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Laplacian Spectral Characterization of a Kind of Unicyclic Graphs

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Abstract Let $H(n;q,n_1,n_2,n_3,n_4)$ be a unicyclic graph with n vertices containing a cycle C_q and four hanging paths P_{n_1+1}, P_{n_2+1}, P_{n_3+1} and P_{n_4+1} attached at the same vertex of the cycle. In this paper, it is proved that all unicyclic graphs $H(n;q,n_1,n_2,n_3,n_4)$ are determined by their Laplacian spectra.

Keywords Laplacian spectrum; unicyclic graphs; Laplacian matrix.

MR(2010) Subject Classification 05C50; 15A18

1. Introduction

In this paper, we assume that all graphs are simple and undirected. Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. Let the matrix $A(G)$ be the $(0,1)$-adjacency matrix of G and $d_i=d(v_i)=d_i(G)$ the degree of the vertex v_i. Assume that $d_1\geq d_2\geq\cdots\geq d_n$. The matrix $L(G)=D(G)-A(G)$ is called the Laplacian matrix of G, where $D(G)=\mathrm{diag}(d_1,d_2,\ldots,d_n)$ denotes the diagonal matrix of vertex degrees of G. The polynomial $\phi(G)=\phi(G,x)=\det(xI_n-L(G))=l_0x^n+l_1x^{n-1}+\cdots+l_n$ is defined as the Laplacian characteristic polynomials of the graph G, where I_n is the $n\times n$ identity matrix. Since $A(G)$ and $L(G)$ are real and symmetric, their eigenvalues are real numbers and called the adjacency and the Laplacian eigenvalues of G, respectively. Assume that $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ and $\mu_1\geq\mu_2\geq\cdots\geq\mu_n(=0)$ are the adjacency and the Laplacian eigenvalues of G. The multiset of eigenvalues of $A(G)$ ($L(G)$) is called the adjacency (Laplacian) spectrum of G. Two graphs are said to be adjacency (Laplacian) cospectral if they have equal adjacency (Laplacian) spectrum. A graph is said to be determined by the adjacency (Laplacian) spectrum if there is no other non-isomorphic graph with the same adjacency (Laplacian) spectrum.

Which graphs are determined by their (Laplacian) spectra is an interesting problem in the theory of graph spectra. Characterizing such graphs seems also to be a difficult problem. Recently, this problem has attracted some researchers' attention. Up until now, only some graphs with special structure are proved to be determined by their (Laplacian) spectra. Some results

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on these special graphs determined by their adjacency or Laplacian spectra can be found in [10, 15, 16, 18, 24, 26, 27] or [1, 10, 17–20, 22, 23, 25, 28], respectively. The reader can also consult the books [3, 5] and the surveys [6, 7].

Let P_n and C_n be the path and cycle with n vertices, respectively. Let a kind of unicyclic graphs $H(n; q, n_1, n_2, \ldots, n_t)$ with n vertices be shown in Figure 1. They are some unicyclic graphs containing a cycle C_q and t hanging paths $P_{n_1+1}, P_{n_2+1}, P_{n_3+1}, \ldots, P_{n_t+1}$ attached at the same vertex of the cycle. For the case $t = 1$, Haemers et al. in [10] have proved all lollipop graphs $H(n; q, n-q)$ to be determined by their Laplacian spectra. For the case $t = 2$, Liu et al. in [18] has proved all graphs $H(n; q, n_1, n_2)$ to be determined by their Laplacian spectra. For the case $t = 3$, Lu et al. in [20] have proved all graphs $H(n; q, n_1, n_2, n_3)$ to be determined by their Laplacian spectra. In this paper, we prove all unicyclic graphs $H(n; q, n_1, n_2, n_3, n_4)$ (shown in Figure 2) to be determined by their Laplacian spectra.

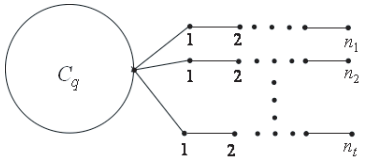

Figure 1 Graph $H(n; q, n_1, n_2, \ldots, n_t)$

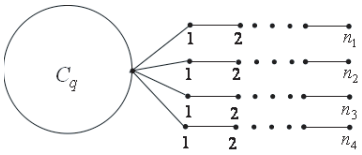

Figure 2 Graph $H(n; q, n_1, n_2, n_3, n_4)$

2. Preliminaries

In this section, some known lemmas about the (Laplacian) spectrum of a graph are given, and these lemmas will play important roles throughout the paper.

Lemma 2.1 ([6, 21]) *For the Laplacian matrix of a graph G, the following results can be deduced from the Laplacian spectrum.*

(i) The number of vertices.

(ii) The number of edges.

(iii) The number of components.

(iv) The number of spanning trees.

(v) The sum of the squares of degrees of vertices.

Lemma 2.2 ([5]) *Assume that N is an $n \times n$ symmetric matrix, its eigenvalues are $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. The eigenvalues of the principal submatrix of order m of N are $\alpha'_1 \geq \alpha'_2 \geq \cdots \geq \alpha'_m$, then $\alpha_i \geq \alpha'_i \geq \alpha_{n-m+i}$, $i = 1, 2, \ldots, m$.*

Lemma 2.3 ([4]) *Let $A = [a_{ij}]$ be a matrix of order n. Let*

$$r_i = \sum_{j=1}^{n} |a_{ij}|, \quad 1 \leq i \leq n$$

be the sum of the absolute values of the entries in row i of A. Then

$$\rho(A) \leq \max(r_1, r_2, \ldots, r_n)$$

where $\rho(A)$ is the maximum of the absolute values of all eigenvalues of A. A similar inequality holds for the sum of the absolute values of the entries in columns of A.

