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Normality Criteria of Meromorphic Functions Sharing a Meromorphic Function

Pai YANG[1,*], Xue WANG[2], Xuecheng PANG[3]

1. *College of Applied Mathematics, Chengdu University of Information Technology, Sichuan* 610225*, P. R. China;*
2. *School of Mathematics and Finance, Fuyang Teachers College, Anhui* 236041*, P. R. China;*
3. *Department of Mathematics, East China Normal University, Shanghai* 200062*, P. R. China*

Abstract Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $\psi(\not\equiv 0)$ be a meromorphic function in D all of whose poles are simple. Suppose that, for each $f \in \mathcal{F}$, $f \neq 0$ in D. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D.

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1. Introduction and main result

We use the following notations. Let $\mathbb{C}$ be complex plane and D be a domain in $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z \mid |z - z_0| < r\}$, $\Delta'(z_0, r) = \{z \mid 0 < |z - z_0| < r\}$, $\Delta = \Delta(0, 1)$ and $\Gamma(z_0, r) = \{z \mid |z - z_0| = r\}$. Let $n(r, f)$ denote the number of poles of $f(z)$ in $\Delta(0, r)$ (counting multiplicity).

We write $f_n \overset{\chi}{\Longrightarrow} f$ in D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ in D if the convergence is in the Euclidean metric.

A family $\mathcal{F}$ of functions meromorphic in D is normal in D if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges spherically uniformly on compact subsets of D.

Let f, g and ψ be meromorphic functions in D. If $f = \psi$ whenever $g = \psi$ and $g = \psi$ whenever $f = \psi$ in D, we say f and g share ψ IM (ignoring multiplicity) [1] in D, or we just say f and g share ψ in D for short.

In 1979, Gu [2] proved the following result.

Theorem A *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let k be a positive integer and a be a nonzero constant. If for each $f \in \mathcal{F}$ and $z \in D$, $f(z) \neq 0$ and $f^{(k)}(z) \neq a$, then $\mathcal{F}$ is normal in D.*

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* Corresponding author
E-mail address: yangpai@cuit.edu.cn (Pai YANG); fywxue@163.com (Xue WANG); xcpang@math.ecnu.edu.cn (Xuecheng PANG)

Yang [3] and Schwick [4] proved that Theorem A still holds if a is replaced by a holomorphic function $\Psi(\not\equiv 0)$ in Theorem A.

Xu [5] improved Theorem A by the ideas of shared values and obtained the following result.

Theorem B *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $\psi(\not\equiv 0)$ be a holomorphic function in D all of whose zeros are simple. Suppose that, for each $f \in \mathcal{F}$, f has only multiple poles and $f \neq 0$. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D.*

Xu did not know whether the conditions ψ has only simple zero and f has only multiple poles in D are necessary or not in Theorem B.

It is natural to ask whether Theorem B still holds when ψ is meromorphic. In this paper, we investigate the problem and obtain the following result.

Theorem 1.1 *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $\psi(\not\equiv 0)$ be a meromorphic function in D. Suppose that*

(a) ψ has only simple poles in D,

(b) for each $f \in \mathcal{F}$, $f \neq 0$ in D,

(c) for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D.

Then $\mathcal{F}$ is normal in D.

Remark 1.2 The condition (a) is necessary. Let $f_n(z) = \frac{1}{nz}$, $n = 1, 2, 3, \ldots$, $\psi(z) = -\frac{1}{z^2}$ and $D = \Delta$. Obviously, (b) and (c) are satisfied, but $\{f_n(z)\}$ fails to be normal at 0.

Remark 1.3 The condition (b) is necessary. Let $f_n(z) = e^z + \frac{1}{nz}$, $n = 1, 2, 3, \ldots$, $\psi(z) = e^z$ and $D = \Delta$. Obviously, (a) and (c) are satisfied. Since $f_n(0) = \infty$ and $f_n(-\frac{1}{n}) = e^{-\frac{1}{n}} - 1 \to 0$ as $n \to \infty$, $\{f_n(z)\}$ fails to be normal at 0.

