 0$, we have

$$\det(w_{ij}) \leq 2^n \beta^n c^n \bar{c}(-w)^{n-\frac{n+1}{n}}[(2\beta - 1)R^2c^{\frac{1}{n}} + 2R^2(1 - \beta)c^{\frac{1}{n}}]$$

$$= 2^n R^2 \beta^n c^{\frac{n+1}{n}}(-w)^{n-\frac{n+1}{n}}$$

$$\leq 2^n R^2 c^{\frac{n+1}{n}}(-w)^{n-\frac{n+1}{n}}.$$  

(2.5)

For $2\beta - 1 \leq 0$, we have

$$\det(w_{ij}) \leq 2^n R^2(1 - \beta)c^{\frac{n+1}{n}}(-w)^{n-\frac{n+1}{n}}$$

$$\leq 2R^2 c^{\frac{n+1}{n}}(-w)^{n-\frac{n+1}{n}}.$$  

(2.6)

Let $S(x)$ be the function as in (1.1), and $k = \frac{n+1}{\beta} - n$. Now we choose the constant $c$ as follows

$$c = (2^n R^2 \cdot \min_{x \in \Omega} S(x))^{\frac{1}{n+1}}.$$  

(2.7)

Then from (2.5) and (2.6), we know

$$\det(w_{ij}) \leq S(x)(-w)^{-k}.$$  

(2.8)

Next we give a comparison result in [4].

**Lemma 2.1** ([4]) Let $\Omega$ be a bounded convex domain, and let $v_k \in C^2(\Omega) \cap C(\bar{\Omega})$ for $k = 1, 2.$
Let \( f(x, \xi) \) be defined for \( x \in \Omega \) and \( \xi \) in some interval containing the ranges of \( v_1 \) and \( v_2 \) and assume that \( f(x, \xi) \) is strictly increasing in \( \xi \) for all \( x \in \Omega \). If

1. The matrix \( ((v_1)_ij) \) is positive definite in \( \Omega \),
2. \( \det((v_1)_{ij}) \geq f(x, v_1), \forall x \in \Omega \),
3. \( \det((v_2)_{ij}) \leq f(x, v_2), \forall x \in \Omega \),
4. \( v_1 \leq v_2, \forall x \in \partial \Omega \),

then \( v_1 \leq v_2, \forall x \in \Omega \).

3. Second order derivative estimates

Proof We divide two steps to prove Theorem 1.1, and follow the calculations as in Loewner-Nirenberg [5] and Pogorelov [7].

Step 1. For any point \( x^0 = (x^0_1, x^0_2, \ldots, x^0_n) \in \Omega \), set \( d_0 = \text{dist}(x^0, \partial \Omega) \). There is a number \( \delta_0 \) such that for \( \delta \leq \delta_0 \) the domain \( \Omega_\delta \) consisting of the set of points in \( \Omega \) whose distance to the boundary is at least \( \delta \) has a smooth, strictly convex boundary.

Let \( z = \sigma(x) \) be the equation of the tangent hyperplane to the graph \( u(x) \) at the point \((x^0, u(x^0))\). We claim that there is a positive constant \( c_0 \) such that

\[
\max_{\partial \Omega_\delta} \sigma(x) \leq -c_0 \cdot d_0^{n+1}, \quad \text{for } \delta \leq \min\{\delta_0, d_0/2\}. \tag{3.1}
\]

To see this, let \( Q \) be the boundary point of \( \Omega_\delta \) where \( \sigma \) takes its maximum. For \( \delta \leq \delta_0 \), there is a fixed positive number \( R_0 \) independent of \( \delta \) such that there is a closed disc \( D \) in \( \Omega_\delta \) of radius \( R_0 \) touching \( \partial \Omega_\delta \) at \( Q \). We may suppose that \( Q \) is the origin and that the inner normal to \( \partial \Omega_\delta \) at \( Q \) has the direction \((0, 0, \ldots, 1)\). Hence the disc \( D \) has center \((0, 0, \ldots, R_0)\).