Lemma 2.4 ([21]) *Let G be a graph with n vertices and m edges and let $\deg(G) = (d_1, d_2, \ldots, d_n)$ be its non-increasing degree sequence. Then the first four coefficients in $\phi(G, x)$ are*

$$l_0 = 1, l_1 = -2m, l_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2,$$

$$l_3 = \frac{1}{3}(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6n_G(C_3)),$$

where $n_G(C_3)$ is the number of triangles in G.

Lemma 2.5 ([12, 14]) *Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then*

$$d_1 + 1 \leq \mu_1 \leq \max\{\frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j}, v_iv_j \in E(G)\}$$

where d_1 denotes the maximum vertex degree of G and m_i denotes the average of the degrees of the vertices adjacent to vertex v_i in G.

Lemma 2.6 ([13]) *Let G be a connected graph with $n \geq 3$ vertices, and let $\deg(G) = (d_1, d_2, \ldots, d_n)$ be its non-increasing degree sequence. Then $\mu_2 \geq d_2$.*

Lemma 2.7 ([2]) *The coefficients l_i of the polynomial $\phi(G, x)$ are given by the formula*

$$(-1)^i l_i = \sum p(\Phi), \quad 1 \leq i \leq n,$$

where the right hand of the equation is the summation of all sub-forests Φ of G which have i edges, and $p(\Phi)$ denotes the product of the numbers of vertices in the components of the forest Φ.

Lemma 2.8 ([8]) *Let B_n be the matrix of order n obtained from $L(P_{n+1})$ by deleting the row and column corresponding to some end vertices of P_{n+1}, and let H_n be the matrix of order n obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to the two end vertices of P_{n+2}. Set $\phi(P_0) = 0, \phi(B_0) = 1, \phi(H_0) = 1$. Then we have the following conclusions.*

(i) $\phi(P_{n+1}) = (x-2)\phi(P_n) - \phi(P_{n-1}), (n \geq 1)$.

(ii) $x\phi(B_n) = \phi(P_{n+1}) + \phi(P_n)$.

(iii) $\phi(P_n) = x(\phi(H_{n-1}))$ $(n \geq 1)$.

(iv) $\phi(C_n) = \frac{1}{x}\phi(P_{n+1}) - \frac{1}{x}\phi(P_{n-1}) + 2(-1)^{n+1}$ $(n \geq 3, x \neq 0)$.

From Lemma 2.8, if $x = 4$, then we can get the following lemma.

Lemma 2.9 ([20]) *(i)* $\phi(P_n; 4) = 4n$.

(ii) $\phi(B_n; 4) = 2n + 1$.

(iii) $\phi(H_n; 4) = n + 1$.

(iv) $\phi(C_n; 4) = 2 + 2(-1)^{n+1}$.

From Lemma 2.8, if $x \neq 4$, then we can get the following lemma.

Lemma 2.10 *For $\phi(G) = \phi(G, x)$ of a graph G, if $x \neq 4$, and y satisfies the equation $y^2 - (x - 2)y + 1 = 0$. Then we have the following results.*

(i) $\phi(P_n) = \frac{(y+1)(y^{2n}-1)}{y^{n+1}-y^n}$.

(ii) $\phi(B_n) = \frac{y^{2n+1}-1}{y^{n+1}-y^n}$.

(iii) $\phi(H_n) = \frac{y^{2n+2}-1}{y^{n+2}-y^n}$.

(iv) $\phi(C_n) = y^n + \frac{1}{y^n} + 2(-1)^{n+1}$.

If v is a vertex of G, let $L_v(G)$ be the principal submatrix of $L(G)$ obtained by deleting the row and column corresponding to vertex v from $L(G)$.

Lemma 2.11([9]) *Let G_1 and G_2 be two vertex-disjoint graphs and $G = G_1u : vG_2$ be a graph obtained by joining the vertex u of the graph G_1 to the vertex v of the graph G_2 by an edge. Then $\phi(G) = \phi(G_1)\phi(G_2) - \phi(G_1)\phi(L_v(G_2)) - \phi(G_2)\phi(L_u(G_1))$.*

Lemma 2.12 ([18]) *Let G be a connected unicyclic graph with n vertices and its cycle C_q. If G' is Laplacian cospectral to G, then G' must be a connected unicyclic graph with n vertices and one cycle C_q. Moreover,*

$$\sum_{i=1}^{n} d_i(G)^3 = \sum_{i=1}^{n} d_i(G')^3.$$

3. Laplacian spectral characterization of graphs $H(n; q, n_1, n_2, n_3, n_4)$

In this section, we prove that no two non-isomorphic graphs of the form $H(n; q, n_1, n_2, n_3, n_4)$ are Laplacian cospectral, and the graph $H(n; q, n_1, n_2, n_3, n_4)$ is determined by its Laplacian spectrum. To prove Theorems 3.3 and 3.4, we first prove the following lemmas.