Remark 1.4 Obviously, the condition (c) is necessary.

2. Some lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 2.1 ([6, Lemma 2]) *Let $\mathcal{F}$ be a family of functions meromorphic in D, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0 \leq \alpha \leq k$,*

(a) points z_n, $z_n \to z_0$;

(b) functions $f_n \in \mathcal{F}$; and

(c) positive numbers $\rho_n \to 0$

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \stackrel{\chi}{\Longrightarrow} g(\zeta)$ in $\mathbb{C}$, where g is a nonconstant meromorphic function in $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$.

Lemma 2.2 ([7, Theorem 1]) *Let f be a meromorphic function in $\mathbb{C}$. If $f(z) \neq 0$ and $f'(z) \neq 1$*

for all $z \in \mathbb{C}$, then f is constant.

Lemma 2.3 ([8, Theorem 3]) *Let $\psi \not\equiv 0$ be a meromorphic function in D and $k \in \mathbb{N}$. Let $\mathcal{F}$ be a family of meromorphic functions in D, such that f and $f^{(k)} - \psi$ have no zeros and f and ψ have no common poles for each $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

Lemma 2.4 ([5, Lemma 6]) *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $\psi(\neq 0)$ be a holomorphic function in D. Suppose that, for each $f \in \mathcal{F}$, $f \neq 0$. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D.*

Lemma 2.5 ([9, Theorem 1]) *Let f be a transcendental meromorphic function in $\mathbb{C}$, all but finitely many of whose zeros are multiple, and let $R \not\equiv 0$ be a rational function. Then $f' - R$ has infinitely many zeros.*

Lemma 2.6 *Let $k \geq 1$ be an integer, and let $Q(z)$ be a polynomial in $\mathbb{C}$, where $Q(0) \neq 0$. Then $H'(z) = \frac{1}{z}$ has at least one non-zero solution, where $H(z) = \frac{1}{z^k Q(z)}$.*

Proof Let

$$T(z) = kQ(z) + zQ'(z) + z^k Q^2(z), \quad s = k + 2\deg(Q(z)),$$

where $\deg(Q(z))$ is the degree of $Q(z)$. Obviously, $s \geq 1$, $T(z)$ is a polynomial of degree s, and $T(z)$ has exactly s zeros. Let z_0 be a zero of $T(z)$. Since $T(0) = kQ(0) \neq 0$, we have $z_0 \neq 0$. Now,

$$T(z_0) = kQ(z_0) + z_0 Q'(z_0) + z_0^k Q^2(z_0) = 0, \tag{1}$$

and hence

$$-\frac{kQ(z_0) + z_0 Q'(z_0)}{z_0^{k+1} Q^2(z_0)} = \frac{1}{z_0}.$$

Observing that $H'(z_0) = -\frac{kQ(z_0)+z_0Q'(z_0)}{z_0^{k+1}Q^2(z_0)}$, we have z_0 is a non-zero solution of $H'(z) = \frac{1}{z}$. $\square$

3. Proof of Theorem 1.1

Proof Since normality is a local property, it suffices to show that $\mathcal{F}$ is normal in a neighborhood of each point in D. By Lemma 2.4, we only need to prove that $\mathcal{F}$ is normal in a neighborhood of each pole of $\psi(z)$ in D.

Without loss of generality, we may assume $D = \Delta$ and, for $z \in \Delta$,

$$\psi(z) = \frac{\varphi(z)}{z},$$

where $\varphi(0) = 1$ and $\varphi(z) \neq 0, \infty$ in Δ.

If $f \in \mathcal{F}$, $f(0) \neq \infty$, then there exists $\delta > 0$ such that $f'(z) \neq \psi(z)$ in $\Delta(0, \delta)$. By the conditions of Theorem 1.1, for each $h \in \mathcal{F}$, we have $h'(z) \neq \psi(z)$ in $\Delta(0, \delta)$. By Lemma 2.3, $\mathcal{F}$ is normal in $\Delta(0, \delta)$.