Let \( w \) be the function

\[
w = -c(R_0^2 - \sum_{1 \leq i \leq n-1} x_i^2 - (x_n - R_0^2) \frac{n+1}{n+2}) = -c \left( \sum_{1 \leq i \leq n} x_i^2 + 2R_0x_n \right)^{\frac{n+1}{n+2}}. \tag{3.2}
\]

From Section 2, we can choose the constant \( c \) such that

\[
\begin{align*}
\det(w_{ij}) &\leq S(x)(-w)^{-k} \quad \text{in } D, \\
w &\equiv 0 \quad \text{on } \partial D.
\end{align*}
\]

By Lemma 2.1 we conclude \( \sigma \leq u \leq w \). Since the maximum of \( \sigma \) on \( \bar{D} \) occurs at \( Q \), we see that the tangent hyperplane has the form \( \sigma = u(x_0) + u_n(x_0)(x_n - x_n^0) \). Hence

\[
\sigma(0, \ldots, 0, x_n) \leq u(0, \ldots, 0, x_n) \leq w(0, \ldots, 0, x_n) = -c(2R_0x_n - x_n^0)^{\frac{n+1}{n+2}}. \tag{3.3}
\]

Since for some constant \( N \)

\[
\sigma(0, \ldots, 0, x_n) \leq -c(2R_0x_n - x_n^0)^{\frac{n+1}{n+2}} \leq -Nx_n^{\frac{n+1}{n+2}}, \quad 0 \leq x_n \leq R_0, \tag{3.4}
\]

we may therefore assert that \( \sigma(0, \ldots, 0, x_n) \leq \tau(x_n) \), where \( z = \tau(x_n) \) is the equation in the \((x_n, z)\) plane of the tangent line from the point \((x_n^0, u(x^0))\) to the curve \( z = -Nx_n^{\frac{n+1}{n+2}} \), further-
more, this line touches the curve at a point \( (t, -Nt^{\frac{n+1}{n+k}}) \) with \( t \leq R_0 \).

The slope of \( \tau(x_n) \) is \(-\frac{N(n+1)}{n+k} t^{\frac{k-1}{n+k}}\) and therefore
\[
\sigma(Q) \leq \tau(0) = -N t^{\frac{n+1}{n+k}} + \frac{N(n+1)}{n+k} t^{\frac{k-1}{n+k}} = -N k - \frac{1}{n+k} t^{\frac{n+1}{n+k}}.
\]
(3.5)

We wish to find a lower bound for \( t \). From the definition of \( \tau(x_n) \), we have
\[
-\frac{N(n+1)}{n+k} t^{\frac{k-1}{n+k}} = \frac{u(x^0) + N t^{\frac{n+1}{n+k}}}{x_n^0 - t},
\]
or
\[
N \frac{k-1}{n+k} t + u(x^0) \cdot t^{\frac{k-1}{n+k}} + N^{\frac{n+1}{n+k}} x_n^0 = 0.
\]
(3.7)

Put \( s = t^{-1/(n+k)} \), then from (3.7) we get
\[
N^{\frac{n+1}{n+k}} x_n^0 \cdot s^{n+k} + u(x^0) \cdot s^{n+1} + N \frac{k-1}{n+k} = 0.
\]
(3.8)

Consider a function
\[
f(y) = N^{\frac{n+1}{n+k}} x_n^0 \cdot y^{n+k} + u(x^0) \cdot y^{n+1} + N \frac{k-1}{n+k},
\]
and a point
\[
s^* = \left[ \frac{-(n+k)u(x^0)}{N(n+1)x_n^0} \right]^{\frac{1}{n+1}}.
\]
(3.10)