Lemma 3.1 *The antepenultimate coefficient l_{n-2} of $\phi(H(n; q, n_1, n_2, n_3, n_4), x)$ is given by*

$$\begin{aligned}(-1)^{n-2}l_{n-2} =& q[\frac{1}{6}(n_1+n_2+n_3+n_4)^3 - (n_1n_2n_3 + n_1n_2n_4 + n_1n_3n_4 + n_2n_3n_4) + \\ & \frac{1}{2}q(n_1+n_2+n_3+n_4)^2 + (1-q)(n_1n_2 + n_1n_3 + n_1n_4 + n_2n_3 + \\ & n_2n_4 + n_3n_4) + (\frac{1}{2}q - \frac{1}{6})(n_1+n_2+n_3+n_4)] + \sum p(\Phi),\end{aligned}$$

where the summation is over all sub-forests Φ of $H(n; q, n_1, n_2, n_3, n_4)$ which have $n - 2$ edges obtained by deleting two edges both from C_q in $H(n; q, n_1, n_2, n_3, n_4)$.

Proof Let $G = H(n; q, n_1, n_2, n_3, n_4)$. Since G has n edges, the $(n-2)$-subforests of G can be obtained from G by deleting two edges in which at least one comes from the cycle C_q. Then by Lemma 2.7, we can get that

$$\begin{aligned}(-1)^{n-2}l_{n-2} =& q\sum_{i=0}^{n_1-1}(q+n_2+n_3+n_4+i)(n_1-i) + q\sum_{i=0}^{n_2-1}(q+n_1+n_3+n_4+i)(n_2-i) + \\ & q\sum_{i=0}^{n_3-1}(q+n_1+n_2+n_4+i)(n_3-i) + q\sum_{i=0}^{n_4-1}(q+n_1+n_2+n_3+i)(n_4-i) +\end{aligned}$$

$$\sum p(\Phi), \tag{1}$$

where the summation is over all sub-forests Φ of $G = H(n; q, n_1, n_2, n_3, n_4)$ which have $n-2$ edges obtained by deleting two edges both from C_q in G.

Because

$$\begin{aligned}
&\sum_{i=0}^{n_1-1}(q+n_2+n_3+n_4+i)(n_1-i)\\
&=\sum_{i=0}^{n_1-1}(q+n_2+n_3+n_4)n_1+\sum_{i=0}^{n_1-1}(n_1-q-n_2-n_3-n_4)i-\sum_{i=0}^{n_1-1}i^2\\
&=(q+n_2+n_3+n_4)n_1^2+(n_1-q-n_2-n_3-n_4)\frac{n_1(n_1-1)}{2}-\\
&\quad\frac{1}{6}n_1(n_1-1)(2n_1-1)\\
&=\frac{1}{6}n_1^3+\frac{1}{2}n_1^2n_2+\frac{1}{2}n_1^2n_3+\frac{1}{2}n_1^2n_4+\frac{1}{2}n_1n_2+\frac{1}{2}n_1n_3+\frac{1}{2}n_1n_4+\frac{1}{2}n_1q-\\
&\quad\frac{1}{6}n_1+\frac{1}{2}n_1^2q,
\end{aligned} \tag{2}$$

similarly, we obtain the following results:

$$\begin{aligned}
&\sum_{i=0}^{n_2-1}(q+n_1+n_3+n_4+i)(n_2-i)\\
&=\frac{1}{6}n_2^3+\frac{1}{2}n_2^2n_1+\frac{1}{2}n_2^2n_3+\frac{1}{2}n_2^2n_4+\frac{1}{2}n_2n_1+\frac{1}{2}n_2n_3+\frac{1}{2}n_2n_4+\frac{1}{2}n_2q-\\
&\quad\frac{1}{6}n_2+\frac{1}{2}n_2^2q,
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\sum_{i=0}^{n_3-1}(q+n_1+n_2+n_4+i)(n_3-i)\\
&=\frac{1}{6}n_3^3+\frac{1}{2}n_3^2n_1+\frac{1}{2}n_3^2n_2+\frac{1}{2}n_3^2n_4+\frac{1}{2}n_3n_1+\frac{1}{2}n_3n_2+\frac{1}{2}n_3n_4+\frac{1}{2}n_3q-\\
&\quad\frac{1}{6}n_3+\frac{1}{2}n_3^2q,
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\sum_{i=0}^{n_4-1}(q+n_1+n_2+n_3+i)(n_4-i)\\
&=\frac{1}{6}n_4^3+\frac{1}{2}n_4^2n_1+\frac{1}{2}n_4^2n_2+\frac{1}{2}n_4^2n_3+\frac{1}{2}n_4n_1+\frac{1}{2}n_4n_2+\frac{1}{2}n_4n_3+\frac{1}{2}n_4q-\\
&\quad\frac{1}{6}n_4+\frac{1}{2}n_4^2q.
\end{aligned} \tag{5}$$

Then, substituting (2)–(5) into (1), we have

$$\begin{aligned}
(-1)^{n-2}l_{n-2}=&q[\frac{1}{6}(n_1+n_2+n_3+n_4)^3-(n_1n_2n_3+n_1n_2n_4+n_1n_3n_4+n_2n_3n_4)+\\
&\frac{1}{2}q(n_1+n_2+n_3+n_4)^2+(1-q)(n_1n_2+n_1n_3+n_1n_4+n_2n_3+\\
&n_2n_4+n_3n_4)+(\frac{1}{2}q-\frac{1}{6})(n_1+n_2+n_3+n_4)]+\sum p(\Phi),
\end{aligned}$$

where the summation is over all sub-forests Φ of $G = H(n; q, n_1, n_2, n_3, n_4)$ which have $n-2$ edges obtained by deleting two edges both from C_q in G. □