Now, we consider $f(0) = \infty$.

We claim that there exists $\delta > 0$ such that $f'(z) \neq \psi(z)$ in $\Delta'(0, \delta)$, and hence by the conditions of Theorem 1.1, for each $h \in \mathcal{F}$, we have $h'(z) \neq \psi(z)$ in $\Delta'(0, \delta)$. Otherwise, $f'(z) \equiv$

$\psi(z)$ in $\Delta(0,\delta)$, and hence $z=0$ is a multiple pole of $\psi(z)$. A contradiction.

Next, we will prove $\mathcal{F}$ is normal at $z=0$. Suppose that $\mathcal{F}$ is not normal at $z=0$. For each $h\in\mathcal{F}$, we have that $h'(z)\neq\psi(z)$ and $h(z)\neq 0$ in $\Delta'(0,\delta)$. By Lemma 2.4, $\mathcal{F}$ is normal in $\Delta'(0,\delta)$. Then there exists a sequence of functions $\{f_n(z)\}\subset\mathcal{F}$ such that

(a) $f_n(z)\stackrel{\chi}{\Longrightarrow}f(z)$ in $\Delta'(0,\delta)$,

(b) no subsequence of $\{f_n(z)\}$ is normal at 0,

where $f(z)$ is a meromorphic function or $f(z)\equiv\infty$ in $\Delta'(0,\delta)$.

We claim that $f(z)\equiv 0$ in $\Delta'(0,\delta)$. Suppose that $f(z)\not\equiv 0$ in $\Delta'(0,\delta)$. Since $f_n\neq 0$, we have $\frac{1}{f_n}\stackrel{\chi}{\Longrightarrow}\frac{1}{f}$ in $\Delta'(0,\delta)$. Clearly for each n, $\frac{1}{f_n}$ is holomorphic function in $\Delta(0,\delta)$. By the maximum principle and Weierstrass' theorem, we get that $\{\frac{1}{f_n}\}_{n=1}^{\infty}$ converges to a certain holomorphic function in $\Delta(0,\delta)$, and hence $\mathcal{F}$ is normal at $z=0$. A contradiction.

Set $\{g_n(z)\}=\{g_n(z)|\ g_n(z)=zf_n(z), z\in\Delta(0,\delta), n=1,2,3,\ldots\}$. Since $f_n(z)\neq 0$ in $\Delta(0,\delta)$ and $f_n(0)=\infty$ for each n, we have $g_n(z)\neq 0$ in $\Delta(0,\delta)$ for each n.

We first prove that $\{g_n\}$ is normal at 0. Suppose that $\{g_n\}$ is not normal at 0. By Lemma 2.1, there exist points $z_n\to 0$, positive numbers $\rho_n\to 0$ and a subsequence (we continue to call $\{g_n\}$) such that

$$G_n(\zeta)=\frac{g_n(z_n+\rho_n\zeta)}{\rho_n}\stackrel{\chi}{\Longrightarrow}G(\zeta),$$

where $G(\zeta)$ is a nonconstant meromorphic function and $G(\zeta)\neq 0$ in $\mathbb{C}$.

We distinguish two cases.