Then \( f(s^*) = N^{\frac{k-1}{n+k}} x_n^0 > 0 \), and the derivative of \( f(y) \) is \( f'(y) = N(n+1)x_n^0 \cdot y^{n+k-1} + (n+1)u(x^0) \cdot y^n \). Obviously for \( y > s^* \), \( f'(y) > 0 \). It follows that \( s \leq s^* \), and hence
\[
t = s^{-(n+k)} \geq (s^*)^{-(n+k)} = \left( \frac{-(n+k)u(x^0)}{N(n+1)x_n^0} \right)^{\frac{n+1}{n+k}}.
\]
(3.11)

Since \( \Omega_{\delta} \) is convex, we get
\[
x_n^0 \geq \text{dist}(x_n^0, \partial \Omega_{\delta}) \geq d_0/2.
\]
(3.12)

From (1.2), we get
\[
-u(x^0) \leq C_0 \cdot d_0^{\frac{n+1}{n+k}}.
\]
(3.13)

Combining (3.5) and (3.11)-(3.13), we get
\[
\sigma(Q) \leq -N \frac{k-1}{n+k} t^{\frac{n+1}{n+k}} \leq -N \frac{k-1}{n+k} \left( \frac{-(n+k)u(x^0)}{N(n+1)x_n^0} \right)^{\frac{n+1}{n+k}} \leq -N \frac{k-1}{n+k} \left( \frac{n+k}{N(n+1)} \right)^{\frac{n+1}{n+k}} (2C_0)^{\frac{n+1}{n+k}} \cdot d_0^{\frac{n+1}{n+k}},
\]
(3.14)

hence (3.1) is proved.

From (1.2) we see that
\[
-u(x) \leq C_0 \cdot \delta^{\frac{n+1}{n+1}}, \quad x \in \partial \Omega_{\delta}.
\]
(3.15)

Note that \( N \) and \( C_0 \) are constants independent of \( \delta \), we now fix
\[
\delta = \min\{d_0, \frac{1}{2}d_0, \left( \frac{C_0}{2C_0 \cdot d_0^{\frac{n+1}{n+k}}} \right)\}.
\]
(3.16)
Then we have

\[-u(x) \leq \frac{c_0}{2} \frac{u_{ii}}{d_0^{n+1}}, \quad x \in \partial \Omega_3. \quad (3.17)\]

It follows from (3.1) and (3.17) that

\[\alpha := \max_{\partial \Omega_3}(\sigma - u(x)) \leq -\frac{c_0}{2} \frac{u_{ii}}{d_0^{n+1}}. \quad (3.18)\]

By formula (3.17) and the convexity of \(u\), there exists a constant \(c_1 > 0\) such that

\[|\nabla u|, |\nabla (\sigma - u)| \leq c_1 d_0^{\frac{n-k}{n+k}}, \quad \text{in } \Omega_3. \quad (3.19)\]

Step 2. Set

\[\gamma = u - \sigma + \alpha, \quad \text{in } \Omega_3, \quad (3.20)\]

then \(\gamma \geq 0\) on \(\partial \Omega_3\). With

\[
\tau = \frac{2}{(c_1)^2 d_0^{\frac{n-k}{n+k}}},
\]

we consider in the region \(\Omega_0 = \) the points in \(\Omega_3\) where \(\gamma < 0\), the function

\[W = -\gamma[\exp(\tau u_1^2/2)]u_{rr}, \quad (3.22)\]

where the index \(r\) denotes differentiation in a fixed direction. In \(\Omega_0\), which contains \(x^0\), the function \(W\) attains a maximum at some point \(O\) and some direction. By a unimodular transformation, we can make our choice of coordinate system in such a way that the direction is \((1, 0, \ldots, 0)\) and at the point \(O\) we have \(u_{ij} = 0\) for \(i \neq j\).