Lemma 3.2 *Let G be the graph $H(n; q, n_1, n_2, n_3, n_4)$ as shown in Figure 2. Then we have the following result:*

$$\begin{aligned}\phi(G,4) =&(32-128q-32(-1)^q)n_1n_2n_3n_4+(16-48q-16(-1)^q)\cdot\\ &(n_1n_2n_3+n_1n_2n_4+n_1n_3n_4+n_2n_3n_4)+(8-16q-8(-1)^q)\cdot\\ &(n_1n_2+n_1n_3+n_1n_4+n_2n_3+n_2n_4+n_3n_4)+(4-4q-\\ &4(-1)^q)\cdot(n_1+n_2+n_3+n_4)+2-2(-1)^q.\end{aligned}$$

Proof Firstly, we assume that $G_2 = H(n-n_1; q, n_2, n_3, n_4)$, $G_3 = H(n-n_1-n_2; q, n_3, n_4)$ and $G_4 = H(n-n_1-n_2-n_3; q, n_4)$. Then, from Lemma 2.11, we obtain

$$\begin{aligned}\phi(G,x) =&\phi(G_2,x)\phi(P_{n_1},x)-\phi(G_2,x)\phi(B_{n_1-1},x)-\\ &\phi(P_{n_1},x)\phi(H_{q-1},x)\phi(B_{n_2},x)\phi(B_{n_3},x)\phi(B_{n_4},x). \qquad (6)\\ \phi(G_2,x) =&\phi(G_3,x)\phi(P_{n_2},x)-\phi(G_3,x)\phi(B_{n_2-1},x)-\\ &\phi(P_{n_2},x)\phi(H_{q-1},x)\phi(B_{n_3},x)\phi(B_{n_4},x).\\ \phi(G_3,x) =&\phi(G_4,x)\phi(P_{n_3},x)-\phi(G_4,x)\phi(B_{n_3-1},x)-\\ &\phi(P_{n_3},x)\phi(H_{q-1},x)\phi(P_{n_4},x).\\ \phi(G_4,x) =&\phi(C_q,x)\phi(P_{n_4},x)-\phi(C_q,x)\phi(B_{n_4-1},x)-\\ &\phi(H_{q-1},x)\phi(B_{n_4},x).\end{aligned}$$

When $x=4$, by Lemma 2.9, we obtain the following results:

$$\phi(G_4,4) =4n_4-4(-1)^qn_4-4qn_4+2-2(-1)^q, \qquad (7)$$

$$\begin{aligned}\phi(G_3,4) =&8n_3n_4-8(-1)^qn_3n_4-16n_3n_4q+4n_3+4n_4-4(-1)^qn_3-\\ &4(-1)^qn_4-4n_4q-4n_3q+2-2(-1)^q, \qquad (8)\end{aligned}$$

$$\begin{aligned}\phi(G_2,4) =&16n_2n_3n_4-16(-1)^qn_2n_3n_4-48n_2n_3n_4q+8n_2n_3+8n_2n_4+\\ &8n_3n_4-8(-1)^qn_2n_3-8(-1)^qn_2n_4-8(-1)^qn_2n_4-\\ &8(-1)^qn_3n_4-16n_2n_4q-16n_2n_3q-16n_3n_4q+4n_2+\\ &4n_3+4n_4-4(-1)^qn_2-4(-1)^qn_3-4(-1)^qn_4-\\ &4n_2q-4n_3q-4n_4q+2-2(-1)^q. \qquad (9)\end{aligned}$$

Then, substituting (7)–(9) into (6), we can get that

$$\begin{aligned}\phi(G,4) =&(32-128q-32(-1)^q)n_1n_2n_3n_4+(16-48q-16(-1)^q)\cdot\\ &(n_1n_2n_3+n_1n_2n_4+n_1n_3n_4+n_2n_3n_4)+(8-16q-8(-1)^q)\cdot\\ &(n_1n_2+n_1n_3+n_1n_4+n_2n_3+n_2n_4+n_3n_4)+(4-4q-\\ &4(-1)^q)\cdot(n_1+n_2+n_3+n_4)+2-2(-1)^q. \ \square \qquad (10)\end{aligned}$$