Case I $\frac{z_n}{\rho_n}\to\infty$. Obviously,

$$\begin{aligned}
g_n'(z)&=f_n(z)+zf_n'(z),\\
G_n'(\zeta)&=g_n'(z_n+\rho_n\zeta)=f_n(z_n+\rho_n\zeta)+(z_n+\rho_n\zeta)f_n'(z_n+\rho_n\zeta)\\
&=\frac{(z_n+\rho_n\zeta)f_n(z_n+\rho_n\zeta)}{\rho_n}\,\frac{\rho_n}{z_n+\rho_n\zeta}+(z_n+\rho_n\zeta)f_n'(z_n+\rho_n\zeta).
\end{aligned}$$

Observing that

$$\frac{\rho_n}{z_n+\rho_n\zeta}\Rightarrow 0,\quad \frac{(z_n+\rho_n\zeta)f_n(z_n+\rho_n\zeta)}{\rho_n}=\frac{g_n(z_n+\rho_n\zeta)}{\rho_n}\stackrel{\chi}{\Longrightarrow}G(\zeta)$$

in $\mathbb{C}$, we have

$$\begin{aligned}
(z_n+\rho_n\zeta)f_n'(z_n+\rho_n\zeta)-\varphi(z_n+\rho_n\zeta)&=(z_n+\rho_n\zeta)f_n'(z_n+\rho_n\zeta)-(z_n+\rho_n\zeta)\psi(z_n+\rho_n\zeta)\\
&=(z_n+\rho_n\zeta)[f_n'(z_n+\rho_n\zeta)-\psi(z_n+\rho_n\zeta)]\\
&\Rightarrow G'(\zeta)-1
\end{aligned}$$

in $\mathbb{C}\backslash E_1$, where $E_1=\{z|\,G(z)=\infty\}$. Clearly, $f_n'(z_n+\rho_n\zeta)-\psi(z_n+\rho_n\zeta)\neq 0$ for sufficiently large n. By Hurwitz's theorem, we have that either $G'(\zeta)-1\equiv 0$ or $G'(\zeta)-1\neq 0$ in $\mathbb{C}$. If $G'(\zeta)-1\equiv 0$ in $\mathbb{C}$, then $G(\zeta)$ has at least one zero which contradicts $G(\zeta)\neq 0$. If $G'(\zeta)-1\neq 0$, then by the fact that $G(\zeta)\neq 0$ and Lemma 2.2, $G(\zeta)$ is constant in $\mathbb{C}$. A contradiction.

Case II $\frac{z_n}{\rho_n}\not\to\infty$. Taking a subsequence and renumbering, we may assume that $\frac{z_n}{\rho_n}\to\alpha$, where

α is a finite complex number. Then

$$0 \neq \frac{g_n(\rho_n\zeta)}{\rho_n} = \frac{g_n(z_n + \rho_n(\zeta - \frac{z_n}{\rho_n}))}{\rho_n} = G_n(\zeta - \frac{z_n}{\rho_n}) \Rightarrow G(\zeta - \alpha) = \widetilde{G}(\zeta).$$

Obviously, $\widetilde{G}(\zeta) \neq 0$.

Set $H_n(\zeta) = f_n(\rho_n\zeta)$. We have

$$H_n(\zeta) = \frac{\rho_n\zeta f_n(\rho_n\zeta)}{\rho_n}\frac{1}{\zeta} = \frac{g_n(\rho_n\zeta)}{\rho_n}\frac{1}{\zeta} \stackrel{\chi}{\Longrightarrow} \frac{\widetilde{G}(\zeta)}{\zeta} = H(\zeta) \tag{2}$$

in $\mathbb{C}$. Obviously, $H(0) = \infty$ and $H(\zeta) \neq 0$. By (2),

$$H_n'(\zeta) - \rho_n\psi(\rho_n\zeta) = \rho_n(f_n'(\rho_n\zeta) - \psi(\rho_n\zeta)) \Rightarrow H'(\zeta) - \frac{1}{\zeta} \tag{3}$$

in $\mathbb{C}\backslash E_2$, where $E_2 = \{z | \, H(z) = \infty\}$.