In \(\Omega_0\), the function \(-\gamma[\exp(\tau u_1^2/2)]u_{11}\) takes a maximum at the point \(O\). Take the logarithm and differentiate it twice with respect to \(x_1\), then at the point \(O\)

\[
\frac{\gamma_{ii}}{\gamma} + \tau u_{i1}u_{11} + \frac{u_{1ii}}{u_{11}} = 0, \quad (3.23)\]

\[
\frac{\gamma_{ii}}{\gamma} - \frac{\gamma_1^2}{\gamma^2} + \tau u_{11} + \tau u_{i1} + \frac{u_{1ii}}{u_{11}} - \frac{u_{11}}{u_{11}^2} \leq 0. \quad (3.24)\]

Multiplying (3.24) by \(u_{ii}u_{11}\) and summing over \(i\), with the aid of (3.23), we obtain (Summation convention is used)

\[u_{ii}u_{11} - u_{ii}u_{11}^2 - \sum_{i>1} u_{ii}u_{11i} + \frac{n}{\gamma} u_{11} - \frac{\gamma_1^2}{\gamma^2} + \tau u_{11} + \tau u_{11} + u_{1i1}u_{i1} \leq 0. \quad (3.25)\]

We now differentiate (1.1) twice with respect to \(x_1\)

\[u_{ij}u_{ij} = -k \frac{u_{ij}}{u} + \frac{S_{ij}}{S}. \quad (3.26)\]

From (3.27), we get

\[u_{ii}u_{11} = -k \frac{u_{11}}{u} + k \frac{u_{11}^2}{u^2} + \frac{S_{11}}{S} - \frac{S_{11}^2}{S^2}, \quad (3.28)\]
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Note that
\[ u^{ii}u^{ij}u_{ij}^2 - u^{ii}u_{ii}^2 - \sum_{i>1} u^{ii}u_{ii}^2 \geq 0. \] (3.29)

On the other hand, by (3.26) we have
\[ \tau u_{ii}u_{ii}^2 = -k^2 u_{ii}^2 + \tau u_{ii}^2 \frac{S_1}{S} \geq \tau u_{ii} \frac{S_1}{S}. \] (3.30)

Combining (3.25) and (3.28)–(3.30), we get
\[ \tau u_{ii}^2 + \left( \frac{n}{\gamma} + \tau \frac{S_1}{S} u_{ii} \right) u_{ii}^2 + \frac{S_1}{S} - \frac{S_1^2}{S^2} - \frac{\gamma^2}{\gamma^2} \leq 0. \] (3.31)

Multiplying inequality (3.31) by \( \gamma^2 \exp(\tau u_{ii}^2) \), we obtain
\[ \tau \cdot W^2 - (n + \gamma u_1 \frac{S_1}{S}) e^{\gamma u_{ii}^2} \cdot W + (\gamma^2 \left( \frac{S_1}{S} - \frac{S_1^2}{S^2} \right) - \frac{\gamma^2}{\gamma^2}) e^{\gamma u_{ii}^2} \leq 0. \] (3.32)

Applying (3.19) and (3.21), there exist constants \( c_2, c_3 \) and \( c_4 \) such that
\[ \frac{2(k-1)}{k} W^2 - c_3 W - c_4 d_0^{n+k} \leq 0. \] (3.33)

It follows that
\[ W(O) \leq c_5 \cdot d_0^{n+k}, \quad \text{for constant } c_5 > 0. \] (3.34)

Hence at the point \( x^0 \), we have
\[ -\gamma \left[ \exp \left( \tau u_{ii}^2/2 \right) \right] u_{rr} \leq c_5 \cdot d_0^{2(1-k)}. \] (3.35)

Since
\[ -\gamma(x^0) = -\alpha \geq \frac{c_0}{2} d_0^{\frac{n+1}{n+k}}, \] (3.36)
we can find a positive constant \( c_0 \) so that
\[ |u_{ii}(x^0)| \leq c_6 d_0^{\frac{n+1}{n+k} - 2}, \quad 1 \leq i \leq n. \] (3.37)

By (3.37) and the convexity of \( u \), we obtain (1.3).

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