Note that when $x \neq 4$ and y satisfies the equation $y^2-(x-2)y+1=0$, by Lemma 2.10, we can get the following expressions of $\phi(G)=\phi(H(n;q,n_1,n_2,n_3,n_4))$ by using Mathematica:

$$\begin{aligned}\phi(G_4)=&\frac{1}{(y-1)^2}[-2(-1)^{1+q}y^{1-n_4}+2(-1)^{1+q}y^{-n_4}-2(-1)^{1+q}y^{1+n_4}+\\&2(-1)^{1+q}y^{2+n_4}+y^{-n_4-q}-2y^{1-n_4-q}+y^{2+n_4-q}+y^{-n_4+q}-\\&2y^{1+n_4+q}+y^{2+n_4+q}],\end{aligned}\tag{11}$$

$$\begin{aligned}\phi(G_3)=&\frac{1}{(y-1)^3}[-2(-1)^{1+q}y^{-n_3-n_4}+2(-1)^{1+q}y^{1-n_3-n_4}\\&+2(-1)^{1+q}y^{1+n_3-n_4}-2(-1)^{1+q}y^{2+n_3-n_4}+2(-1)^{1+q}y^{1-n_3+n_4}-\\&2(-1)^{1+q}y^{2-n_3+n_4}-2(-1)^{1+q}y^{2+n_3+n_4}+2(-1)^{1+q}y^{3+n_3+n_4}\\&-y^{-n_3-n_4-q}+3y^{1-n_3-n_4-q}-2y^{2+n_3-n_4-q}-2y^{2-n_3+n_4-q}+\\&y^{2+n_3+n_4-q}+y^{3+n_3+n_4-q}-y^{-n_3-n_4+q}-y^{1-n_3-n_4+q}+\\&2y^{1+n_3-n_4+q}+2y^{1-n_3+n_4+q}-3y^{2+n_3+n_4+q}+y^{3+n_3+n_4+q}],\end{aligned}\tag{12}$$

$$\begin{aligned}\phi(G_2)=&\frac{1}{(y-1)^4}[2(-1)^{1+q}y^{-n_2-n_3-n_4}-2(-1)^{1+q}y^{1-n_2-n_3-n_4}-\\&2(-1)^{1+q}y^{1+n_2-n_3-n_4}+2(-1)^{1+q}y^{2+n_2-n_3-n_4}-\\&2(-1)^{1+q}y^{1-n_2+n_3-n_4}+2(-1)^{1+q}y^{2+n_2+n_3-n_4}-\\&2(-1)^{1+q}y^{3+n_2+n_3-n_4}-2(-1)^{1+q}y^{1-n_2-n_3+n_4}+\\&2(-1)^{1+q}y^{2-n_2+n_3+n_4}-2(-1)^{1+q}y^{2+n_2-n_3+n_4}-\\&2(-1)^{1+q}y^{3+n_2-n_3+n_4}+2(-1)^{1+q}y^{2-n_2+n_3+n_4}-\\&2(-1)^{1+q}y^{3-n_2+n_3+n_4}-2(-1)^{1+q}y^{3+n_2+n_3+n_4}+\\&2(-1)^{1+q}y^{4+n_2+n_3+n_4}+y^{-n_2-n_3-n_4-q}-4y^{1-n_2-n_3-n_4-q}+\\&3y^{2+n_2-n_3-n_4-q}+3y^{2-n_2+n_3-n_4-q}-y^{2+n_2+n_3-n_4-q}-\\&2y^{3+n_2+n_3-n_4-q}+3y^{2-n_2-n_3+n_4-q}-y^{2+n_2-n_3+n_4-q}-\\&2y^{3+n_2-n_3+n_4-q}-y^{2-n_2+n_3+n_4-q}-2y^{3-n_2+n_3+n_4-q}+\\&2y^{3+n_2+n_3+n_4-q}+y^{4+n_2+n_3+n_4-q}+y^{-n_2-n_3-n_4+q}-\\&y^{2-n_2+n_3-n_4+q}+3y^{2+n_2+n_3-n_4+q}-2y^{1-n_2-n_3+n_4+q}-\\&y^{2-n_2-n_3+n_4+q}+3y^{2+n_2-n_3+n_4+q}+3y^{2-n_2+n_3+n_4+q}-\\&4y^{3+n_2+n_3+n_4+q}+y^{4+n_2+n_3+n_4+q}].\end{aligned}\tag{13}$$

Then, substituting (11)–(13) into (6), we can get that

$$\begin{aligned}&(y-1)^5\phi(G)+1-5y+y^{2q}+3y^{1+2q}+5y^{4+2n}-y^{5+2n}-3y^{4+2(n-q)}-\\&y^{5+2(n-q)}+2(-1)^{1+q}y^q-2(-1)^{1+q}y^{1+q}+2(-1)^{1+q}y^{4+2n-q}-\\&2(-1)^{1+q}y^{5+2n-q}=f(n_1,n_2,n_3,n_4;y)\end{aligned}\tag{14}$$