We claim that $H'(\zeta) = \frac{1}{\zeta}$ if and only if $\zeta = 0$. For sufficiently large n, $f_n(\rho_n\zeta) - \psi(\rho_n\zeta) \neq 0$ in $\mathbb{C}\backslash\{0\}$. By Hurwitz's theorem and (3), we have that either $H'(\zeta) \equiv \frac{1}{\zeta}$ or $H'(\zeta) \neq \frac{1}{\zeta}$ in $\mathbb{C}\backslash\{0\}$. If $H'(\zeta) \equiv \frac{1}{\zeta}$ in $\mathbb{C}\backslash\{0\}$, then $H(\zeta)$ is a multi-valued function. A contradiction. If $H'(\zeta) \neq \frac{1}{\zeta}$ in $\mathbb{C}\backslash\{0\}$, then $H'(\zeta) = \frac{1}{\zeta}$ if and only if $\zeta = 0$.

Since $H'(\zeta) = \frac{1}{\zeta}$ if and only if $\zeta = 0$ and $H(\zeta) \neq 0$, by Lemma 2.5, $H(\zeta)$ is a rational function. Since $H(0) = \infty$ and $H(\zeta) \neq 0$ in $\mathbb{C}$, we have $H(\zeta) = \frac{1}{\zeta^k Q(\zeta)}$, where $k \geq 1$ is an integer, $Q(z)$ is a polynomial in $\mathbb{C}$ and $Q(0) \neq 0$. By Lemma 2.6, $H'(z) = \frac{1}{z}$ has at least one non-zero solution, which results in a contradiction. Now, we have shown that $\{g_n\}$ is normal at 0.

We claim that 0 is a pole of order 2 of $f_n'(z) - \psi(z)$ for sufficiently large n, and hence $f_n'(z) - \psi(z) \neq 0$ in $\Delta(0, \delta)$ for sufficiently large n. Since $\{g_n\}$ is normal at 0 and $g_n(z) = zf_n(z) \Rightarrow zf(z) \equiv 0$ in $\Delta'(0, \delta)$, we have $g_n(z) \Rightarrow 0$ in $\Delta(0, \delta)$, and then

$$g_n(0) \to 0 \text{ as } n \to \infty. \tag{4}$$

By the fact that $f_n(0) = \infty$, we have

$$g_n(0) = zf_n(z)|_{z=0} \neq 0. \tag{5}$$

By (4) and (5), we have $z = 0$ is a simple pole of $f_n(z)$ for sufficiently large n. Obviously, $z = 0$ is a pole of order 2 of $f_n'(z) - \psi(z)$ for sufficiently large n.

Now, we have

$$f_n'(z) - \psi(z) = \{\frac{g_n(z)}{z}\}' - \psi(z) = \frac{zg_n'(z) - g_n(z)}{z^2} - \frac{\varphi(z)}{z} \Rightarrow -\frac{\varphi(z)}{z}$$

on $\Gamma(0, \frac{\delta}{2})$. By argument principle, for sufficiently large n,

$$n\big(\frac{\delta}{2}, \frac{1}{f_n'(z) - \psi(z)}\big) - n\big(\frac{\delta}{2}, f_n'(z) - \psi(z)\big) = n\big(\frac{\delta}{2}, \frac{1}{-\frac{\varphi(z)}{z}}\big) - n\big(\frac{\delta}{2}, -\frac{\varphi(z)}{z}\big).$$

On the one hand, since $g_n(z) = zf_n(z) \Rightarrow 0$ in $\Delta(0, \delta)$, we have, for sufficiently large n, $f_n(z)$ has only one simple pole in $\Delta(0, \delta/2)$, and then for sufficiently large n,

$$n\big(\frac{\delta}{2}, \frac{1}{f_n'(z) - \psi(z)}\big) - n\big(\frac{\delta}{2}, f_n'(z) - \psi(z)\big) = 0 - 2 = -2.$$

On the other hand,

$$n\Big(\frac{\delta}{2},\frac{1}{-\frac{\varphi(z)}{z}}\Big)-n\Big(\frac{\delta}{2}.-\frac{\varphi(z)}{z}\Big)=0-1=-1.$$

This leads to a contradiction. This completes the proof of the theorem. □

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