where $n=q+n_1+n_2+n_3+n_4$,

$f(n_1,n_2,n_3,n_4;y)$

$$
\begin{aligned}
&= -4y^{2+2n_1} - 4y^{2+2n_2} - 4y^{2+2n_3} - 4y^{2+2n_4} + y^{2+2n_1+2n_2} + y^{2+2n_1+2n_3} + \\
&\quad y^{2+2n_1+2n_4} + y^{2+2n_2+2n_3} + y^{2+2n_2+2n_4} + y^{2+2n_3+2n_4} + 3y^{3+2n_1+2n_2} + \\
&\quad 3y^{3+2n_1+2n_3} + 3y^{3+2n_1+2n_4} + 3y^{3+2n_2+2n_3} + 3y^{3+2n_2+2n_4} + 3y^{3+2n_3+2n_4} - \\
&\quad 2y^{3+2n_1+2n_2+2n_3} - 2y^{3+2n_1+2n_2+2n_4} - 2y^{3+2n_1+2n_3+2n_4} - 2y^{3+2n_2+2n_3+2n_4} - \\
&\quad 2y^{4+2n_1+2n_2+2n_3} - 2y^{4+2n_1+2n_2+2n_4} - 2y^{4+2n_1+2n_3+2n_4} - 2y^{4+2n_2+2n_3+2n_4} + \\
&\quad 2(-1)^{1+q}y^{1+2n_1+q} + 2(-1)^{1+q}y^{1+2n_2+q} + 2(-1)^{1+q}y^{1+2n_3+q} + \\
&\quad 2(-1)^{1+q}y^{1+2n_4+q} - 2(-1)^{1+q}y^{2+2n_1+q} - 2(-1)^{1+q}y^{2+2n_2+q} - \\
&\quad 2(-1)^{1+q}y^{2+2n_3+q} - 2(-1)^{1+q}y^{2+2n_4+q} - 2(-1)^{1+q}y^{2+2n_1+2n_2+q} - \\
&\quad 2(-1)^{1+q}y^{2+2n_1+2n_3+q} - 2(-1)^{1+q}y^{2+2n_1+2n_4+q} - 2(-1)^{1+q}y^{2+2n_2+2n_3+q} - \\
&\quad 2(-1)^{1+q}y^{2+2n_2+2n_4+q} - 2(-1)^{1+q}y^{2+2n_3+2n_4+q} + 2(-1)^{1+q}y^{3+2n_1+2n_2+q} + \\
&\quad 2(-1)^{1+q}y^{3+2n_1+2n_3+q} + 2(-1)^{1+q}y^{3+2n_1+2n_4+q} + 2(-1)^{1+q}y^{3+2n_2+2n_3+q} + \\
&\quad 2(-1)^{1+q}y^{3+2n_2+2n_4+q} + 2(-1)^{1+q}y^{3+2n_3+2n_4+q} - 2(-1)^{1+q}y^{4+2n_1+2n_2+2n_3+q} - \\
&\quad 2(-1)^{1+q}y^{4+2n_1+2n_2+2n_4+q} - 2(-1)^{1+q}y^{4+2n_1+2n_3+2n_4+q} - \\
&\quad 2(-1)^{1+q}y^{4+2n_2+2n_3+2n_4+q} + 2(-1)^{1+q}y^{3+2n_1+2n_2+2n_3+q} + \\
&\quad 2(-1)^{1+q}y^{3+2n_1+2n_2+2n_4+q} + 2(-1)^{1+q}y^{3+2n_1+2n_3+2n_4+q} + \\
&\quad 2(-1)^{1+q}y^{3+2n_2+2n_3+2n_4+q} + 2y^{1+2n_1+2q} + 2y^{1+2n_2+2q} + 2y^{1+2n_3+2q} + \\
&\quad 2y^{1+2n_4+2q} + 2y^{2+2n_1+2q} + 2y^{2+2n_2+2q} + 2y^{2+2n_3+2q} + 2y^{2+2n_4+2q} - \\
&\quad 3y^{2+2n_1+2n_2+2q} - 3y^{2+2n_1+2n_3+2q} - 3y^{2+2n_1+2n_4+2q} - 3y^{2+2n_2+2n_3+2q} - \\
&\quad 3y^{2+2n_2+2n_4+2q} - 3y^{2+2n_3+2n_4+2q} - y^{3+2n_1+2n_2+2q} - y^{3+2n_1+2n_3+2q} - \\
&\quad y^{3+2n_1+2n_4+2q} - y^{3+2n_2+2n_3+2q} - y^{3+2n_2+2n_4+2q} - y^{3+2n_3+2n_4+2q} + \\
&\quad 4y^{3+2n_1+2n_2+2n_3+2q} + 4y^{3+2n_1+2n_2+2n_4+2q} + 4y^{3+2n_1+2n_3+2n_4+2q} + \\
&\quad 4y^{3+2n_2+2n_3+2n_4+2q}.
\end{aligned}
$$

Theorem 3.3 *No two non-isomorphic graphs of the form $H(n; q, n_1, n_2, n_3, n_4)$ are Laplacian cospectral.*

Proof Suppose that $G' = H(n; q', n_1', n_2', n_3', n_4')$ is Laplacian cospectral to $G = H(n; q, n_1, n_2, n_3, n_4)$. From Lemma 2.12, we have $q = q'$, then $n_1 + n_2 + n_3 + n_4 = n_1' + n_2' + n_3' + n_4'$. In the following, we prove that G and G' are isomorphic.

We consider the coefficient l_{n-2} of $\phi(G, x)$ and the coefficient l'_{n-2} of $\phi(G', x)$. From Lemma 3.1, we can get l_{n-2} of $\phi(G)$ and l'_{n-2} of $\phi(G')$.

$$
\begin{aligned}
(-1)^{n-2}l_{n-2} =& q[\frac{1}{6}(n_1+n_2+n_3+n_4)^3 - (n_1n_2n_3 + n_1n_2n_4 + n_1n_3n_4 + n_2n_3n_4) + \\
& \frac{1}{2}q(n_1+n_2+n_3+n_4)^2 + (1-q)(n_1n_2 + n_1n_3 + n_1n_4 + n_2n_3 + \\
& n_2n_4 + n_3n_4) + (\frac{1}{2}q - \frac{1}{6})(n_1+n_2+n_3+n_4)] + \sum p(\Phi),
\end{aligned}
$$

$$\begin{aligned}(-1)^{n-2}l'_{n-2} =&q'[\frac{1}{6}(n'_1+n'_2+n'_3+n'_4)^3-(n'_1n'_2n'_3+n'_1n'_2n'_4+n'_1n'_3n'_4+n'_2n'_3n'_4)+\\ &\frac{1}{2}q'(n'_1+n'_2+n'_3+n'_4)^2+(1-q')(n'_1n'_2+n'_1n'_3+n'_1n'_4+n'_2n'_3+\\ &n'_2n'_4+n'_3n'_4)+(\frac{1}{2}q'-\frac{1}{6})(n'_1+n'_2+n'_3+n'_4)]+\sum p(\Phi'),\end{aligned}$$

where the summation is over all sub-forests Φ (resp., Φ') of G (G') with $n-2$ (or $n'-2$) edges obtained by deleting two edges both from C_q in G (G').

Because $q=q'$,$n_1+n_2+n_3+n_4=n'_1+n'_2+n'_3+n'_4$, $\sum p(\Phi)=\sum p(\Phi')$, and $(-1)^{n-2}l_{n-2}=(-1)^{n-2}l'_{n-2}$, we have

$$\begin{aligned}&(n_1n_2n_3+n_1n_2n_4+n_1n_3n_4+n_2n_3n_4)-(1-q)(n_1n_2+n_1n_3+n_1n_4+\\ &n_2n_3+n_2n_4+n_3n_4)=(n'_1n'_2n'_3+n'_1n'_2n'_4+n'_1n'_3n'_4+n'_2n'_3n'_4)-\\ &(1-q)(n'_1n'_2+n'_1n'_3+n'_1n'_4+n'_2n'_3+n'_2n'_4+n'_3n'_4).\end{aligned}\tag{15}$$

From Lemma 3.2, we can get $\phi(G,4)$ and $\phi(G',4)$ as follows

$$\begin{aligned}\phi(G,4)=&(32-128q-32(-1)^q)n_1n_2n_3n_4+(16-48q-16(-1)^q)\cdot\\ &(n_1n_2n_3+n_1n_2n_4+n_1n_3n_4+n_2n_3n_4)+(8-16q-8(-1)^q)\cdot\\ &(n_1n_2+n_1n_3+n_1n_4+n_2n_3+n_2n_4+n_3n_4)+(4-4q-\\ &4(-1)^q)\cdot(n_1+n_2+n_3+n_4)+2-2(-1)^q,\\ \phi(G',4)=&(32-128q'-32(-1)^{q'})n'_1n'_2n'_3n'_4+(16-48q'-16(-1)^{q'})\cdot\\ &(n'_1n'_2n'_3+n'_1n'_2n'_4+n'_1n'_3n'_4+n'_2n'_3n'_4)+(8-16q'-8(-1)^{q'})\cdot\\ &(n'_1n'_2+n'_1n'_3+n'_1n'_4+n'_2n'_3+n'_2n'_4+n'_3n'_4)+(4-4q'-\\ &4(-1)^{q'})\cdot(n'_1+n'_2+n'_3+n'_4)+2-2(-1)^{q'}.\end{aligned}$$

Because $\phi(G,4)=\phi(G',4)$, $q=q'$, we have

$$\begin{aligned}&(4-16q-4(-1)^q)n_1n_2n_3n_4+(2-6q-2(-1)^q)(n_1n_2n_3+n_1n_2n_4+\\ &n_1n_3n_4+n_2n_3n_4)+(1-2q-(-1)^q)(n_1n_2+n_1n_3+n_1n_4+n_2n_3+\\ &n_2n_4+n_3n_4)=(4-16q-4(-1)^q)n'_1n'_2n'_3n'_4+(2-6q-2(-1)^q)\cdot\\ &(n'_1n'_2n'_3+n'_1n'_2n'_4+n'_1n'_3n'_4+n'_2n'_3n'_4)+(1-2q-(-1)^q)(n'_1n'_2+\\ &n'_1n'_3+n'_1n'_4+n'_2n'_3+n'_2n'_4+n'_3n'_4).\end{aligned}\tag{16}$$

Because the graphs G and G' have the same Laplacian characteristic polynomials, that is, $\phi(G)=\phi(G')$, by (14) we have $f(n_1,n_2,n_3,n_4;y)=f(n'_1,n'_2,n'_3,n'_4;y)$. Without loss of generality, assume that $n_1\le n_2\le n_3\le n_4$ and $n'_1\le n'_2\le n'_3\le n'_4$, then the smallest degrees of all items of the polynomials $f(n_1,n_2,n_3,n_4;y)$ and $f(n'_1,n'_2,n'_3,n'_4;y)$ on variable y are $2+2n_1$ and $2+2n'_1$, respectively. Then $n_1=n'_1$.

Because $n_1+n_2+n_3+n_4=n'_1+n'_2+n'_3+n'_4$, by (15) and (16), we can deduce the following equations (17) and (18):

$$n_2n_3n_4+(n_1-1+q)(n_2n_3+n_2n_4+n_3n_4)$$

$$= n_2'n_3'n_4' + (n_1 - 1 + q)(n_2'n_3' + n_2'n_4' + n_3'n_4'), \tag{17}$$

$$\begin{aligned}&[n_1(4 - 4(-1)^q - 16q) + (2 - 2(-1)^q - 16q)]n_2n_3n_4 + [n_1(2 - 2(-1)^q - \\ &6q) + (1 - (-1)^q - 2q)](n_2n_3 + n_2n_4 + n_3n_4) = [n_1(4 - 4(-1)^q - 16q) + \\ &(2 - 2(-1)^q - 16q)]n_2'n_3'n_4' + [n_1(2 - 2(-1)^q - 6q) + (1 - (-1)^q - 2q)]\cdot \\ &(n_2'n_3' + n_2'n_4' + n_3'n_4').\end{aligned} \tag{18}$$

By solving the above two equations, we get that

$$\begin{cases} n_2n_3n_4 = n_2'n_3'n_4', \\ n_2n_3 + n_2n_4 + n_3n_4 = n_2'n_3' + n_2'n_4' + n_3'n_4'. \end{cases} \tag{19}$$

Let

$$A = \prod_{i=2}^{4} n_i = \prod_{i=2}^{4} n_i', \quad B = \sum_{2\le i\le j\le 4} n_in_j = \sum_{2\le i\le j\le 4} n_i'n_j'.$$

Obviously, n_2, n_3, n_4 and n_2', n_3', n_4' are the roots of the following equation, respectively,

$$x^3 - (n - q - n_1)x^2 + Bx - A = 0.$$

Hence the graph G' and G are isomorphic. Thus this theorem is proved. □

Theorem 3.4 *Graph $H(n; q, n_1, n_2, n_3, n_4)$ is determined by its Laplacian spectrum.*

Proof Let $G = H(n; q, n_1, n_2, n_3, n_4)$. Assume that G' is Laplacian cospectral to G. By Lemma 2.12, G' is a connected unicyclic graph with n vertices, n edges and cycle C_q. Suppose that G' has x_j' vertices of degree j, for $j = 1, 2, \ldots, \Delta$, where Δ is the maximum degree of G'. From Lemma 2.5, we obtain $7 \le \mu_1(G) = \mu_1(G') \le 7 + \frac{1}{2}$. Then $\Delta = d_1(G') \le 6$. Because $\max\{r_i(L_v(G))\} = 4$ for $i = 1, 2, \ldots, n-1$, by Lemma 2.3, then $\mu_1(L_v(G)) \le 4$. By Lemma 2.2, we can get $\mu_2(L(G)) \le \mu_1(L_v(G)) \le \mu_1(L(G))$, that is $\mu_2(L(G)) \le 4$. Then according to Lemma 2.6, $d_2(G') \le \mu_2(L(G')) = \mu_2(L(G)) \le 4$. So, G' has at most one vertex of degree greater than 4. Therefore, by (i), (ii) and (v) of Lemmas 2.1 and 2.12, we can get the following equations:

$$\begin{cases} x_1' + x_2' + x_3' + x_4' + 1 = n, \\ x_1' + 2x_2' + 3x_3' + 4x_4' + \Delta = 2n, \\ x_1' + 2^2x_2' + 3^2x_3' + 4^2x_4' + \Delta^2 = 4 + 6^2 + (n-5)\cdot 2^2, \\ x_1' + 2^3x_2' + 3^3x_3' + 4^3x_4' + \Delta^3 = 4 + 6^3 + (n-5)\cdot 2^3. \end{cases} \tag{20}$$

By solving Eqs. (20), we get the following result.

$$\begin{cases} x_1' = \frac{1}{6}\Delta^3 - \frac{3}{2}\Delta^2 + \frac{13}{3}\Delta - 4, \\ x_2' = n - \frac{1}{2}\Delta^3 + 4\Delta^2 - \frac{19}{2}\Delta + 16, \\ x_3' = \frac{1}{2}\Delta^3 - \frac{7}{2}\Delta^2 + 7\Delta - 24, \\ x_4' = 11 - \frac{1}{6}\Delta^3 + \Delta^2 - \frac{11}{6}\Delta. \end{cases} \tag{21}$$

Now we consider the following cases.

Case 1 If $\Delta = 1$, then $x_1' = -1, x_2' = n + 10, x_3' = -20, x_4' = 10$.

Case 2 If $\Delta = 2$, then $x'_1 = 0, x'_2 = n + 9, x'_3 = -20, x'_4 = 10$.

Case 3 If $\Delta = 3$, then $x'_1 = 0, x'_2 = n + 10, x'_3 = -21, x'_4 = 10$.

Case 4 If $\Delta = 4$, then $x'_1 = 0, x'_2 = n + 10, x'_3 = -20, x'_4 = 9$.

Case 5 If $\Delta = 6$, then $x'_1 = 4, x'_2 = n - 5, x'_3 = 0, x'_4 = 0$.

Because x'_1, x'_2, x'_3 and x'_4 are nonnegative integers, we have $\Delta = 6$. Therefore, G' is only the graph $H(n; q', n'_1, n'_2, n'_3, n'_4)$. According to Theorem 3.3, we can know that the graph $G' = H(n; q', n'_1, n'_2, n'_3, n'_4)$ is isomorphic to the graph $G = H(n; q, n_1, n_2, n_3, n_4)$. Thus, the theorem is proved. □

For a graph, its Laplacian eigenvalues determine the eigenvalues of its complement [11], so the complements of all unicyclic graphs $H(n; q, n_1, n_2, n_3, n_4)$ are determined by their Laplacian spectra.

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